Nonlinear Functional Analysis and Applications
Vol. 23, No. 3 (2018), pp. 575-583
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME PROPERTIES OF NONCONVEX FUNCTIONS

Muhammad Aslam Noor

Department of Mathematics
COMSATS University Islamabad, Islamabad, Pakistan
e-mail: noormaslam@gmail.com

Abstract. In this paper, we introduce and study a new class of convex functions with respect to an arbitrary function, which is called the k-convex function. These functions are nonconvex functions and include the convex function and φ-convex convex as special cases. We study some properties of k-convex functions. It is shown that the minimum of k-convex functions on the k-convex sets can be characterized by a class of variational inequalities, which is called the k-directional variational inequalities. Some open problems are also suggested for future research.

1. Introduction

In recent years, several extensions and generalizations of the convex sets and convex functions have been considered and investigated, see, for example [1-13] and the references therein. In this paper, we consider a new class of convex sets and convex functions which are called modified k-convex sets and convex functions. These new class of convex sets and convex functions include the φ-convex sets [9], Toader type convex sets and φ-convex functions [9] as special cases. Several new concepts are defined and their properties have been studied. We prove that the minimum of the differential k-convex functions on the k-convex set can be characterized by a class of variational inequalities. In order to convey the flavour of these new concepts, we have tried to emphasize the basic characteristic of these new classes of nonconvex functions. Some

[0]Received March 1, 2018. Revised April 26, 2018.
[0]2010 Mathematics Subject Classification: 54E70, 47H25.
[0]Keywords: Directional derivatives, k-convex functions, k-convex sets, variational inequalities.

basic properties of these nonconvex functions along with some open problems are discussed.

2. PRELIMINARIES

Let K_k be a nonempty closed set in a normed space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm, respectively.

Definition 2.1. The set K_k is said to be k-convex with respect to arbitrary function k, if

$$g(u)+k(t)(v-u)\in K_k, \quad \forall u,v\in K_k, \quad t\in[0,1].$$

Clearly, for $k(t)=t$, the set K_k is convex.

If $k(t)=t^s, s\in[0,1]$ then the k-convex set K_k reduces to:

$$u+t^s(v-u)\in K, \quad \forall u,v\in K_k, \quad t\in[0,1],$$

which is known as Toader type convex set.

We would like to point that the k-convex set was introduced and studied in [2, 3]. A set D_k is said to be k-convex if, for all $u,v\in D_k, t\in(0,1)$ such that

$$k(1-t)u+k(t)v\in D_k.$$

Note that, if $k(1-t)+k(t)=1$, then the k-convex set K_k and k-convex set D_k are equivalent. However, it is worth mentioning that these two different convex sets have distinctly different properties.

From now onwards, the set K_k is a k-convex set, unless otherwise specified. We now introduce the concept of k-convex function with respect to an arbitrary function.

Definition 2.2. The function f on K_k is called k-convex, if there exists an arbitrary function k such that

$$f(u+k(t)(v-u))\leq(1-t)f(u)+tf(v), \quad \forall u,v\in K_k,\ t\in[0,1].$$

Obviously every convex function with $k(t)=t$ is k-convex, but the converse may not be true. Also for $t=1$, the k-convex function reduces to:

$$f(u+k(1)(v-u))\leq f(v), \quad \forall u,v\in K. \tag{2.1}$$

If $k(t)=T^s, s\in[0,1]$, then we have a new class of convex functions, which is called Toader's type convex functions.

Definition 2.3. The function f on K_k is said to be quasi k-convex, if there exists a function k such that

$$f(u+k(t)(v-u))\leq\max\{f(u),f(v)\}, \quad \forall u,v\in K_k, \quad t\in[0,1].$$

Definition 2.4. The function f on K_k is said to be logarithmic k-convex, if there exists a function k such that

$$f(u+k(t)(v-u)) \leq (f(u)^{1-t}(f(v))^t, \quad \forall u,v \in K_k, \quad t \in [0,1],$$

where $f(\cdot) > 0$.

