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SOME CONVERGENCE RESULTS FOR MIXED TYPE TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study a new two-step iteration scheme of mixed type for two total asymptotically nonexpansive self-mappings and two total asymptotically nonexpansive non-self mappings and establish some strong convergence theorems in the framework of Banach spaces. Our results extend and generalize several results from the current existing literature.

1. Introduction

Let C be a nonempty subset of a real Banach space E. Let $T: C \to C$ be a nonlinear mapping. Then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of four mappings S_1, S_2, T_1 and T_2 will be denoted by $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$. Throughout this paper $\mathbf{N}$

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denotes the set of all positive integers and $\mathbb{R}^+$ denotes the set of all positive real numbers.

Definition 1.1. A mapping $T\colon C \to C$ is said to be asymptotically nonexpansive [6] if there exists a positive sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n(x) - T^n(y)\| \quad \leq \quad k_n\|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a generalization of the class of nonxpansive mappings. They proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C, then T has a fixed point.

Definition 1.2. A mapping $T\colon C \to C$ is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \Big(\|T^n(x) - T^n(y)\| - \|x - y\|\Big) \leq 0. \tag{1.2}$$

Observe that if we define

$$c_n = \limsup_{n\to\infty} \sup_{x,y\in C} \Big(\|T^n(x) - T^n(y)\| - \|x - y\|\Big) \text{ and } \nu_n = \max\{0, c_n\},$$

then $\nu_n \to 0$ as $n \to \infty$. It follows that (1.2) is reduced to

$$\|T^n(x) - T^n(y)\| \quad \leq \quad \|x - y\| + \nu_n \tag{1.3}$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2]. It is known [9], that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense mapping, T has a fixed point. It is worth mentioning that the class of mapping which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Albert *et al.* [1] introduced the concept of total asymptotically nonexpansive mappings in 2006.

Definition 1.3. A mapping $T\colon C \to C$ is said to be total asymptotically nonexpansive [1] if

$$\|T^n(x) - T^n(y)\| \quad \leq \quad \|x - y\| + \mu_n\psi(\|x - y\|) + \nu_n, \tag{1.4}$$

for all $x, y \in C$ and $n \in \mathbf{N}$, where $\{\mu_n\}$ and $\{\nu_n\}$ are nonnegative real sequences such that $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$.

From the above definition, we see that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings as a special case; see also [5] for more details.

Remark 1.4. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive with $\nu_n = 0$, $\mu_n = k_n - 1$ for all $n \geq 1$, $\psi(t) = t$, $t \geq 0$.

Definition 1.5. A subset C of a Banach space E is said to be a retract of E if there exists a continuous mapping $P\colon E \to C$ (called a retraction) such that $P(x) = x$ for all $x \in C$. If, in addition P is nonexpansive, then P is said to be a nonexpansive retract of E.

If $P\colon E \to C$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

In 2003, Chidume *et al.* [3] defined non-self asymptotically nonexpansive mappings.

Definition 1.6. A non-self mapping $T\colon C \to E$ is said to be asymptotically nonexpansive [3] if there exists a positive sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq k_n\|x - y\| \tag{1.5}$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

Recently, Yolacan and Kiziltunc [18] defined non-self total asymptotically nonexpansive mappings.

Definition 1.7. Let C be a nonempty closed and convex subset of a Banach space E. A non-self mapping $T\colon C \to E$ is said to be total asymptotically nonexpansive [18] if there exist sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0,\infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq \|x - y\| + \mu_n\psi(\|x - y\|) + \nu_n, \tag{1.6}$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

Chidume *et al.* [3] studied the following iteration process for non-self asymptotically nonexpansive mappings:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n + (1-\alpha_n)x_n),\ n \geq 1, \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Chidume *et al.* [4] studied the following iteration scheme:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n + (1-\alpha_n)x_n),\ n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and C is a nonempty closed convex subset of a real uniformly convex Banach space E, P is a nonexpansive retraction of E onto C, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings in the intermediate sense in the framework of uniformly convex Banach spaces.

