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ON APPLICATIONS OF MULTIDIMENSIONAL FIXED POINT THEOREMS

H. Akhadkulov[1], A.B. Saaban[2], M.F. Alipiah[3] and A.F. Jameel[4]

[1]School of Quantitative Sciences, University Utara Malaysia
CAS 06010, UUM Sintok, Kedah Darul Aman, Malaysia
e-mail: habibulla@uum.edu.my

[2]School of Quantitative Sciences, University Utara Malaysia
CAS 06010, UUM Sintok, Kedah Darul Aman, Malaysia
e-mail: azizan.s@uum.edu.my

[3]School of Quantitative Sciences, University Utara Malaysia
CAS 06010, UUM Sintok, Kedah Darul Aman, Malaysia
e-mail: fathilah@uum.edu.my

[4]School of Quantitative Sciences, University Utara Malaysia
CAS 06010, UUM Sintok, Kedah Darul Aman, Malaysia
e-mail: alifareed@uum.edu.my

Abstract. The purpose of this paper is to present the applications of multidimensional fixed point theorems. For this, we provide two multidimensional fixed point theorems and then using these theorems, we prove the existence and uniqueness of solution of a nonlinear systems of matrix equations.

1. Introduction and Preliminaries

The notion of multidimensional Υ-*fixed point* was introduced by Roldán *et. al.* [10, 11] in 2012. This notion covers the concepts of *coupled, tripled* and *quadruple* fixed point (see for instance [1, 3, 6, 7, 9, 12]). Due to wide potential application of fixed point results in various branches of mathematics, such

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[0]Corresponding author: H. Akhadkulov(habibulla@uum.edu.my).

as differential equations, mathematical economics, game theory, dynamics, optimal control, functional analysis, operator theory etc.

In this work we focus to applications of multidimensional fixed points. More precisely, we prove the existence and uniqueness of solution of the following system of matrix equations, in a space of $n \times n-$ Hermitian matrices $\mathcal{H}(n)$.

$$X_j = Q + \sum_{i=1}^{2m} (-1)^{i-1} A_i^* \mathcal{F}(X_{k^\sharp(i+j-1)}) A_i, \quad j \in \Lambda_{2m} = \{1, 2, ..., 2m\} \tag{1.1}$$

where Q is a positive matrix, A_i, $i \in \Lambda_{2m}$ are $n \times n$ matrices, $\mathcal{F} : \mathcal{H}(n) \to \mathcal{H}(n)$ is a matrix function and $k^\sharp : \Lambda_{4m} \to \Lambda_{2m}$ is a mapping defined as

$$k^\sharp(i) = \begin{cases} i, & \text{if } 1 \le i \le 2m \\ i - 2m, & \text{if } 2m < i \le 4m. \end{cases}$$

Let us briefly recall some necessary notions in order to formulate our main results. These notions can also be found in [10, 11]. Here and further we denote by $(X, d, \preceq)$ a *partially ordered metric space*.

Definition 1.1. An ordered metric space $(X, d, \preceq)$ is called *regular* if it satisfies the following:

- if $\{x_m\}$ is a nondecreasing sequence and $\{x_m\} \xrightarrow{d} x$, then $x_m \preceq x$ for all m;
- if $\{y_m\}$ is a nonincreasing sequence and $\{y_m\} \xrightarrow{d} y$, then $y_m \succeq y$ for all m.

Taking a natural number $k \ge 2$ we consider the set $\Lambda_k = \{1, 2, \ldots, k\}$. Let $\{\mathcal{A}, \mathcal{B}\}$ be a partition of Λ_k that is $\mathcal{A} \cup \mathcal{B} = \Lambda_k$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Using this partition and partially ordered metric space $(X, d, \preceq)$ we define a k-dimensional partially ordered metric space $(X^k, \mathbf{d}_k, \preceq_k)$ as follows:

- the k-cartesian power of a set X

$$X^k = \underbrace{X \times X \times \cdots \times X}_{k-times} = \{(\mathbf{x} = (x_1, x_2, ..., x_k)) : |x_i \in X \text{ for all } i \in \Lambda_k\};$$