Lemma 2.5. *Let f be a k-convex function. Then any local minimum of f on K_k is a global minimum.*

Proof. Let the k-convex function f have a local minimum at $u \in K_k$. Assume the contrary, that is, $f(v) < f(u)$ for some $v \in K_k$. Since f is a k-convex function, so

$$f(u+k(t)(v-u)) \leq f(u) + t(f(v) - f(u)),$$

which implies that

$$f(u+k(t)(v-u))f(u) < 0,$$

for arbitrary small $t > 0$, contradicting the local minimum. □

Essentially using the technique and ideas of the classical convexity, one can easily prove the following results.

Theorem 2.6. *If f is a k-convex function on K_k, then the level set $L_\alpha = \{u \in K_k : f(u) \leq \alpha, \alpha \in \mathbb{R}\}$ is k-convex with respect to k.*

Theorem 2.7. *A function f is a k-convex function if and only if epi$(f) = \{(u,\alpha) : u \in K_k, \alpha \in \mathbb{R}, f(u) \leq \alpha\}$ is a k-convex set with respect to k.*

Theorem 2.8. *A function f is a quasi k-convex function if and only if the level set $L_\alpha = \{u \in K_k : f(u) \leq \alpha, \alpha \in \mathbb{R}\}$ is a k-convex set with respect to k.*

Definition 2.9. A function f is said to be a pseudo k-convex function with respect to k, if there exists a strictly positive function $b(\cdot,\cdot)$ such that

$$\begin{aligned} f(v) &< f(u) \Rightarrow f(u+k(t)(v-u)) \\ &\leq f(u) + t(t-1)b(u,v), \quad \forall u,v \in K_k,\ t \in (0,1). \end{aligned}$$

Theorem 2.10. *If the function f is a k-convex function with respect to k, then f is pseudo k-convex function with respect to k.*

Proof. Without loss of generality, we assume that $f(v) < f(u)$, for all $u, v \in K_k$. For every $t \in [0,1]$, we have

$$\begin{aligned} f(u+k(t)(v-u)) &\leq (1-t)f(u)+tf(v) \\ &< f(u)+t(t-1)\{f(u)-f(v)\} \\ &= f(u)+t(t-1)b(u,v), \end{aligned}$$

where $b(u,v) = f(u) - f(v) > 0$. Thus, it follows the function f a is pseudo k-convex function with respect to k, the required result. □

Theorem 2.11. *Let f be a k-convex function with respect to k. If $g : L \rightarrow \mathbb{R}$ is a nondecreasing function, then $g \circ f$ is a k-convex function with respect to the function k.*

Proof. Since f is a k-convex function and g is decreasing, we have, for all $u, v \in K_k$ and $t \in [0,1]$

$$\begin{aligned} g \circ f(u+k(t)(v-u)) &= g[f(u)+k(t)(f(v)-f(u)))] \\ &\leq g[(1-t)f(u)+tf(v)] \\ &\leq (1-t)g \circ f(u)+tg \circ f(v), \end{aligned}$$

from which it follows that $g \circ f$ is a k-convex function with respect to k. □

3. Main Results

We now introduce the concept of k-directional derivative.

Definition 3.1. We define the k-directional derivative of f at a point $u \in K_k$ in the direction $v \in K_k$ by

$$D_{\varphi}f(u,v) := f'_k(u;v) = \lim_{t \rightarrow 0^+} \frac{f(u+k(t)v)-f(u)}{t}.$$

Note that for $k(t) = t$, the k-directional derivative of f at u in the direction v coincides with the usual directional derivative of f at u in a direction v given by

$$Df(u,v) := f'(u;v) = \lim_{t \rightarrow 0^+} \frac{f(u+tv)-f(u)}{t}.$$

It is well known that the function $v \rightarrow f'_k(u;v)$ is subadditive, positively homogeneous and $|f'_k(u;v)| \leq \nu \parallel v \parallel$, where $\nu > 0$ is a constant.

