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COMMON FIXED POINT RESULTS FOR NONCOMMUTING MAPPINGS IN GENERALIZED CONE METRIC SPACES

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Abstract. Sufficient conditions for existence of fixed point in the setting of generalized cone metric spaces are obtained and then several common fixed point theorems are proved for two maps. These results generalize several well known comparable results in the literature.

1. Introduction and Preliminaries.

Recently the study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. Sessa [15] introduced the notion of weakly commuting maps. Jungck [5] coined the term compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. Afterwards, Jungck [7] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. For details on coincidence point theory, its applications, comparison of different

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contractive conditions and related results, we refer to ([2, 8, 6] and references contained therein).

To overcome fundamental flaws in Dhage's theory of generalized metric spaces [4], Mustafa and Sims [12] introduced a more appropriate generalization of metric spaces, that of $G-$ metric spaces. Afterwards, Mustafa et. el [11] obtained fixed point theorems for mappings satisfying different contractive conditions in $G-$ metric spaces. Abbas and Rhoades [1] obtained common fixed point results for noncommuting mappings without continuity in generalized metric spaces.

Guang and Xian [10] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Rezapour and Hamlbarani [13] showed the existence of a non normal cone metric space and obtained some fixed point results in cone metric spaces (See also [14]). In this paper, common fixed point theorems for two pairs of weakly compatible map, which are more general than $R-$ weakly commuting, and compatible mappings are obtained, in the setting of non normal generalized cone metric spaces, without exploiting the notion of continuity. Our results extend [1, Theorem 2.1-2.6] to a generalized cone metric space.

We first state following definitions which are needed in the sequel.

Let E be a real Banach space. A subset P of E is called a *cone* if and only if:

(a) P is closed, non empty and $P \neq \{0\}$;
(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$;
(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).

Rezapour [13] proved that there is no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$.

Definition 1.1. [3] Let X be a nonempty set. Suppose that the mapping $G : X \times X \times X \to E$ satisfies:

(a) $0 \leq G(x, y, z)$ for all $x, y, z \in X$ and $G(x, y, z) = 0$ if and only if $x = y = z$,
(b) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,
(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$,

(d) $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$, (symmetric in all three variables)
(e) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$. (rectangle inequality)

Then G is called a *generalized cone metric* on X or G- cone metric on X and (X,G) is called a G-cone *metric space*. The concept of a G-cone metric space is more general than that of G-metric spaces and cone metric spaces.

Definition 1.2. A G-cone metric space X is said to be symmetric if

$$G(x,y,y) = G(y,x,x) \text{ for all } x,y \in X.$$

Let X be a $G-$ cone metric space, define $d_G : X \times X \to E$ by

$$d_G(x,y) = G(x,y,y) + G(y,x,x).$$

Then it is noted that d_G is a cone metric on X. Also note that if X is symmetric $G-$ cone metric space, then

$$d_G(x,y) = 2G(x,y,y),$$

for all $x,y \in X$.

Definition 1.3. Let X be a G-cone metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is:

(a) a *Cauchy* sequence if, for every $c \in E$ with $0 \ll c$, there is N such that for all $n,m,l > N$, $G(x_n,x_m,x_l) \ll c$.
(b) a *convergent* sequence if, for every c in E with $0 \ll c$, there is N such that for all $n,m > N$, $G(x_n,x_m,x) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$.

A G-cone metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

Remark 1.4. [3]

(a) If $x \ll y \ll z$, then $x \ll z$.
(b) If $x \ll y \leq z$, then $x \ll z$.
(c) If $x \leq y \ll z$, then $x \ll z$.
(d) If E is a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P$ and $\lambda \in [0,1)$ then $a = 0$.

For the sake of completeness, we now state following basic facts (Lemmas 1.5 and 1.6) in a generalized cone metric space, proof is an easy exercise.

Lemma 1.5. [3] Let X be a G-cone metric space then the following are equivalent.

(i) $\{x_n\}$ is converges to x.

(ii) $G(x_n, x_n, x) \ll c$, as $n \to \infty$.
(iii) $G(x_n, x, x) \ll c$, as $n \to \infty$.
(iv) $G(x_n, x_m, x) \ll c$, as $m, n \to \infty$.

Lemma 1.6. [3] Let X be a G-cone metric space.

