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COUPLED COMMON FIXED POINT RESULTS IN ORDERED S-METRIC SPACES

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Abstract. In this paper, we prove a coupled coincidence and common fixed point theorems for commutating with mixed g-monotone property in the setting of a partially ordered S-metric space. Examples are given to support the usability of our results and to distinguish them from the existing ones.

1. Introduction

In 1922, Banach [4] proposed a theorem, which is well-known as Banach's Fixed Point Theorem (or Banach's Contraction Principle, BCP for short) to establish the existence of solutions for nonlinear operator equations and integral equations. Since then, because of simplicity and usefulness, it has become a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Later, a huge amount

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of literature is witnessed on applications, generalizations and extensions of this theorem. They are carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups.

Many mathematics problems require one to find a distance between two or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric structure. Due to the need to construct a suitable framework to model several distinguished problems of practical nature, the study of metric spaces has attracted and continue to attract the interest of many authors. Over last few decades, a numbers of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, partial metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [6, 10, 11, 12, 13, 16, 22]. Sedghi *et al.* [20] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

In recent years, fixed point theory has developed rapidly in metric spaces endowed with a partial order. Fixed point problems have also been considered in partially ordered probabilistic metric spaces [9], partially ordered G-metric spaces [3, 18], partially ordered cone metric spaces [7, 15, 23], partially ordered fuzzy metric spaces and partially ordered non-Archimedean fuzzy metric spaces [1, 2].

Mixed monotone operators were introduce by Guo and Lakshmikantham in [14]. Their study has not only important theoretical meaning but also wide applications in engineering, nuclear physics, biological chemistry technology, etc. Particularly, a coupled fixed point result in partially ordered metric spaces was established by Bhaskar and Lakshmikantham [5]. After the publication of this work, several coupled fixed point and coincidence point results have appeared in the recent literature.

In [5], Bhaskar and Lakshmikantham introduced the notions of a mixed monotone mapping and a coupled fixed point. Lakshmikantham et al. [8] Introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X and studied fixed point theorems in partially ordered metric spaces.

The aim of this paper is to prove a coupled coincidence and common fixed point theorems for commutating mappings with mixed g-monotone property in partially ordered S-metric spaces.

2. Preliminaries

Throughout this paper, we denotes $(X, \preceq)$ be a partially ordered set with the partial order $\preceq$. We also write

$x \prec y$ means $x \preceq y$ and $x \neq y$,
$x \succeq y$ means $y \preceq x$,
$x \succ y$ means $y \prec x$.

Definition 2.1. ([20]) Let X be a nonempty set. A real valued function S defined on $X \times X \times X$, *i.e.* ordered pairs of elements in X, is called an S-metric on X if it satisfies the following conditions: for each $x, y, z, a \in X$,

(S_1) $S(x, y, z) \geq 0$,
(S_2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(S_3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

We give some examples of such S-metric spaces:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, defined by

$$S(x, y, z) = ||y + z - 2x|| + ||y - z||.$$

Then S is an S-metric on X.

(2) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, defined by

$$S(x, y, z) = ||x - z|| + ||y - z||.$$

Then S is an S-metric on X.

(3) Let X be a nonempty set, d is ordinary metric on X, defined by

$$S(x, y, z) = d(x, y) + d(y, z).$$

Then S is an S-metric on X.

Lemma 2.2. ([20]) *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Definition 2.3. ([20]) Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows respectively:

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

Example 2.4. ([20]) Let $X = \mathbb{R}$. Define $S : X \times X \times X \to [0, \infty)$ by

$$S(x, y, z) = |y + z - 2x| + |y - z|$$

for all $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} B_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} \\ &= \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} \\ &= (0,2). \end{aligned}$$

Definition 2.5. ([20]) Let (X, S) be an S-metric space and $A \subset X$.

(1) A is called an open subset of X, if for every $x \in A$, there exists $r > 0$ such that

$$B_S(x,r) \subset A.$$

(2) A is called S-bounded, if there exists $r > 0$ such that

$$S(x,x,y) < r$$

for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x, denoted by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad S(x_n, x_n, x) \to 0$$

if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \text{implies} \quad S(x_n, x_n, x) < \varepsilon.$$

(4) A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.