In 2006, Wang [16] generalized the iteration process (1.8) as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n),\ n \geq 1, \end{aligned} \tag{1.9}$$

where $T_1, T_2 \colon C \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings.

In 2012, Guo *et al.* [7] generalized the iteration process (1.9) as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n),\ n \geq 1, \end{aligned} \tag{1.10}$$

where $S_1, S_2 \colon C \to C$ are two asymptotically nonexpansive self mappings and $T_1, T_2 \colon C \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for mixed type asymptotically nonexpansive mappings.

Now, we define the mixed type iteration scheme.

Let E be a real Banach space, C be a nonempty closed convex subset of E and $P \colon E \to C$ be a nonexpansive retraction of E onto C. Let $S_1, S_2 \colon C \to C$ be two total asymptotically nonexpansive self mappings and $T_1, T_2 \colon C \to E$

are two total asymptotically nonexpansive non-self mappings. Then the mixed type iteration scheme for the mentioned mappings is as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\ y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n),\ n \geq 1, \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$.

The purpose of this paper is to study newly defined mixed type iteration scheme (1.11) and establish some strong convergence theorems in the setting of real Banach spaces.

2. Preliminaries

A mapping $T\colon C \to C$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) [14] if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0)=0$, $f(t)>0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where

$$d(x, F(T)) = \inf\{\|x-p\| : p \in F(T)\}.$$

A mapping $T\colon C \to C$ is called:

(1) *demicompact* if any bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ is convergent, then it has a convergent subsequence $\{x_{n_i}\}$;

(2) semi-compact (or hemicompact) if any bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\} \to 0$ as $n \to \infty$ has a convergent subsequence.

Every demicompact mapping is semi-compact but the converse is not true in general.

Senter and Dotson [14] have approximated fixed point of a nonexpansive mapping T by Mann iterates whereas Maiti and Ghosh [10] and Tan and Xu [15] have approximated the fixed points using Ishikawa iterates under *condition* (A) of [14]. Tan and Xu [15] pointed out that *condition* (A) is weaker than the compactness of C.

Proposition 2.1. *Let C be a nonempty subset of a Banach space E which is also a nonexpansive retract of E, and $T_1, T_2\colon C \to E$ be two total asymptotically nonexpansive non-self mappings. Then there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0,\infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0)=0$ such that*

$$\|T_1(PT_1)^{n-1}(x) - T_1(PT_1)^{n-1}(y)\| \leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n, \tag{2.1}$$

and

$$\|T_2(PT_2)^{n-1}(x) - T_2(PT_2)^{n-1}(y)\| \leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n, \tag{2.2}$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

Proof. Since $T_1, T_2 \colon C \to E$ are two total asymptotically nonexpansive non-self mappings, there exist nonnegative real sequences $\{\mu'_n\}$, $\{\mu''_n\}$, $\{\nu'_n\}$ and $\{\nu''_n\}$ in $[0, \infty)$ with $\mu'_n, \mu''_n \to 0$ and $\nu'_n, \nu''_n \to 0$ as $n \to \infty$ and strictly increasing continuous functions $\psi_1, \psi_2 \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi_1(0) = 0$ and $\psi_2(0) = 0$ such that

$$\|T_1(PT_1)^{n-1}(x) - T_1(PT_1)^{n-1}(y)\| \le \|x - y\| + \mu'_n \psi_1(\|x - y\|) + \nu'_n, \quad (2.3)$$

and

$$\|T_2(PT_2)^{n-1}(x) - T_2(PT_2)^{n-1}(y)\| \le \|x - y\| + \mu''_n \psi_2(\|x - y\|) + \nu''_n, \quad (2.4)$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