- the maximum metric $\mathbf{d}_k : X^k \times X^k \to [0, +\infty)$, given by

$$\mathbf{d}_k(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le k} \{d(x_i, y_i)\},$$

  where $\mathbf{x} = (x_1, x_2, ..., x_k), \mathbf{y} = (y_1, y_2, ..., y_k) \in X^k$;
- the partial order w.r.t $\{\mathcal{A}, \mathcal{B}\}$ that is, for any $\mathbf{x} = (x_1, x_2, ..., x_k)$ and $\mathbf{y} = (y_1, y_2, ..., y_k) \in X^k$ we have

$$\mathbf{x} \preceq_k \mathbf{y} \Leftrightarrow \begin{cases} x_i \preceq y_i, & \text{if } \quad i \in \mathcal{A}, \\ x_i \succeq y_i, & \text{if } \quad i \in \mathcal{B}. \end{cases}$$

It is easy to see that if (X,d) is a complete metric space, then $(X^k,\mathbf{d}_k)$ is also a complete metric space.

Definition 1.2. We say that a mapping $F : X^k \to X$ has the *mixed monotone* property w.r.t partition $\{\mathcal{A},\mathcal{B}\}$, if F is monotone nondecreasing in arguments of $\mathcal{A}$ and monotone nonincreasing in arguments of $\mathcal{B}$.

We define the following set of mappings:

$$\Omega_{\mathcal{A},\mathcal{B}} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(\mathcal{A}) \subseteq \mathcal{A},\ \sigma(\mathcal{B}) \subseteq \mathcal{B}\},$$

$$\Omega'_{\mathcal{A},\mathcal{B}} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(\mathcal{A}) \subseteq \mathcal{B},\ \sigma(\mathcal{B}) \subseteq \mathcal{A}\}.$$

Let $\Upsilon = (\sigma_1,\sigma_2,\ldots,\sigma_k)$ be k-tuple of mappings of $\sigma_i : \Lambda_k \to \Lambda_k$ such that $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$. In the sequel we consider only such kind of k-tuple of mappings.

Definition 1.3. A point $\mathbf{x} = (x_1,x_2,\ldots,x_k) \in X^k$ is called an Υ-*fixed point* of a mapping $F : X^k \to X$ if

$$F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(k)}) = x_i$$

for all $i \in \Lambda_k$.

2. Roldàn's multidimensional fixed point theorems

In this section we provide relations between one and multidimensional fixed point theorems. Define $T_\Upsilon : X^k \to X^k$ as follows:

$$T_\Upsilon(x_1,x_2,\ldots,x_k) = \Big(F(x_{\sigma_1(1)},x_{\sigma_1(2)},\ldots,x_{\sigma_1(k)}),F(x_{\sigma_2(1)},x_{\sigma_2(2)},\ldots,x_{\sigma_2(k)}) \\ \ldots,F(x_{\sigma_k(1)},x_{\sigma_k(2)},\ldots,x_{\sigma_k(k)})\Big)$$

for all $\mathbf{x} = (x_1,x_2,\ldots,x_k) \in X^k$.

Now we are ready to formulate Roldàn's theorems.

Theorem 2.1. ([11]) *Let $(X,d,\preceq)$ be a complete partially ordered metric space. Let $\Upsilon : \Lambda_k \to \Lambda_k$ be a k-tuple of mappings $\Upsilon = (\sigma_1,\sigma_2,...,\sigma_k)$ which is verifying $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$.*

- *If F has the mixed monotone property, then T_Υ is monotone nondecreasing w.r.t $\preceq_k$.*
- *If F is continuous, then T_Υ is also continuous.*
- *A point $\mathbf{x} = (x_1,x_2,\ldots,x_k) \in X^k$ is a Υ-fixed point of F, if and only if $\mathbf{x} = (x_1,x_2,\ldots,x_k)$ is a fixed point of T_Υ.*

We need the following definition which was introduced by Khan *et. al.* in [8].

Definition 2.2. A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an *altering distance function*, if ψ is continuous, monotonically increasing and $\psi(\{0\}) = \{0\}$.