(5) An S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

(6) Let $\mathcal{T}$ be the set of all subset A of X which satisfies for each $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$. Then $\mathcal{T}$ is a topology on X(induced by the S-metric S).

Definition 2.6. ([21]) Let (X, S) and (X', S') be two S-metric spaces, and let $f : (X, S) \to (X', S')$ be a function. Then f is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X, $S(x_n, x_n, a) \to 0$ implies $S'(f(x_n), f(x_n), f(a)) \to 0$. The function f is continuous on X if it is continuous at all $a \in X$.

Lemma 2.7. ([21]) *Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Definition 2.8. ([5]) Let $(X, \preceq)$ be a patially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.9. ([5]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if

$$F(x, y) = x \quad and \quad F(y, x) = y.$$

Definition 2.10. ([5]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$. The mapping F is said to has the mixed monotone property if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$,

$$x_1 \preceq x_2 \Longrightarrow F(x_1, y) \preceq F(x_2, y), \quad for \;\; x_1, x_2 \in X,$$

and

$$y_1 \preceq y_2 \Longrightarrow F(x, y_2) \preceq F(x, y_1), \quad for \;\; y_1, y_2 \in X.$$

This definition coincides with the notion of a mixed monotone function on $\mathbb{R}^2$ when $\leq$ represents the usual total order on $\mathbb{R}$.

Definition 2.11. ([8]) Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. The mapping F is said to has the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for any $x, y \in X$,

$$g(x_1) \preceq g(x_2) \Longrightarrow F(x_1, y) \preceq F(x_2, y), \quad for \;\; x_1, x_2 \in X \tag{2.1}$$

and

$$g(y_1) \preceq g(y_2) \Longrightarrow F(x, y_2) \preceq F(x, y_1), \quad for \;\; y_1, y_2 \in X. \tag{2.2}$$

Definition 2.12. ([8]) An element $(x, y) \in X \times X$ is said to be a coupled coincidence point of the mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if

$$F(x, y) = g(x) \quad and \quad F(y, x) = g(y).$$

It is a common coupled fixed point of F and g if

$$F(x, y) = g(x) = x \quad and \quad F(y, x) = g(y) = y.$$

Definition 2.13. ([8]) Let (X, d) be a metric space and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are mappings. We say that F and g are commutative if

$$F(g(x), g(y)) = g(F(x, y))$$

for all $x, y \in X$.

Definition 2.14. ([8]) Let (X, S) be an S-metric space. A mapping $F : X \times X \longrightarrow X$ is said to be continuous if for any two sequences $\{x_n\}$ and $\{y_n\}$ S-converging to x and y respectively, $\{F(x_n, y_n)\}$ is S-convergent to $F(x, y)$.

3. Main results

Theorem 3.1. *Let $(X, S, \preceq)$ be a partially ordered S-metric space. Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists a $k \in [0, \frac{1}{2})$ such that for $x, y, z, u, v, w \in X$, the following inequality holds:*

$$S(F(x,y), F(u,v), F(z,w)) \leq k[S(gx, gu, gz) + S(gy, gv, gw)], \tag{3.1}$$

for all $gx \succeq gu \succeq gz$ and $gy \preceq gv \preceq gw$ where either $gu \neq gz$ or $gv \neq gw$. We assume the following hypotheses:

(i) *$F(X \times X) \subseteq g(X)$,*
(ii) *$g(X)$ is S-complete,*
(iii) *g is S-continuous and commutates with F.*

Then F and g have a coupled coincidence point. If $gu = gz$ and $gv = gw$, then F and g have common fixed point, that is, there exist $x \in X$ such that

$$g(x) = F(x, x) = x.$$

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$.

Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad \textit{and} \quad g(y_{n+1}) = F(y_n, x_n), \quad \forall\, n \geq 0. \tag{3.2}$$

Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{3.3}$$

and

$$g(y_n) \succeq g(y_{n+1}). \tag{3.4}$$

We shall use the mathematical induction. Let $n = 0$. Since $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (3.3) and (3.4) hold for $n = 0$. We presume that (3.3) and (3.4) hold for some $n > 0$. As F has the mixed

g-monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (3.2), we get

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \tag{3.5}$$

and

$$F(y_{n+1}, x_n) \preceq F(y_n, x_n) = g(y_{n+1}). \tag{3.6}$$

Also for the same reason we have

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n)$$

and

$$F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g(y_{n+2}).$$

Then from (3.2) and (3.3), we obtain

$$g(x_{n+1}) \preceq g(x_{n+2}) \quad \textit{and} \quad g(y_{n+1}) \succeq g(y_{n+2}).$$

Thus by the mathematical induction, we conclude that (3.3) and (3.4) hold for all $n \geq 0$.

Continuing this process, one can easily verify that

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_{n+1}) \preceq \cdots$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq \cdots \succeq g(y_{n+1}) \succeq \cdots .$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then F and g have a coupled coincidence point. So we assume

$$(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$$

for all $n \geq 0$, that is, we assume that either

$$g(x_{n+1}) = F(x_n, y_n) \neq g(x_n) \quad \text{or} \quad g(y_{n+1}) = F(y_n, x_n) \neq g(y_n).$$

Next, we claim that, for all $n \geq 0$,

$$S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{2}(2k)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \tag{3.7}$$

For $n = 1$, we have

$$\begin{aligned} S(gx_1, gx_1, gx_2) &= S(F(x_0, y_0), F(x_0, y_0), F(x_1, y_1)) \\ &\leq k[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\ &= \frac{1}{2}(2k)^1[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \end{aligned}$$

Thus (3.7) holds for $n = 1$. Therefore, we presume that (3.7) holds $n > 0$. Since $g(x_{n+1}) \succeq g(x_n)$and $g(y_{n+1}) \preceq g(y_n)$, from (3.1) and (3.2), we have

$$\begin{aligned} &S(gx_n, gx_n, gx_{n+1}) \\ &= S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)]. \end{aligned} \tag{3.8}$$

From

$$\begin{aligned} &S(gx_{n-1}, gx_{n-1}, gx_n) \\ &= S(F(x_{n-2}, y_{n-2}), F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1})) \\ &\leq k[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})] \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &S(gy_{n-1}, gy_{n-1}, gy_n) \\ &= S(F(y_{n-2}, x_{n-2}), F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1})) \\ &\leq k[S(gy_{n-2}, gy_{n-2}, gy_{n-1}) + S(gx_{n-2}, gx_{n-2}, gx_{n-1})]. \end{aligned} \tag{3.10}$$

By combining (3.9) and (3.10), we have

$$\begin{aligned} &S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) \\ &\leq 2k[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})] \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, from (3.8)

$$\begin{aligned} &S(gx_n, gx_n, gx_{n+1}) \\ &\leq k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)] \\ &\leq 2k^2[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})] \\ &\vdots \\ &\leq \frac{1}{2}(2k)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \end{aligned}$$

Thus for each $n \in \mathbb{N}$, we have

$$\begin{aligned} &S(gx_n, gx_n, gx_{n+1}) \\ &\leq \frac{1}{2}(2k)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \end{aligned} \tag{3.11}$$

Let $m, n \in \mathbb{N}$ with $m > n$. By condition (S_3) in definition of S-metric, we have

$$\begin{aligned} S(gx_n, gx_n, gx_m) &\leq 2S(gx_n, gx_n, gx_{n+1}) + 2S(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\ &\quad + \cdots + S(gx_{m-1}, gx_{m-1}, gx_m) \\ &\leq 2\sum_{i=n}^{m-1} S(gx_i, gx_i, gx_{i+1}). \end{aligned}$$