(i) If $\{x_m\}$, $\{y_n\}$, and $\{z_l\}$ are sequences in X such that $x_m \to x$, $y_n \to y$, and $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$ as m, n, $l \to \infty$.
(ii) Let $\{x_n\}$ be a sequence in X and $x \in X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.
(iii) Let $\{x_n\}$ be a sequence in X and $x \in X$. If $\{x_n\}$ converges to x, then $G(x_m, x_n, x) \to 0$ as m, $n \to \infty$.
(iv) Let $\{x_n\}$ be a sequence in X and $x \in X$. If $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is a Cauchy sequence.
(v) Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is a Cauchy sequence in X, then $G(x_m, x_n, x_l) \to 0$, as $m, n, l \to \infty$.

Definition 1.7. Let f and g be self maps of a set X. If $u = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g, and u is called a *point of coincidence* of f and g.

Lemma 1.8. ([1]**)** Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Definition 1.9. Let (X, d) is said be a cone metric space and f be a mapping of X into itself. An orbit of f at the point x in X is the set

$$O(x, f) = \{x, fx, f^2x, ..., f^nx, ...\}.$$

A cone metric space (X, d) is said to be f-*orbitally complete* if f is self-mapping of X and if any Cauchy subsequence $\{f^{n_i}x\}$ in orbit $O(x, f)$, $x \in X$, converges in X.
A mapping $f : X \to X$ is said to be *orbitally continuous* if $f^{n_i}x \to p \Rightarrow f(f^{n_i}x) \to fp$ as $i \to \infty$.

2. Common Fixed Points.

In this section we first obtain a fixed point theorem for a single map, and then obtain coincidence and common fixed point theorems for mappings defined on a G-cone metric space.

Theorem 2.1. Let X be a complete G-cone metric space. If there exists a point $u \in X$ and a $\lambda \in [0, 1)$ with $\overline{O(u)}$ complete and

$$G(fx, fy, fy) \leq \lambda G(x, y, z), \tag{2.1}$$

for each x, $z = y = fx \in O(u)$, then $\{f^n u\}$ converges to some point $p \in X$ and, for all $m, n \in \mathbb{N}, m > n$,

$$G(x_n, x_m, x_m) \leq \frac{\lambda^n}{1-\lambda} G(u, fu, fu),$$

for $n \geq 1$. Further, if f is orbitally continuous at p or if (2.1) holds for all $x \in \overline{O(u)}$, then p is a fixed point of f.

Proof. If G is symmetric, then

$$d_G(x, y) = 2G(x, y, y),$$

(2.1) becomes

$$d_G(fx, fy) \leq \lambda d_G(x, y)$$

for all $y \in O(x)$, and the result follows from [14, Theorem 2.3]. Suppose that G is not symmetric. With $x_n = f^n u$, one has from (2.1), that

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \lambda G(x_n, x_{n+1}, x_{n+1}) \\ &\leq \cdots \\ &\leq \lambda^n G(u, fu, fu). \end{aligned}$$

For all $m, n \in \mathbb{N}$, $m > n$, it follows that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) G(u, fu, fu) \\ &\leq \frac{\lambda^n}{1-\lambda} G(u, fu, fu). \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Also, choose a natural number N_1 such that $\frac{\lambda^n}{1-\lambda} G(u, fu, fu) \in N_\delta(0)$, for all $m \geq N_1$. Then, $\frac{\lambda^n}{1-\lambda} G(u, fu, fu) \ll c$, for all $m \geq N_1$. So we have $G(x_n, x_m, x_m) \ll c$, for all $m > n$. Thus $\{x_n\}$ is a Cauchy sequence, so there exist $p \in X$ such that $\{x_n\}$ converges to p.

If f is orbitally continuous at $x = p$, then $\lim f^n u = p$ implies that $\lim(f^{n+1}u) = fp$, and p is a fixed point of f. If (2.1) holds for all $x, y = z \in \overline{O(u)}$, then we have

$$G(fp, f^{n+1}u, f^{n+1}u) \leq \lambda G(p, f^n u, f^n u) \ll c,$$

whenever $n > N$. Thus $G(fp, f^{n+1}u, f^{n+1}u) \ll c$. By Lemma 1.6(ii) implies $fp = p$. $\square$

Example 2.2. Let $X = [0,1]$ and

$$G(x,y,z) = (|x-y| + |y-z| + |z-x|, \alpha(|x-y| + |y-z| + |z-x|)),$$

where α is a positive constant, be a $G-$cone metric on X. Define $f : X \to X$ as $f(x) = \dfrac{x}{2}$. Take $u = \frac{1}{2} \in X$, then $\overline{O(u)} = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\}$ and it may be verified that $G(fx, fy, fy) \leq \lambda G(x,y,z)$ for each x, $z = y = fx \in O(u)$, where $\lambda = \dfrac{1}{2}$. Obviously $\{f^n u\}$ converges to 0 and f is orbitally continuous at $0 \in X$. Moreover, 0 is a fixed point of f.