Setting

$$\mu_n = \max\{\mu'_n, \mu''_n\}, \ \nu_n = \max\{\nu'_n, \nu''_n\}$$

and

$$\psi(a) = \max\{\psi_1(a), \psi_2(a)\}, \text{ for } a \ge 0,$$

then we get that, there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0, \infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\begin{aligned} \|T_1(PT_1)^{n-1}(x) - T_1(PT_1)^{n-1}(y)\| &\le \|x - y\| + \mu'_n \psi_1(\|x - y\|) + \nu'_n \\ &\le \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n \end{aligned}$$

and

$$\begin{aligned} \|T_2(PT_2)^{n-1}(x) - T_2(PT_2)^{n-1}(y)\| &\le \|x - y\| + \mu''_n \psi_2(\|x - y\|) + \nu''_n \\ &\le \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n, \end{aligned}$$

for all $x, y \in C$ and $n \in \mathbf{N}$. This completes the proof. □

Proposition 2.2. *Let C be a nonempty subset of a Banach space E and let $S_1, S_2 \colon C \to C$ be two total asymptotically nonexpansive self mappings. Then there exist nonnegative real sequences $\{\mu_{n_1}\}$ and $\{\nu_{n_1}\}$ in $[0, \infty)$ with $\mu_{n_1} \to 0$ and $\nu_{n_1} \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that*

$$\|S_1^n(x) - S_1^n(y)\| \le \|x - y\| + \mu_{n_1} \psi(\|x - y\|) + \nu_{n_1}, \quad (2.5)$$

and

$$\|S_2^n(x) - S_2^n(y)\| \le \|x - y\| + \mu_{n_1} \psi(\|x - y\|) + \nu_{n_1}, \quad (2.6)$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

Proof. Since $S_1, S_2 \colon C \to C$ are two total asymptotically nonexpansive self mappings, there exist nonnegative real sequences $\{\mu'_{n_1}\}$, $\{\mu''_{n_1}\}$, $\{\nu'_{n_1}\}$ and $\{\nu''_{n_1}\}$ in $[0, \infty)$ with $\mu'_{n_1}, \mu''_{n_1} \to 0$ and $\nu'_{n_1}, \nu''_{n_1} \to 0$ as $n \to \infty$ and strictly increasing continuous functions $\psi_3, \psi_4 \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi_3(0) = 0$ and $\psi_4(0) = 0$ such that

$$\|S_1^n(x) - S_1^n(y)\| \leq \|x - y\| + \mu'_{n_1}\psi_3(\|x - y\|) + \nu'_{n_1} \tag{2.7}$$

and

$$\|S_2^n(x) - S_2^n(y)\| \leq \|x - y\| + \mu''_{n_1}\psi_4(\|x - y\|) + \nu''_{n_1}, \tag{2.8}$$

for all $x, y \in C$ and $n \in \mathbf{N}$.

Setting

$$\mu_{n_1} = \max\{\mu'_{n_1}, \mu''_{n_1}\},\ \nu_{n_1} = \max\{\nu'_{n_1}, \nu''_{n_1}\}$$

and

$$\psi(a) = \max\{\psi_3(a), \psi_4(a)\},\ \text{for } a \geq 0,$$

then we get that, there exist nonnegative real sequences $\{\mu_{n_1}\}$ and $\{\nu_{n_1}\}$ in $[0, \infty)$ with $\mu_{n_1} \to 0$ and $\nu_{n_1} \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\begin{aligned} \|S_1^n(x) - S_1^n(y)\| &\leq \|x - y\| + \mu'_{n_1}\psi_3(\|x - y\|) + \nu'_{n_1} \\ &\leq \|x - y\| + \mu_{n_1}\psi(\|x - y\|) + \nu_{n_1} \end{aligned}$$

and

$$\begin{aligned} \|S_2^n(x) - S_2^n(y)\| &\leq \|x - y\| + \mu''_{n_1}\psi_4(\|x - y\|) + \nu''_{n_1} \\ &\leq \|x - y\| + \mu_{n_1}\psi(\|x - y\|) + \nu_{n_1}, \end{aligned}$$

for all $x, y \in C$ and $n \in \mathbf{N}$. This completes the proof. □

Next, we need the following useful lemma to prove our main results.