Since $2k < 1$, by (3.11), we get

$$\begin{aligned} S(gx_n, gx_n, gx_m) &\leq 2 \times \frac{1}{2}\sum_{i=n}^{m-1}(2k)^i[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\ &\leq \frac{(2k)^n}{(1-2k)}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \end{aligned}$$

Letting $n, m \to +\infty$, we have

$$\lim_{n,m\to+\infty} S(gx_n, gx_n, gx_m) = 0.$$

Thus $\{gx_n\}$ is S-Cauchy in $g(X)$. Similarly, we may show that $\{gy_n\}$ is an S-Cauchy in $g(X)$. Since $g(X)$ is S-complete, we get $\{gx_n\}$ and $\{gy_n\}$ are S-convergent to some $x \in X$ and $y \in X$, respectively. Since g is S-continuous, we have $\{g(gx_n)\}$ is S-convergent to gx and $\{g(gy_n)\}$ is S-convergent to gy, *i.e.*,

$$\lim_{n\to+\infty} g(g(x_n)) = g(x) \quad \text{and} \quad \lim_{n\to+\infty} g(g(y_n)) = g(y). \tag{3.12}$$

Also, from commutativity of F and g, we have

$$F(g(x_n), g(y_n)) = g(F(x_n, y_n)) = g(g(x_{n+1})) \tag{3.13}$$

and

$$F(g(y_n), g(x_n)) = g(F(y_n, x_n)) = g(g(y_{n+1})). \tag{3.14}$$

Next, we claim that (x, y) is a coupled coincidence point of F and g. Now, from the condition (3.1), we have:

$$\begin{aligned} S(ggx_{n+1}, ggx_{n+1}, F(x, y)) &= S(F(gx_n, gy_n), F(gx_n, gy_n), F(x, y)) \\ &\leq k[S(ggx_n, ggx_n, gx) + S(ggy_n, ggy_n, gy)]. \end{aligned}$$

Letting $n \to +\infty$, by Lemma 2.7 yields

$$S(gx, gx, F(x, y)) \leq k[S(gx, gx, gx) + S(gy, gy, gy)] = 0.$$

Hence $gx = F(x, y)$. Similarly, we can show that $gy = F(y, x)$.

Finally, we claim that x is a common fixed point of F and g. Since (x, y) is a coupled coincidence point of the mappings F and g, we have $gx = F(x, y)$ and $gy = F(y, x)$. Assume $gx \neq gy$. Then by (3.1), we get

$$\begin{aligned} S(gx, gx, gy) &= S(F(x, y), F(x, y), F(y, x)) \\ &\leq k[S(gx, gx, gy) + S(gy, gy, gx)]. \end{aligned}$$

Also, by (3.1), we have

$$\begin{aligned} S(gy, gy, gx) &= S(F(y, x), F(y, x), F(x, y)) \\ &\leq k[S(gy, gy, gx) + S(gx, gx, gy)]. \end{aligned}$$

Therefore

$$S(gx, gx, gy) + S(gy, gy, gx) \leq 2k[S(gx, gx, gy) + S(gy, gy, gx)].$$

Since $2k < 1$, we get

$$S(gx, gx, gy) + S(gy, gy, gx) < S(gx, gx, gy) + S(gy, gy, gx).$$

This is a contradiction. So $gx = gy$. Hence

$$F(x, y) = gx = gy = F(y, x).$$

Since $\{gx_{n+1}\}$ is subsequence of $\{gx_n\}$, we have $\{gx_{n+1}\}$ is S-convergent to x. Thus

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx) &= S(gx_{n+1}, gx_{n+1}, F(x, y)) \\ &= S(F(x_n, y_n), F(x_n, y_n), F(x, y)) \\ &\leq k[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)]. \end{aligned}$$

Letting $n \to +\infty$, by Lemma 2.7 yields

$$S(x, x, gx) \leq k[S(x, x, gx) + S(y, y, gy)].$$

Similarly, we can show that

$$S(y, y, gy) \leq k[S(x, x, gx) + S(y, y, gy)].$$

Thus

$$S(x, x, gx) + S(y, y, gy) \leq 2k[S(x, x, gx) + S(y, y, gy)].$$

Since $2k < 1$, the last inequality happenes only if

$$S(x, x, gx) = 0 \quad \text{and} \quad S(y, y, gy) = 0.$$

Hence $x = gx$ and $y = gy$. Thus we get $gx = F(x, x) = x$. This means that F and g have a common fixed point. This completes the proof. □