Definition 3.2. A differential function f on K_k is said to be k-invex, if

$$f(v)-f(u) \geq f'_k(u;v-u), \qquad \forall u,v \in K_k,$$

where $f_k'(u;v)$ is the k-directional derivative of f at $u \in K_k$ in the direction of $v \in K_k$

Theorem 3.3. *Let f be a differential k-convex function on K_k. Then the function $v \to f_k'(u;v)$ is positively homogeneous and k-convex.*

Proof. It is follow from the definition of the k-directional derivative that $f_k'(u;\lambda v) = \lambda f_k'(u;v)$, whenever $v \in K_k$ and $\lambda \geq 0$, hence the function $v \to f_k'(u;v)$ is positively homogeneous.

To prove the k-convexity of the function $v \to f_k'(u;v)$, we consider for all $u, v, z \in K_k, \quad t \geq 0,\ \lambda \in (0,1)$,

$$\begin{aligned}
&\frac{1}{t}[f(u + k(t)(\lambda v + (1-\lambda)z)) - f(u)] \\
&= \frac{1}{t}[f(\lambda(u + k(t)v) + (1-\lambda)(u + k(t)z)) - f(u)] \\
&\leq \frac{1}{t}[\lambda f(u + k(t)v) + (1-\lambda)f(u + k(t)z) - f(u)] \\
&= \lambda \frac{f(u + k(t)v) - f(u)}{t} + (1-\lambda)\frac{f(u + k(t)z) - f(u)}{t}.
\end{aligned} \tag{3.1}$$

Taking the limit as $t \to 0^+$ in (3.1), we have

$$f_k'(u;\lambda v + (1-\lambda)z) \leq \lambda f_k'(u;v) + (1-\lambda)f_k'(u;z),$$

which shows that the function $v \to f_k'(u;v)$ is k-convex. □

For $k(t) = t$, the k-convex function f becomes the convex function and the k-convex set K_k is a convex set. Consequently, Theorem 3.3 reduces to the well-known result in convexity, (see [2]).

Theorem 3.4. *Let K_k be a k-convex set. If the function $f : K_k \to \mathbb{R}$ is differentiable k-convex such that $k(0) = 0$ and (2.1) holds, then the following statements are equivalent.*

(1) *f is k-invex.*
(2) *φ-directional derivative $f_k'(\cdot,\cdot)$ of f is monotone, that is,*

$$f_k'(u;v-u) + f_k'(v;u-v) \leq 0, \quad \forall u, v \in K_k.$$

Proof. Let f be a k-convex function. Then

$$f(u + k(t)(v-u)) \leq f(u) + t\{f(v) - f(u)\} \quad \forall u, v \in K,\ t \in [0,1],$$

which can be written as

$$f(v) - f(u) \geq \frac{f(u + k(t)(v-u)) - f(u)}{t}. \tag{3.2}$$

Taking the limit as $t \to 0^+$ in (3.2), we have

$$f(v) - f(u) \geq f'_k(u; v-u), \quad \forall u, v \in K, \tag{3.3}$$

showing that the k-convex function f is a k-invex function.

Changing the role of u and v in (3.3), we have

$$f(u) - f(v) \geq f'_k(v; u-v), \quad \forall u, v \in K, \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$f'_k(u; v-u) + f'_k(v; u-v) \leq 0, \quad \forall u, v \in K, \tag{3.5}$$

which shows that the k-directional derivative $f'_k(\cdot, \cdot)$ is monotone.

Conversely, let (3.5) hold. Since K_k is a k-convex set, so

$$\forall u, v \in K_k, \quad t \in [0,1], \quad v_t = u + k(t)(v-u) \in K_k.$$

Replacing v by v_t in (3.5) and simplifying,, we have

$$f'_k(v_t; v-u) \geq f'_k(u; v_-u), \quad \forall u, v \in K_k. \tag{3.6}$$

Consider the auxiliary function

$$g(t) = f(u + k(t)(v-u)) - f(u) + t f'_k(u; v-u), \quad \forall u, v \in K_k. \tag{3.7}$$