Lemma 2.3. ([15]) *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of non-negative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n,\ \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then

(i) $\lim_{n\to\infty} \alpha_n$ *exists;*

(ii) *In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

In this section, we prove some strong convergence theorems of iteration scheme (1.11) for two total asymptotically nonexpansive self mappings and two total asymptotically nonexpansive non-self mappings in the framework of real Banach spaces. First, we shall need the following lemma.

Lemma 3.1. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $S_1, S_2 \colon C \to C$ be two total asymptotically nonexpansive self mappings with sequences $\{\mu_{n_1}\}$ and $\{\nu_{n_1}\}$ as defined in Proposition 2.2 and $T_1, T_2 \colon C \to E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition 2.1 and*

$$F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset.$$

Let $\{x_n\}$ be the iteration scheme defined by (1.11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$ and the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \mu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
(ii) *there exists a constant $M > 0$ such that $\psi(t) \leq M\,t$, $t \geq 0$.*

Then $\lim_{n\to\infty} \|x_n - q\|$ and $\lim_{n\to\infty} d(x_n, F)$ both exist for all $q \in F$.

Proof. Let $q \in F$ and let $h_n = \max\{\mu_{n_1}, \mu_n\}$, $l_n = \max\{\nu_{n_1}, \nu_n\}$ with $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty} l_n < \infty$. From (1.11), we have

$$\begin{aligned} \|y_n - q\| &= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) - P(q)\| \\ &\leq \|(1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n - q\| \\ &= \|(1-\beta_n)(S_2^n x_n - q) + \beta_n (T_2(PT_2)^{n-1}x_n - q)\| \\ &\leq (1-\beta_n)\|S_2^n x_n - q\| + \beta_n\|T_2(PT_2)^{n-1}x_n - q\| \\ &\leq (1-\beta_n)[\|x_n - q\| + \mu_{n_1}\psi(\|x_n - q\|) + \nu_{n_1}] + \beta_n[\|x_n - q\| \\ &\quad +\mu_n\psi(\|x_n - q\|) + \nu_n] \\ &\leq (1-\beta_n)[\|x_n - q\| + h_n M\|x_n - q\| + l_n] + \beta_n[\|x_n - q\| \\ &\quad +h_n M\|x_n - q\| + l_n] \\ &= (1-\beta_n)[(1+h_n M)\|x_n - q\| + l_n] \\ &\quad +\beta_n[(1+h_n M)\|x_n - q\| + l_n] \\ &\leq (1+h_n M)\|x_n - q\| + l_n. \end{aligned} \tag{3.1}$$

Again using (1.11), we have

$$\begin{aligned}
\|x_{n+1}-q\| &= \|P((1-\alpha_n)S_1^n x_n+\alpha_n T_1(PT_1)^{n-1}y_n)-P(q)\| \\
&\le \|(1-\alpha_n)S_1^n x_n+\alpha_n T_1(PT_1)^{n-1}y_n-q\| \\
&= \|(1-\alpha_n)(S_1^n x_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)\| \\
&\le (1-\alpha_n)\|S_1^n x_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-q\| \\
&\le (1-\alpha_n)[\|x_n-q\|+\mu_{n_1}\psi(\|x_n-q\|)+\nu_{n_1}]+\alpha_n[\|y_n-q\| \\
&\quad +\mu_n\psi(\|y_n-q\|)+\nu_n] \\
&\le (1-\alpha_n)[\|x_n-q\|+h_nM\|x_n-q\|+l_n]+\alpha_n[\|y_n-q\| \\
&\quad +h_nM\|y_n-q\|+l_n] \\
&= (1-\alpha_n)[(1+h_nM)\|x_n-q\|+l_n] \\
&\quad +\alpha_n[(1+h_nM)\|y_n-q\|+l_n] \\
&= (1-\alpha_n)(1+h_nM)\|x_n-q\| \\
&\quad +\alpha_n(1+h_nM)\|y_n-q\|+l_n. \qquad (3.2)
\end{aligned}$$