Theorem 2.3. ([11]) *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $\Upsilon : \Lambda_k \to \Lambda_k$ be a k-tuple of mappings $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_k)$ which is verifying $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$. Suppose $F : X^k \to X$ satisfies the following conditions:*

(i) *There exist altering distance functions ψ, φ such that for all $\mathbf{x} = (x_1, x_2, \ldots, x_k), \mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ with $\mathbf{x} \preceq_k \mathbf{y}$*

$$\psi(d(F(x_1, x_2, \ldots, x_k), F(y_1, y_2, \ldots, y_k))) \leq \psi(\boldsymbol{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\boldsymbol{d}_k(\mathbf{x}, \mathbf{y}));$$

(ii) *There exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ such that*
  (a) *$x_i^0 \preceq F(x^0_{\sigma_i(1)}, x^0_{\sigma_i(2)}, \ldots, x^0_{\sigma_i(k)})$ if $i \in \mathcal{A}$ and*
  (b) *$x_i^0 \succeq F(x^0_{\sigma_i(1)}, x^0_{\sigma_i(2)}, \ldots, x^0_{\sigma_i(k)})$ if $i \in \mathcal{B}$;*

(iii) *F has the mixed monotone property w.r.t $\{\mathcal{A}, \mathcal{B}\}$;*

(iv) *For all $i \in \Lambda_k$, the mapping σ_i is a permutation of Λ_k;*

(v) (a) *F is continuous or*
  (b) *$(X, d, \preceq)$ is regular.*

Then F has at least one Υ-fixed point.

Note that, in this theorem, the uniqueness of Υ-fixed point can easily be proven under the following additional condition.

Remark 2.4. *If for any $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ there exists a $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in X^k$, such that $\mathbf{x} \preceq_k \mathbf{z}$ and $\mathbf{y} \preceq_k \mathbf{z}$ $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*) \in X^k$. F has a unique Υ-fixed point.*

Moreover, in above theorem the authors required to be a permutation of the mapping σ_i for all $i \in \Lambda_k$ (i.e. condition (iv)). It runs out this condition would not be necessary, if we change the contractive condition of Theorem 2.3. More precisely, we have:

Theorem 2.5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $\Upsilon : \Lambda_k \to \Lambda_k$ be a k-tuple of mappings $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_k)$ which is verifying $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$. Suppose $F : X^k \to X$ be a mapping which obeys the following conditions:*

(i) *There exist altering distance functions ψ, θ and a monotonically decreasing continuous function $\varphi : [0, \infty) \to \mathbb{R}$ such that for all $\mathbf{x} = (x_1, x_2, \ldots, x_k), \mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ with $\mathbf{x} \preceq_k \mathbf{y}$*

$$\psi(d(F(x_1, x_2, \ldots, x_k), F(y_1, y_2, \ldots, y_k))) \leq \theta(\boldsymbol{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\boldsymbol{d}_k(\mathbf{x}, \mathbf{y}))$$

where $\theta(0) = \varphi(0) = 0$ and $\psi(x) - \theta(x) + \varphi(x) > 0$ for all $x > 0$;

(ii) *There exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ such that*

(a) $x_i^0 \preceq F(x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0)$ *if* $i \in \mathcal{A}$ *and*
(b) $x_i^0 \succeq F(x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0)$ *if* $i \in \mathcal{B}$;

(iii) *F has the mixed monotone property w.r.t* $\{\mathcal{A}, \mathcal{B}\}$;

(iv) (a) *F is continuous or*
(b) $(X, d, \preceq)$ *is regular.*

Then F has at least one Υ*-fixed point. Moreover*

(v) *if for any* $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ *there exists a* $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in X^k$, *such that* $\mathbf{x} \preceq_k \mathbf{z}$ *and* $\mathbf{y} \preceq_k \mathbf{z}$, *then F has a unique* Υ*-fixed point* $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*) \in X^k$.