Theorem 3.2. *In Theorem* 3.1, *we assume the following conditions in the complete S-metric space X instead of the condition* (ii), *namely,*

(j) *if* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \longrightarrow x$ *in* X*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*,*

(jj) *if* $\{y_n\} \subset X$ *is a nondecreasing sequence with* $y_n \longrightarrow y$ *in* X*, then* $y_n \succeq y$ *for all* $n \in \mathbb{N}$*.*

If g *is nondecreasing, then there exist* $x \in X$ *such that*

$$g(x) = F(x, x) = x.$$

Proof. According to the Theorem 3.1, we have $\{gx_n\}$ and $\{gy_n\}$ are S-Cauchy in X. Since (X, S) is a complete metric space, there exists $(x, y) \in X \times X$ such that

$$\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} g(x_n) = x \tag{3.15}$$

and

$$\lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} g(y_n) = y. \tag{3.16}$$

Therefore, from (iii), we can get (3.12), (3.13) and (3.14). Since $\{g(x_n)\}$ is a nondecreasing sequence and $g(x_n) \to x$, and as $\{g(y_n)\}$ is a nonincreasing sequence and $g(y_n) \to y$, by conditions (j) and (jj), we have $g(gx_n) \preceq g(x)$ and $g(gy_n) \succeq g(y)$ for all $n \geq 0$. If $g(gx_n) = g(x)$ and $g(gy_n) = g(y)$ for some n, then, by construction, $g(gx_{n+1}) = g(x)$, $g(gy_{n+1}) = g(y)$ and (x, y) is a coupled fixed point. So we assume either $g(gx_n) \neq g(x)$ or $g(gy_n) \neq g(y)$. Applying the contractive condition (3.1), we have

$$\begin{aligned}
&S(F(x, y), F(x, y), gx)\\
&\leq 2S(F(x, y), F(x, y), F(g(x_n), g(y_n)))\\
&\quad + S(F(g(x_n), g(y_n)), F(g(x_n), g(y_n)), gx)\\
&= 2S(F(g(x_n), g(y_n)), F(g(x_n), g(y_n)), F(x, y))\\
&\quad + S(gF(x_n, y_n), gF(x_n, y_n), gx)\\
&\leq 2k[S(g(gx_n), g(gx_n), gx) + S(g(gy_n), g(gy_n), gy)]\\
&\quad + S(g(gx_{n+1}), g(gx_{n+1}), gx).
\end{aligned}$$

Taking $n \to +\infty$ in the above inequality, we obtain $S(F(x, y), F(x, y), gx) = 0$, that is,

$$F(x, y) = g(x).$$

Similarly, we have that $F(y, x) = g(y)$. Remaining part of the proof follows from Theorem 3.1. Hence, we have

$$g(x) = F(x, x) = x.$$

This completes the proof of the Theorem 3.2. $\square$

Corollary 3.3. *Let $(X, S, \preceq)$ be a partially ordered S-metric space. Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists a $k \in [0, \frac{1}{2})$ such that for $x, y, u, v \in X$, the following inequality holds:*

$$S(F(x,y), F(x,y), F(u,v)) \leq k[S(gx, gx, gu) + S(gy, gy, gv)] \tag{3.17}$$

for all $gx \succeq gu$ and $gy \preceq gv$. We assume the following hypotheses:

(1) *$F(X \times X) \subseteq g(X)$,*
(2) *g is S-continuous and commutates with F.*

Either of the following conditions is satisfied:

(3) *$g(X)$ is S-complete, or*
(3') *g is a nondecreasing with*
   (i) *if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow x$ in X, then $x_n \preceq x$ for all $n \in \mathbb{N}$,*
   (ii) *if $\{y_n\} \subset X$ is a nondecreasing sequence with $y_n \longrightarrow y$ in X, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