Using equation (3.1) in (3.2), we obtain

$$\begin{aligned}
\|x_{n+1}-q\| &\le (1-\alpha_n)(1+h_nM)\|x_n-q\| \\
&\quad +\alpha_n(1+h_nM)[(1+h_nM)\|x_n-q\|+l_n]+l_n \\
&\le [(1-\alpha_n)+\alpha_n](1+h_nM)^2\|x_n-q\|+(2+h_nM)l_n \\
&= (1+h_nM)^2\|x_n-q\|+(2+h_nM)l_n \\
&\le (1+M_1h_n)\|x_n-q\|+M_2l_n \qquad (3.3)
\end{aligned}$$

for some $M_1, M_2>0$. Since $\sum_{n=1}^{\infty}h_n<\infty$ and $\sum_{n=1}^{\infty}l_n<\infty$, it follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n-q\|$ exists.

Now, taking the infimum over all $q\in F$ in (3.3), we have

$$d(x_{n+1},F) \le [1+M_1h_n]d(x_n,F)+M_2l_n \qquad (3.4)$$

for all $n\in\mathbf{N}$. It follows from $\sum_{n=1}^{\infty}h_n<\infty$, $\sum_{n=1}^{\infty}l_n<\infty$ and Lemma 2.3 that $\lim_{n\to\infty}d(x_n,F)$ exists. This completes the proof. □

Theorem 3.2. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let S_1, $S_2\colon C\to C$ be two total asymptotically nonexpansive self mappings with sequences $\{\mu_{n_1}\}$ and $\{\nu_{n_1}\}$ as defined in Proposition 2.2 and T_1, $T_2\colon C\to E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition 2.1 and*

$$F=F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)\neq\emptyset.$$

Let $\{x_n\}$ be the iteration scheme defined by (1.11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$ and the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty}\mu_{n_1}<\infty$, $\sum_{n=1}^{\infty}\mu_{n}<\infty$, $\sum_{n=1}^{\infty}\nu_{n_1}<\infty$, $\sum_{n=1}^{\infty}\nu_{n_1}<\infty$;
(ii) *there exists a constant* $M>0$ *such that* $\psi(t)\le M\,t$, $t\ge 0$.

Then $\{x_n\}$ *converges strongly to a common fixed point of* S_1, S_2, T_1 *and* T_2 *if and only if* $\liminf_{n\to\infty} d(x_n,F)=0$, *where* $d(x,F)=\inf\{\|x-p\|:p\in F\}$.

Proof. The necessity is obvious. Indeed, if $x_n\to q\in F$ as $n\to\infty$, then

$$d(x_n,F)=\inf_{q\in F} d(x_n,q)\le \|x_n-q\|\to 0\,(n\to\infty).$$

Thus $\liminf_{n\to\infty} d(x_n,F)=0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F)=0$. By Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n,F)$ exists. Further, by assumption $\liminf_{n\to\infty} d(x_n,F)=0$, from (3.4) and Lemma 2.3(ii), we conclude that $\lim_{n\to\infty} d(x_n,F)=0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in E. Indeed, from (3.3), we have

$$\|x_{n+1}-q\| \le [1+M_1h_n]\|x_n-q\|+M_2l_n$$

for each $n\in\mathbf{N}$, where h_n and l_n be taken as in Lemma 3.1 and $q\in F$. For any $m,n,m>n\in\mathbf{N}$, we have

$$\begin{aligned}\|x_m-q\| &\le [1+M_1h_{m-1}]\|x_{m-1}-q\|+M_2l_{m-1}\\ &\le e^{M_1h_{m-1}}\|x_{m-1}-q\|+M_2l_{m-1}\\ &\ \vdots\\ &\le \Big(e^{\sum_{i=n}^{m-1}M_1h_i}\Big)\|x_n-q\|+M_2\Big(e^{\sum_{i=n+1}^{m-1}M_ih_i}\Big)\sum_{i=n}^{m-1}l_i\\ &\le M'\|x_n-q\|+M'M_2\sum_{i=n}^{m-1}l_i\end{aligned}$$