Using $k(0) = 0$, we have

$$g(0) = 0, \quad g(1) = f(u + k(t)(v-u)) - f(u) + f'_k(u : v-u). \tag{3.8}$$

Since f is differentiable, so the function $g(t)$ is also differentiable. Hence, using (3.6), we have

$$\begin{aligned} g'(t) &= f'(u + k(t)(v-u)), v-u) \\ &\geq 2 f'_k(u; v-u). \end{aligned} \tag{3.9}$$

Integrating the inequality (3.9) on the interval $[0,1]$ and using (3.8), we have

$$\begin{aligned} f(u + k(t)(v-u)) - f(u) + f'_k(u : v-u) &= g(1) - g(0) \\ &\geq 2 \int_0^1 f'(u; v-u) dt \\ &= 2 f'_k(u; v-u), \end{aligned}$$

from which, using (2.1), we obtain

$$f'_k(u; v-u) \leq f(u + k(t)(v-u)) - f(u) \leq f(v) - f(u).$$

which is the required (3.3). □

Theorem 3.5. *Let the differential $f'(.;.)$ of the k-convex function f be Lipschitz continuous with constant $\beta \geq 0$. If $k(0) = 0$, then*

$$f(u + k(1)(v-u)) - f(u)$$
$$\leq f'_k(u; v-u) + \beta\|v-u\|^2 \int_0^{\prime} k(t)dt, \quad \forall u, v \in K_k. \tag{3.10}$$

Proof. Since K_k is a k-convex set, for all $u, v \in K_k,\ t \in [0,1]$, we consider the function

$$\varphi(t) = f(u + k(t)(v-u)) - f(u) - t{f'}_k(u; v-u).$$

Using $k(0) = 0$, we obtain

$$\varphi(0) = 0, \quad \varphi(1) = f(u + k(1)(v-u)) - f(u) - f'_k(u; v-u).$$

Also

$$\varphi'(t) = f'_k(u + k(t)(v-u); v-u) - {f'}_k(u; v-u). \tag{3.11}$$

Integrating (3.11) on the integral $[0,1]$ and using the Lipschitz continuity of $f'_k(.;.)$ with constant $\beta \geq 0$, we have

$$\begin{aligned}
\varphi(1) &= f(u + k(1)(v-u)) - f(u) - f'_k(u; v-u) \\
&\leq \int_0^1 |\varphi'(t)|dt \\
&= \int_0^{\prime} |f'_k(u + k(t)(v-u); v-u) - {f'}_k(u; v-u)|dt \\
&\leq \beta \int_0^{\prime} k(t)\|v-u\|^2 dt \\
&= \beta\|v-u\|^2 \int_0^{\prime} k(t)dt,
\end{aligned}$$

the required result. □

It is well known that the minimum of the differentiable convex function on the convex set can be characterized by a class of variational inequalities. We proved $u \in K_k$ is the minimum of the differentiable k-convex functions can be characterized by a class of variational inequality, which is known as the directional variational inequality. This is the main motivation of our next result.

Theorem 3.6. *Let f be a differentiable k-convex function on K_k. Then the $u \in K_k$ is the minimum of the differentiable k-convex function f on K_k, if and only if, $u \in K_k$ satisfies the inequality*

$$f'_k(u; v-u) \geq 0, \quad \forall u, v \in K_k. \tag{3.12}$$

Proof. Let $u \in K$ be a minimum of the k-convex function f. Then

$$f(u) \le f(v), \quad \forall v \in K_k. \tag{3.13}$$

Since K is a k-convex set, so, for all $u, v, \in K_k, \quad t \in [0,1]$, $v_t = u + k(t)(v-u) \in K$. Taking $v = v_t$ in (3.13), we have

$$f(u) \le f(v_t) = f(u + k(t)(v-u)),$$

which implies that

$$\frac{f(u + k(t)(v-u)) - f(u)}{t} \ge 0.$$

Taking the limit as $t \to 0^+$ in the above inequality, we have

$$f_k'(u; v-u) \ge 0 \quad \forall v \in K_k,$$

the required (3.12).