Theorem 2.3. Let X be a $G-$cone metric space. Suppose that the mappings $f, g : X \longrightarrow X$ satisfy one of the following condition

$$\begin{aligned} G(fx, fy, fz) \leq\ & aG(gx, gy, gz) + bG(gx, fx, fx) \\ & + cG(gy, fy, fy) + dG(gz, fz, fz), \end{aligned} \tag{2.2}$$

or

$$\begin{aligned} G(fx, fy, fz) \leq\ & aG(gx, gy, gz) + bG(gx, gx, fx) \\ & + cG(gy, gy, fy) + dG(gz, gz, fz), \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$, where $a + b + c + d < 1$. If the range of g contains the range of f and $g(X)$ is complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Suppose that f and g satisfy condition (2.2). Then for all $x, y \in X$ we have

$$G(fx, fy, fy) \leq aG(gx, gy, gy) + bG(gx, fx, fx) + (c+d)G(gy, fy, fy).$$

Also,

$$G(fy, fx, fx) \leq aG(gy, gx, gx) + bG(gy, fy, fy) + (c+d)G(gx, fx, fx),$$

If G is symmetric, then adding above two given inequalities we have

$$d_G(fx, fy) \leq a d_G(gx, gy) + \frac{b+c+d}{2} d_G(fx, gx) + \frac{b+c+d}{2} d_G(gy, fy). \tag{2.4}$$

Since $a + b + c + d < 1$, the existence and uniqueness of a common fixed point follows from [9, Theorem 2.8]. However, if (X, G) is not symmetric, then (2.4) gives no information about the maps, as in this case, the contractive constant need not be less that 1. In this case, let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the range of

g contains the range of f. Continuing this process, having chosen x_n in X, we obtain an x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq aG(gx_{n-1}, gx_n, gx_n) + bG(gx_{n-1}, fx_{n-1}, fx_{n-1}) \\ &\quad +(c+d)G(gx_n, fx_n, fx_n), \end{aligned}$$

that is,

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &\leq (a+b)G(gx_{n-1}, gx_n, gx_n) \\ &\quad +(c+d)G(gx_n, gx_{n+1}, gx_{n+1}), \end{aligned}$$

which implies that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq kG(gx_{n-1}, gx_n, gx_n),$$

where, $0 \leq k = \dfrac{a+b}{1-c-d} < 1$. Continuing the above process, we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq k^n G(gx_0, gx_1, gx_1).$$

Then, for $m > n$,

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\ &\quad +G(gx_{n+2}, gx_{n+3}, gx_{n+3}) + ... + G(gx_{m-1}, gx_m, gx_m) \\ &\leq (k^n + k^{n+1} + ... + k^{m-1})G(gx_0, gx_1, gx_1) \\ &= \frac{k^n}{1-k} G(gx_0, gx_1, gx_1). \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Also, choose a natural number N_1 such that

$$\frac{k^n}{1-k} G(gx_0, gx_1, gx_1) \in N_\delta(0),$$

for all $n \geq N_1$. Then, $\dfrac{k^n}{1-k} G(gx_0, gx_1, gx_1) \ll c$, for all $n \geq N_1$. So we have $G(gx_n, gx_m, gx_m) \ll c$, for all $m > n$. Hence $\{gx_n\}$ is a Cauchy sequence. Since $g(X)$ is complete, there exists, a point q such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find a point p in X such that $g(p) = q$. We claim that $f(p) = g(p)$. For this consider

$$\begin{aligned} d(gx_n, fp, fp) &= G(fx_{n-1}, fp, fp) \\ &\leq aG(gx_{n-1}, gp, gp) + bG(gx_{n-1}, fx_{n-1}, fx_{x-1}) \\ &\quad +(c+d)G(gp, fp, fp), \end{aligned}$$

which, on taking the limit as $n \to \infty$ implies that

$$G(gp, fp, fp) \leq (c+d)G(gp, fp, fp),$$

Remark 1.4(d) implies that $G(gp, fp, fp) = 0$, and hence $f(p) = g(p)$. Hence f and g have a coincidence point in X.