where $M'=e^{\sum_{i=n}^{\infty}M_1h_i}$. Thus for any $q\in F$, we have

$$\begin{aligned}\|x_n-x_m\| &\le \|x_n-q\|+\|x_m-q\|\\ &\le \|x_n-q\|+M'\|x_n-q\|+M'M_2\sum_{i=n}^{m-1}l_i\\ &\le (M'+1)\|x_n-q\|+M'M_2\sum_{i=n}^{\infty}l_i.\end{aligned}$$

Taking the infimum over all $q\in F$, we obtain

$$\|x_n-x_m\| \le (M'+1)d(x_n,F)+M'M_2\sum_{i=n}^{\infty}l_i.$$

Thus it follows from $\lim_{n\to\infty} d(x_n,F)=0$ and $l_n\to 0$ as $n\to\infty$ that $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of E, the sequence $\{x_n\}$

converges strongly to some $q^* \in C$. Next, we show that $q^* \in F$. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q^*, F) = 0$. Since F is closed, $q^* \in F$. Thus q^* is a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 3.3. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $S_1, S_2\colon C \to C$ be two total asymptotically nonexpansive self mappings with sequences $\{\mu_{n_1}\}$ and $\{\nu_{n_1}\}$ as defined in Proposition 2.2 and $T_1, T_2\colon C \to E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition 2.1 and*

$$F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset.$$

Let $\{x_n\}$ be the iteration scheme defined by (1.11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$ and the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} \mu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
(ii) *there exists a constant $M > 0$ such that $\psi(t) \le M\,t$, $t \ge 0$.*

If one of S_1, S_2, T_1 and T_2 is completely continuous and $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof. Without loss of generality we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 3.1, there exists a subsequence $\{S_1 x_{n_k}\}$ of $\{S_1 x_n\}$ such that $\{S_1 x_{n_k}\}$ converges strongly to some $q_1 \in C$. Moreover, by hypothesis of the theorem we know that

$$\lim_{k\to\infty} \|x_{n_k} - S_1 x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - S_2 x_{n_k}\| = 0$$

and

$$\lim_{k\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - T_2 x_{n_k}\| = 0$$

which implies that

$$\|x_{n_k} - q_1\| \le \|x_{n_k} - S_1 x_{n_k}\| + \|S_1 x_{n_k} - q_1\| \to 0$$

as $k \to \infty$ and so $x_{n_k} \to q_1 \in C$. Thus, by the continuity of S_1, S_2, T_1 and T_2, we have

$$\|q_1 - S_i q_1\| = \lim_{k\to\infty} \|x_{n_k} - S_i x_{n_k}\| = 0$$

and

$$\|q_1 - T_i q_1\| = \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i = 1, 2$. Thus it follows that $q_1 \in F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$. Again, since $\lim_{n\to\infty} \|x_n - q_1\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q_1\| = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 3.4. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let S_1, $S_2 \colon C \to C$ be two total asymptotically nonexpansive self mappings with sequences $\{\mu_{n_1}\}$ and $\{\nu_{n_1}\}$ as defined in Proposition* 2.2 *and T_1, $T_2 \colon C \to E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition* 2.1 *and*

$$F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset.$$

Let $\{x_n\}$ be the iteration scheme defined by (1.11)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in* $[0, 1)$ *and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
(ii) *there exists a constant $M > 0$ such that $\psi(t) \leq M\, t$, $t \geq 0$.*