Conversely, let $u \in K_k$ be a solution of (3.12). Since f is a k-convex function, it follows, using (3.12), that

$$f(v) - f(u) \ge f_k'(u; v-u) \ge 0,$$

which implies that

$$f(u) \le f(v), \quad \forall v \in K_k,$$

showing that $u \in K_k$ is the minimum of the k-convex function f, the required result. □

The inequality of the type (3.12) is called the k-directional variational inequality. For $k(t) = t$, problem (3.12) reduces to the directional variational inequalities. It is worth mentioning that even the directional variational inequalities have not been studied in the literature.

Theorem 3.7. *If the k-directional derivative of f is pseudomonotone and hemicontinuous, then the k-directional variational inequality is equivalent to finding $u \in K$ such that*

$$-f_k'(v; u-v) \ge 0 \quad \forall v \in K_k. \tag{3.14}$$

Proof. Let $u \in K_k$ be a solution of inequality (3.12). Then, using the pseudomonotonicity of the differental $f_k'(u; v)$, we have (3.14). Since k_k is a k-convex set, so, for all $u, v \in K_k, t \in [0,1], v_t = u + k(t)(v-u) \in K_k$.
Replacing v by v_t in (3.14), we obtain

$$f_k'(v_t; u - v_t) = f_k'(u + k(t)(v-u) : v-u) \ge 0 \quad \forall v \in K_k. \tag{3.15}$$

Using the hemicontinuity of the differential $f_k'(u; v)$ and taking the limit, we obtain the inequality (3.12), since $\lim_{t \longrightarrow 0} k(0) = 0$. □

Remark 3.8. We would like to mention that the inequality of the type (3.14) is known as the Minty k-directional variational inequality or dual k-directional variational inequality. Using this equivalent result, one can show that the solution set of the directional variational inequalities is a closed convex set. If $k(t) = 1$, then we have the known results for the directional variational inequalities. We would like to point that for $k(t) = t^s, s \in [0, 1]$, we obtain some new classes of k-convex functions. For the applications, numerical methods and other aspects of variational inequalities, see [4-6] and the references therein. Interested readers are may explore the applications in various branches of pure and applied sciences.

Acknowledgements: The author would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environments.

References

[1] G. Crestescu, M. Gaianu and M.U. Awan, *Regularity properties and integral inequalities realted to* (k, h_1, h_2)*-convexity of functions*, Analele Universit. Vest Timisoara, Ser. Math.-Informat., **LIII**(1) (2015), 19–35.

[2] A. Hazy, *Bernstein-Doetsch type results for* (k, h)*-convex functions*, Miskolc Math. Notes, **13** (2012), 325–336.

[3] B. Micherda and T. Rajba, *On some Hermite-Hadamard-Fejer inequalities for* (k, h)*-convex functions*, Math. Ineq. Appl., **12** (2012), 931–940.

[4] M.A. Noor, *General variational inequalities*, Appl. Math. Letters, **1** (1988), 119–121.

[5] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251** (2000), 217–229.

[6] M.A. Noor, *Fundamental of equilibrium problems*, Math. Inequal. Appl., **9**(3) (2006), 520–566.

[7] M.A. Noor, *Some developments in general variational inequalities*, Appl. Math. Comput., **152** (2004), 199–277.

[8] M.A. Noor, *Some new classes of nonconvex functions*, Nonlinear Funct. Anal. and Appl., **11**(1) (2006), 165–171.

[9] K.I. Noor and M.A. Noor, *Relaxed strongly nonconvex functions*, Appl. Math. E-Notes, **6** (2006), 259–267.

[10] M.A. Noor and K.I. Noor, *Generalized preinvex functions and their properties*, J. Appl. Math. Stochast. Anal., **2006** (2006),Article ID 12736, Pages 1–13.

[11] J.E. Pecaric, F. Proschan and Y.L. Tong, *Conve Functions and Statistical Applications*, Academic Press, New York, 1992.

[12] J.V. Tiel, *Convex Analysis*, John Wiley and Sons, New York, 1984.

[13] D.L. Zhu and P. Marcotte, *Co-coercvity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Optim., **6** (1966), 714–726.