Proof. Using condition (i), we get

$$\begin{aligned}\psi\Big(\mathbf{d}_k(T_\Upsilon(\mathbf{x}), T_\Upsilon(\mathbf{y})\Big) &\leq \max_{i\in\Lambda_k}\Big(\theta(\max_{j\in\Lambda_k} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})) - \varphi(\max_{j\in\Lambda_k} d(x_{\sigma_i(j)}, y_{\sigma_i(j)}))\Big)\\ &\leq \theta(\mathbf{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\mathbf{d}_k(\mathbf{x}, \mathbf{y}))\end{aligned}$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ such that $\mathbf{x} \preceq_k \mathbf{y}$. Thus we have shown that the mapping T_Υ satisfies the contractive condition of Theorem 2.5 in [12]. The rest of the proof follows exactly same way that of Theorem 2.5 in [12]. □

3. Main result

In this section we study the existence and uniqueness of solutions of nonlinear systems of matrix equations. We deal on the set of $n \times n$ matrices and we denote this set by $\mathcal{M}(n)$. Let $\mathcal{P}(n)$ be the set of all $n \times n$ positive definite matrices and $\widetilde{\mathcal{P}}(n)$ be the set of all $n \times n$ positive semidefinite matrices. Let us first define some necessary facts. A partial order $\preceq$ on $\mathcal{H}(n)$ defined by

$$X, Y \in \mathcal{H}(n),\ \ X \preceq Y \Leftrightarrow Y - X \in \widetilde{\mathcal{P}}(n).$$

The set $\mathcal{H}(n)$ is partially ordered and for every $X, Y \in \mathcal{H}(n)$ there is a greatest lower bound and a least upper bound (see [2]). Next we use the following two norms:

- $\|A\| = \sqrt{\lambda_{\max}(A^*A)} = \max_{1\leq i\leq n} s_i(A)$ the spectral norm;
- $\|A\|_1 = tr\Big(\sqrt{A^*A}\Big) = \sum_{i=1}^{n} s_i(A)$ the trace norm, where $s_i(A)$, $i = 1, 2, ..., n$ are the singular values of A and $tr(\cdot)$ is the trace of a matrix.

Further it is convenient us to use metric induced by the trace norm. Since $\mathcal{H}(n)$ is a finite dimensional linear metric space equipped the metric indicate by $\|\cdot\|_1$, complete (see Theorem IX.2.2 in [4]). The following lemma plays a key role for our application.

Lemma 3.1. ([2]) *Let $A, B \in \widetilde{\mathcal{P}}(n)$. Then we have*

$$0 \le tr(AB) \le \|A\| tr(B)$$

where $\|\cdot\|$ is the spectral norm.

3.1. **Hypothesis for the system (1.1).** We suppose:

(a) $A_2^*\mathcal{F}(Q)A_2 + A_4^*\mathcal{F}(Q)A_4 + ... + A_{2m}^*\mathcal{F}(Q)A_{2m} \preceq Q$;
(b) $\mathcal{F}$ is continuous, $\mathcal{F}(0_n) = 0_n$ and preserves the order that is:

$$X \preceq Y \Rightarrow \mathcal{F}(X) \preceq \mathcal{F}(Y)$$

where 0_n is the $n \times n$ zero matrix;
(c) there exists a positive number M such that

$$\sum_{i=1}^{2m} \|A_i A_i^*\| < M;$$

(d) for any $X, Y \in \mathcal{H}(n)$ such that $Y \preceq X$ we have

$$\left| tr\Big(\mathcal{F}(X) - \mathcal{F}(Y)\Big)\right| \le \frac{1}{M} \exp\Big(-\frac{1}{tr(X-Y)}\Big).$$

We are ready to formulate our second result.