If one of S_1, S_2, T_1 and T_2 is semi-compact and $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof. Since by hypothesis $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $q_2 \in C$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, we have $\|q_2 - S_i q_2\| = \lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$ and $\|q_2 - T_i q_2\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for $i = 1, 2$. Thus it follows that $q_2 \in F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$. Since $\lim_{n\to\infty} \|x_n - q_2\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q_2\| = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 3.5. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let S_1, $S_2 \colon C \to C$ be two total asymptotically nonexpansive self mappings with sequences $\{\mu_{n_1}\}$ and $\{\nu_{n_1}\}$ as defined in Proposition* 2.2 *and T_1, $T_2 \colon C \to E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition* 2.1 *and*

$$F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset.$$

Let $\{x_n\}$ be the iteration scheme defined by (1.11)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in* $[0, 1)$ *and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$, $\sum_{n=1}^{\infty} \nu_{n_1} < \infty$;
(ii) *there exists a constant $M > 0$ such that $\psi(t) \leq M\, t$, $t \geq 0$.*

If S_1, S_2, T_1 and T_2 satisfy the following conditions:

(C_1) $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for $i = 1, 2$;*
(C_2) *there exists a continuous function $\varphi \colon [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t \in (0, \infty)$ such that*

$$\varphi(d(x, F)) \leq a_1\|x - S_1 x\| + a_2\|x - S_2 x\| + a_3\|x - T_1 x\| + a_4\|x - T_2 x\|$$

for all $x \in C$, and a_1, a_2, a_3, a_4 are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 = 1$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} \varphi(d(x_n, F)) &\leq a_1.\|x_n - S_1x_n\| + a_2.\|x_n - S_2x_n\| \\ &\quad + a_3.\|x_n - T_1x_n\| + a_4.\|x_n - T_2x_n\| \\ &= 0.\end{aligned}$$

That is,

$$\lim_{n\to\infty} \varphi(d(x_n, F)) = 0.$$

Since $\varphi\colon [0, \infty) \to [0, \infty)$ is a continuous function and $\varphi(0) = 0$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Therefore, Theorem 3.2 implies that $\{x_n\}$ must converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Now, we give some examples in support of our result: take two mappings $T_1 = T_2 = T$ and $S_1 = S_2 = S$.

Example 3.6. Let E be the real line with the usual norm $|.|$, $C = [0, \infty)$ and P be the identity mapping. Assume that $S(x) = x$ and $T(x) = sin\, x$ for all $x \in C$. Let ϕ be the strictly increasing continuous function such that $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$. Let $\{\mu_n\}_{n\geq 1}$ and $\{\nu_n\}_{n\geq 1}$ be two nonnegative real sequences defined by $\mu_n = \frac{1}{n^2}$ and $\nu_n = \frac{1}{n^3}$ for all $n \geq 1$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$. Then S and T are total asymptotically nonexpansive mappings with common fixed point 0, that is, $F = F(S) \cap F(T) = \{0\}$.

Example 3.7. Let $E = \mathbb{R}$ be the real line with the usual norm $\|.\| = |.|$, $C = [-1, 1]$ and P be the identity mapping. For each $x \in C$, define two mappings T, $S\colon C \to C$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0) \end{cases}$$

and

$$S(x) = \begin{cases} \frac{x}{2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T and S are asymptotically nonexpansive mappings with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and are uniformly L-Lipschtzian mappings with $L = \sup_{n\geq 1}\{k_n\}$ and hence are total asymptotically nonexpansive mapping by Remark 1.4. Also $F(T) = \{0\}$ is the unique fixed point of T and

$F(S) = \{0\}$ is the unique fixed point of S, that is, $F = F(S) \cap F(T) = \{0\}$ is the unique common fixed point of S and T.

4. Conclusion

In this paper, we establish some strong convergence theorems for newly defined mixed type two-step iteration scheme for two total asymptotically nonexpansive self mappings and two total asymptotically nonexpansive non-self mappings using completely continuous and semi-compactness conditions in the framework of real Banach spaces. Our results extend and generalize the corresponding results of [3, 4, 7, 8, 11, 12, 13, 15, 16, 17] to the case of more general class of mappings and iteration scheme.

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