Then there exists $x \in X$ such that

$$gx = F(x, x) = x.$$

Proof. Following from Theorem 3.1 by taking $z = u$ and $v = w$. □

Corollary 3.4. *Let $(X, S, \preceq)$ be a partially ordered S-metric space. Let $F : X \times X \longrightarrow X$ be a mapping having mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Assume that there exists a $k \in [0, \frac{1}{2})$ such that for $x, y, u, v \in X$,*

$$S(F(x,y), F(x,y), F(u,v)) \leq k[S(x, x, u) + S(y, y, v)] \tag{3.18}$$

for all $x \succeq u$ and $y \preceq v$. If the following conditions are satisfied;

(i) *if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow x$ in X, then $x_n \preceq x$ for all $n \in \mathbb{N}$,*
(ii) *if $\{y_n\} \subset X$ is a nondecreasing sequence with $y_n \longrightarrow y$ in X, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

Then there exists $x \in X$ such that $F(x, x) = x$.

Proof. Define $g : X \longrightarrow X$ by $gx = x$. Then F and g satisfy all the hypotheses of Corollary 3.3. Hence the result follows from Corollary 3.3. □

Theorem 3.5. *Let $(X, S, \preceq)$ be a partially ordered S-metric space. Let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are mappings such that F has the mixed g-monotone property on X and $F(x, y) \preceq F(y, x)$ whenever $x \preceq y$. Suppose*

(1) $F(X \times X) \subseteq g(X)$,

(2) *g is S-continuous and commutes with F. Assume that there exists a $k \in [0, \frac{1}{2})$ such that for $x, y, z, u, v, w \in X$, the inequality (3.1) holds, whenever $gx \succeq gu \succeq gz$ and $gy \preceq gv \preceq gw$ where either $gu \neq gz$ or $gv \neq gw$. If there exist two elements $x_0, y_0 \in X$ such that*

$$g(x_0) \preceq g(y_0), g(x_0) \preceq F(x_0, y_0) \quad \text{and} \quad g(y_0) \succeq F(y_0, x_0).$$

Either of the following conditions holds:

(3) *$g(X)$ is S-complete,*

(3') *g is a nondecreasing in the complete S-metric space (X, S) with*
  - (i) *if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow x$ in X, then $x_n \preceq x$ for all $n \in \mathbb{N}$,*
  - (ii) *if $\{y_n\} \subset X$ is a nondecreasing sequence with $y_n \longrightarrow y$ in X, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

Then, there exist $x \in X$ such that

$$g(x) = F(x, x) = x.$$

Proof. By the condition of the theorem, there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. We define $x_1, y_1 \in X$ as $g(x_0) \preceq F(x_0, y_0) = g(x_1)$ and $g(y_0) \succeq F(y_0, x_0) = g(y_1)$. Since $g(x_0) \preceq g(y_0)$, we have, by a condition of the theorem, $F(x_0, y_0) \preceq F(y_0, x_0)$. Hence

$$g(x_0) \preceq g(x_1) = F(x_0, y_0) \preceq F(y_0, x_0) = g(y_1) \preceq g(y_0).$$

Continuing the above procedure, we have two sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ recursively as follows:

$$g(x_n) = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad g(y_n) = F(y_{n-1}, x_{n-1}), \quad \forall\, n \geq 1 \tag{3.19}$$

such that

$$\begin{aligned} g(x_0) &\preceq F(x_0, y_0) = g(x_1) \preceq \cdots \preceq F(x_{n-1}, y_{n-1}) = g(x_n) \preceq \cdots \\ &\preceq g(y_n) = F(y_{n-1}, x_{n-1}) \preceq \cdots \preceq g(y_1) = F(y_0, x_0) \preceq g(y_0). \end{aligned} \tag{3.20}$$

In particular, we have

$$g(x_n) \preceq F(x_n, y_n) = g(x_{n+1}) \preceq g(y_{n+1}) = F(y_n, x_n) \preceq g(y_n), \quad \forall\, n \geq 0.$$