Theorem 3.2. *Under assumptions $(a)-(d)$, the system of equations* (1.1) *has a unique solution in $\mathcal{H}(n)$.*

Proof. Let $\Lambda_{2m} = \{1, 2, ..., 2m\}$. Consider a partition $\mathcal{A} = \{1, 3, ..., 2m-1\}$ and $\mathcal{B} = \{2, 4, ..., 2m\}$. We choose $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_{2m})$ as

$$\Upsilon = \begin{pmatrix} 1 & 2 & \dots & 2m-2 & 2m-1 & 2m \\ 2 & 3 & \dots & 2m-1 & 2m & 1 \\ 3 & 4 & \dots & 2m & 1 & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2m & 1 & \dots & 2m-3 & 2m-2 & 2m-1 \end{pmatrix}.$$

Next we consider the operator $\mathbb{B} : \mathcal{H}^{2m}(n) \to \mathcal{H}(n)$

$$\mathbb{B}\Big(X_1, X_2, ..., X_{2m}\Big) = Q + \sum_{i=1}^{2m} (-1)^{i-1} A_i^* \mathcal{F}(X_i) A_i. \tag{3.1}$$

It is clear that the system (1.1) has a solution if and only if $\mathbb{B}$ has a Υ- fixed point. Therefore, further we show that the operator $\mathbb{B}$ satisfies all conditions of Theorem 2.5. Since $\mathcal{F}$ is continuous, $\mathbb{B}$ is continuous. Next we show that $\mathbb{B}$ has the mixed monotone property w.r.t $\{\mathcal{A}, \mathcal{B}\}$. By assumption (b) the mapping $\mathcal{F}$

preserves order, therefore for any $\mathbf{X} = \Big(X_1, ..., X_{2m}\Big), \mathbf{Y} = \Big(Y_1, ..., Y_{2m}\Big)$ such that

$$\mathbf{X} \preceq_{2m} \mathbf{Y} \Leftrightarrow \begin{cases} X_i \preceq Y_i, & \text{if} \quad i \in \mathcal{A}, \\ X_i \succeq Y_i, & \text{if} \quad i \in \mathcal{B} \end{cases}$$

we have

$$\Big(\mathcal{F}(X_1), ..., \mathcal{F}(X_{2m})\Big) \preceq_{2m} \Big(\mathcal{F}(Y_1), ..., \mathcal{F}(Y_{2m})\Big)$$

$$\Leftrightarrow \begin{cases} \mathcal{F}(X_i) \preceq \mathcal{F}(Y_i), & \text{if} \quad i \in \mathcal{A}, \\ \mathcal{F}(X_i) \succeq \mathcal{F}(Y_i), & \text{if} \quad i \in \mathcal{B}. \end{cases}$$

Thus

$$\begin{aligned} &\mathbb{B}\Big(Y_1, Y_2 ..., Y_{2m}\Big) - \mathbb{B}\Big(X_1, X_2 ..., X_{2m}\Big) \\ &= \sum_{i \in \mathcal{A}} A_i^* \Big(\mathcal{F}(Y_i) - \mathcal{F}(X_i)\Big) A_i + \sum_{i \in \mathcal{B}} A_i^* \Big(\mathcal{F}(X_i) - \mathcal{F}(Y_i)\Big) A_i \\ &\succeq 0_n. \end{aligned}$$

Let $\Big(Z_1^0, Z_2^0, ..., Z_{2m}^0\Big) = \Big(Q, 0_n, ..., 0_n\Big)$. Next we show

$$Z_i^0 \preceq \mathbb{B}\Big(Z^0_{\sigma_i(1)}, Z^0_{\sigma_i(2)}, ..., Z^0_{\sigma_i(2m)}\Big) \quad \text{if} \quad i \in \mathcal{A}$$

and

$$Z_i^0 \succeq \mathbb{B}\Big(Z^0_{\sigma_i(1)}, Z^0_{\sigma_i(2)}, ..., Z^0_{\sigma_i(2m)}\Big) \quad \text{if} \quad i \in \mathcal{B}.$$

Indeed

$$Q \preceq Q + \sum_{i \in \mathcal{A}} A_i^* \mathcal{F}(Q) A_i = \mathbb{B}\Big(Q, 0_n, ..., 0_n\Big)$$

and by assumption (a) we have

$$\mathbb{B}\Big(0_n, Q, ..., Q\Big) = Q - \sum_{i \in \mathcal{B}} A_i^* \mathcal{F}(Q) A_i \succeq 0_n.$$

Further, we show that $\mathbb{B}$ satisfies the first condition of Theorem 2.5 with

$$\psi(x) = e^{-1/x} \ x > 0, \ \psi(0) = 0, \ \theta(x) = \lambda \psi(x)$$

for some $\lambda \in (0, 1)$ and $\varphi(x) = 0$.