Let $x_n = y_n = c$ (say) for some n, then $g(c) \preceq F(c, c) \preceq F(c, c) \preceq g(c)$. This shows that

$$g(c) = F(c, c).$$

Thus (c, c) is a coupled fixed point. Hence we assume that

$$g(x_n) \prec g(y_n), \quad \forall\, n \geq 0. \tag{3.21}$$

Further, for the same reason as stated in Theorem 3.1, we assume that $(x_n, y_n) \neq (x_{n+1}, y_{n+1})$. Then, in view of (3.21), for all $n \geq 0$, the inequality (3.1) will hold with

$$x = x_{n+2}, \quad u = x_{n+1}, \quad w = x_n, \quad y = y_n, \quad v = y_{n+1}, \quad z = y_{n+2}.$$

The rest of the proof is completed by repeating the same steps as in Theorem 3.1 and Theorem 3.2. □

Now, we present examples to illustrate our obtained results given by Theorems 3.1 and Theorem 3.2 and to show that they are proper extension of some known results.

Example 3.6. Let $X = \mathbb{R}$ be ordered by the following relation

$$x \preceq y \quad \Longleftrightarrow \quad x = y \quad \text{or} \quad (x, y \in [0,1] \quad \text{and} \quad x \leq y).$$

Let an S-metric on X be defined by

$$S(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Then, $(X, S, \preceq)$ is a complete regular ordered S-metric space.

Let $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$ be defined by

$$g(x) = \begin{cases} \frac{x}{20}, & \text{if } x < 0, \\ \frac{x}{2}, & \text{if } x \in [0,1], \\ \frac{x}{20} + \frac{9}{20}, & \text{if } x > 1 \end{cases}$$

and

$$F(x, y) = \frac{x + y}{20}.$$

Take $k = \frac{1}{10}$. We will check that condition (3.1) of Theorem 3.2 is fulfilled for all $x, y, z, u, v, w \in X$ satisfying

$$[gz \preceq gu \preceq gx \quad \text{and} \quad gw \succeq gv \succeq gy]$$

or

$$[gx \preceq gu \preceq gz \quad \text{and} \quad gy \succeq gv \succeq gw].$$

The only nontrivial case is when $x, y, z, u, v, w \in [0,1]$ and

$$[z \leq u \leq x \quad \text{and} \quad w \geq v \geq y] \quad \text{or} \quad [x \leq u \leq z \quad \text{and} \quad y \geq v \geq w].$$

Then,

$$\begin{aligned}
&S(F(x,y),F(u,v),F(z,w))\\
&=|F(x,y)-F(u,v)|+|F(u,v)-F(z,w)|+|F(z,w)-F(x,y)|\\
&=\left|\frac{x+y}{20}-\frac{u+v}{20}\right|+\left|\frac{u+v}{20}-\frac{z+w}{20}\right|+\left|\frac{z+w}{20}-\frac{x+y}{20}\right|\\
&=\left|\frac{x-u}{20}+\frac{y-v}{20}\right|+\left|\frac{u-z}{20}+\frac{v-w}{20}\right|+\left|\frac{z-x}{20}+\frac{w-y}{20}\right|\\
&\leq\frac{1}{20}\{[|x-u|+|u-z|+|z-x|]+[|y-v|+|v-w|+|w-y|]\}\\
&=\frac{1}{10}[S(gx,gu,gz),+S(gy,gv,gw)]\\
&=k[S(gx,gu,gz)+S(gy,gv,gw)] \qquad (3.22)
\end{aligned}$$

and the condition holds. We conclude that all the conditions of Theorem 3.1 and Theorem 3.2 are satisfied. Obviously, the mappings g and F have unique common coupled fixed point $(0,0)$.