Let $\Big(X_1, X_2, ..., X_{2m}\Big), \Big(Y_1, Y_2, ..., Y_{2m}\Big) \in \mathcal{H}^{2m}(n)$ such that

$$\Big(X_1, X_2, ..., X_{2m}\Big) \preceq_{2m} \Big(Y_1, Y_2, ..., Y_{2m}\Big).$$

Because of $\mathbb{B}\Big(X_1, X_2, ..., X_{2m}\Big) \preceq \mathbb{B}\Big(Y_1, Y_2, ..., Y_{2m}\Big)$ we have

$$
\begin{aligned}
&\|\mathbb{B}\Big(X_1, X_2, ..., X_{2m}\Big) - \mathbb{B}\Big(Y_1, Y_2, ..., Y_{2m}\Big)\|_1 \\
&= \sum_{i\in\mathcal{A}} tr\Big(A_i^*\Big(\mathcal{F}(Y_i) - \mathcal{F}(X_i)\Big)A_i\Big) + \sum_{i\in\mathcal{B}} tr\Big(A_i^*\Big(\mathcal{F}(X_i) - \mathcal{F}(Y_i)\Big)A_i\Big) \\
&\leq \sum_{i=1}^{2m} \|A_iA_i^*\|\|\mathcal{F}(X_i) - \mathcal{F}(Y_i)\|_1.
\end{aligned} \tag{3.2}
$$

Applying assumption (d) we get

$$
\sum_{i=1}^{2m} \|A_iA_i^*\|\|\mathcal{F}(X_i) - \mathcal{F}(Y_i)\|_1 \leq \lambda \exp\Big(-\frac{1}{\max\limits_{1\leq i\leq 2m}\|X_i - Y_i\|_1}\Big) \tag{3.3}
$$

where

$$
\lambda = \frac{\sum\limits_{i=1}^{2m}\|A_iA_i^*\|}{M}.
$$

Assumption (c) implies $\lambda \in (0,1)$. It is obvious that $\exp(-1/x) < x$ for $x > 0$. Taking into account this and inequalities (3.2) and (3.3) we get

$$
\begin{aligned}
&\exp\Big(-\frac{1}{\|\mathbb{B}\Big(X_1, X_2, ..., X_{2m}\Big) - \mathbb{B}\Big(Y_1, Y_2, ..., Y_{2m}\Big)\|_1}\Big) \\
&\leq \lambda \exp\Big(-\frac{1}{\max\limits_{1\leq i\leq 2m}\|X_i - Y_i\|_1}\Big).
\end{aligned} \tag{3.4}
$$

Thus we have shown that the operator $\mathbb{B}$ satisfies the conditions $(i) - (iv)$ of Theorem 2.5. Hence $\mathbb{B}$ has a Υ- fixed point $\Big(\widehat{X}_1, \widehat{X}_2, ..., \widehat{X}_{2m}\Big) \in \mathcal{H}^{2m}(n)$. On the other hand, for all $X, Y \in \mathcal{H}(n)$ there is a greatest lower bound and least upper bound, hence the conditions of Theorem 2.5 hold. Therefore $\mathbb{B}$ has a unique Υ-fixed point $\Big(\widehat{X}_1, \widehat{X}_2, ..., \widehat{X}_{2m}\Big) \in \mathcal{H}^{2m}(n)$ which is also the unique solutions of the system (1.1), that is

$$
\widehat{X}_j = Q + \sum_{i=1}^{2m}(-1)^{i-1}A_i^*\mathcal{F}(\widehat{X}_{k^\sharp(i+j-1)})A_i, \quad j = 1, 2, ..., 2m. \tag{3.5}
$$

□

Remark 3.3. Note that Theorem 3.2 generalizes the main result of [5].

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