However, note that these theorems cannot be used in non-ordered case to reach this conclusion. Indeed, take

$$x=2,\quad u=2 \quad \text{and} \quad y=v=z=w=0.$$

Then condition (3.1) does not hold. Since

$$S(F(2,0),F(2,0),F(0,0))=S\left(\frac{1}{10},\frac{1}{10},0\right)=\frac{1}{5},$$

while

$$\begin{aligned}
k[S(g_2,g_2,g_0)+S(g_0,g_0,g_0)]&=\frac{1}{10}\left[S\left(\frac{11}{20},\frac{11}{20},0\right)+0\right]\\
&=\frac{1}{10}\cdot 2\cdot\frac{11}{20}=\frac{11}{100}<\frac{1}{5}
\end{aligned}$$

and obviously contractive condition (3.1) is not fulfilled.

Example 3.7. Let $X=[0,+\infty)$ be equipped with the S-metric S defined by

$$S(x,y,z)=|x-y|+|y-z|+|z-x|$$

and the order $\preceq$ defined by

$$x\preceq y \quad\Longleftrightarrow\quad x=y \vee (x,y\in[0,1] \wedge x\leq y).$$

Then $(X, S, \preceq)$ is a complete partially ordered S-metric space. Consider the (continuous) mapping $F : X \times X \longrightarrow X$ given by

$$F(x,y) = \begin{cases} \frac{x}{6}, & \text{if } x \in [0,1],\ y \in X, \\ x - \frac{5}{6}, & \text{if } x > 1,\ y \in X, \end{cases}$$

and take $g : X \longrightarrow X$ given by

$$gx = x.$$

Obviously, F has the g-mixed monotone property. Let $x, y, z, u, v, w \in X$ be such that $x \succeq u \succeq z$ and $y \preceq v \preceq w$. Then the following cases are possible.

Case 1. All of these variables belong to $[0, 1]$, hence $x \geq u \geq z$ and $y \leq v \leq w$. If we denote by L and R, the left-hand and right-hand side (with, say, $k = \frac{1}{4}$) of inequality (3.1), respectively. Then we have

$$\begin{aligned} L &= S\left(\frac{1}{6}x, \frac{1}{6}u, \frac{1}{6}z\right) \\ &= \frac{1}{6}(|x-u| + |u-z| + |z-x|) \\ &\leq \frac{1}{4}(|x-u| + |u-z| + |z-x| + |y-v| + |v-w| + |w-y|) \\ &= R. \end{aligned}$$

Case 2. Let $x, u, z \in [0, 1]$ (and $x \geq u \geq z$) and $y, v, w > 1$ (and $y = v = w$). Then we have

$$\begin{aligned} L &= S\left(\frac{1}{6}x, \frac{1}{6}u, \frac{1}{6}z\right) \\ &= \frac{1}{6}(|x-u| + |u-z| + |z-x|) \\ &\leq \frac{1}{4}(|x-u| + |u-z| + |z-x|) \\ &= R. \end{aligned}$$

The case when $x, u, z > 1$ and $y, v, w \in [0, 1]$ is treated similarly.

Case 3. Let $x, u, z, y, v, w > 1$. Then $x = u = z$, $y = v = w$ and $L = R = 0$.

Therefore, all the conditions of Theorem 3.1 are fulfilled and F and g have a common coupled fixed point (which is $(0, 0)$).

However, consider the same S-metric space (X, S) without order. Take

$$(x, y) = (2, 2), \quad (u, v) = (2.3) \quad \text{and} \quad (z, w) = (3, 3).$$

Then we have

$$L = S(F(2,2), F(2,3), F(3,3)) = S\left(\frac{7}{6}, \frac{7}{6}, \frac{13}{6}\right) = 2$$

and

$$R = k[S(2,2,3) + S(2,3,3)] = k[2+2] < 2,$$

i.e., $L > R$ whatever $k \in [0, \frac{1}{2})$ is chosen, and the contractive condition cannot be satisfied.

4. Conclusion

We prove a coupled coincidence and common fixed point theorems for commutating with mixed g-monotone property in the setting of a partially ordered S-metric space. Examples are given to support the usability of our results and to distinguish them from the other results. It is expect that this class will inspire and motivate further research in this area.

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