Assume that f and g are weakly compatible. Now we show that f and g have a unique point of coincidence. For this, suppose that there exist a point q in X such that $f(q) = g(q)$. We need to prove $g(p) = g(q)$. For this

$$\begin{aligned} G(gq, gp, gp) &= G(fq, fp, fp) \\ &\leq aG(gq, gp, gp) + bG(gq, fq, fq) \\ &\quad +(c+d)G(gp, fp, fp), \end{aligned}$$

that is,

$$G(gq, gp, gp) \leq aG(gq, gp, gp).$$

Hence by Remarks 1.4(d) $G(gq, gp, gp) = 0$, and $gq = gp$. From Lemma 1.8, f and g have a unique common fixed point. The proof using (2.3) is similar. □

Theorem 2.4. Let X be a $G-$ cone metric space and $f, g : X \to X$, be two mappings such that for some $m \in \mathbb{N}$, satisfies one of the following condition

$$\begin{aligned} G(f^m x, f^m y, f^m z) &\leq aG(g^m x, g^m y, g^m z) + bG(g^m x, f^m x, f^m x) \\ &\quad +cG(g^m y, f^m y, f^m y) + dG(g^m z, f^m z, f^m z), \quad (2.5) \end{aligned}$$

or

$$\begin{aligned} G(f^m x, f^m y, f^m z) &\leq aG(g^m x, g^m y, g^m z) + bG(g^m x, g^m x, f^m x) \\ &\quad +cG(g^m y, g^m y, f^m y) + dG(g^m z, g^m z, f^m z), \quad (2.6) \end{aligned}$$

for all $x, y, z \in X$, where $a + b + c + d < 1$. If the range of g contains the range of f and $g(X)$ is complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. It follows from Theorem 2.3, that f^m and g^m have a unique common fixed point p. Now $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$, and $g(p) = g(g^m(p)) = g^{m+1}(p) = g^m(g(p))$ implies that $f(p)$ and $g(p)$ are also fixed points for f^m and g^m. Hence $f(p) = g(p) = p$. □

Theorem 2.5. Let X be a $G-$cone metric space. Suppose that the mappings $f, g : X \longrightarrow X$ satisfy either

$$G(fx, fy, fz) \leq ku_{(f,g)}(x, y, z),$$

where

$$u_{(f,g)}(x, y, z) \in \{G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz)\}, \quad (2.7)$$

or

$$u_{(f,g)}(x, y, z) \in \{G(gx, gx, fx), G(gy, gy, fy), G(gz, gz, fz)\}, \quad (2.8)$$

for all $x, y, z \in X$ where $0 \leq k < 1$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Suppose f and g satisfy condition (2.7). Then for all $x, y \in X$, we have

$$G(fx, fy, fy) \leq ku_{(f,g)}(x, y, y),$$

where

$$\begin{aligned} u_{(f,g)}(x, y, y) &\in \{G(gx, fx, fx), G(gy, fy, fy), G(gy, fy, fy)\} \\ &= \{G(gx, fx, fx), G(gy, fy, fy)\}. \end{aligned} \tag{2.9}$$

Now interchanging the role of x and y we obtain

$$G(fy, fx, fx) \leq ku_{(f,g)}(y, x, x),$$

where

$$\begin{aligned} u_{(f,g)}(y, x, x) &\in \{G(gy, fy, fy), G(gx, fx, fx), G(gx, fx, fx)\} \\ &= \{G(gy, fy, fy), G(gx, fx, fx)\}. \end{aligned} \tag{2.10}$$

If G is symmetric, then adding above two given inequalities we have

$$d_G(fx, fy) \leq k[u_{(f,g)}(x, y, y) + u_{(f,g)}(y, x, x)].$$

Now four cases arises:

(1) If $u_{(f,g)}(x, y, y) = G(gx, fx, fx)$ and $u_{(f,g)}(y, x, x) = G(gy, fy, fy)$, then

$$\begin{aligned} d_G(fx, fy) &\leq k[G(gx, fx, fx) + G(gy, fy, fy)] \\ &= \frac{k}{2}[d_G(gx, fx) + d_G(gy, fy)]. \end{aligned}$$

Since $k < 1$, the existence and uniqueness of a common fixed point follows from [9, Corollary 2.4].

(2) If $u_{(f,g)}(x, y, y) = G(gx, fx, fx)$ and $u_{(f,g)}(y, x, x) = G(gx, fx, fx)$, then

$$\begin{aligned} d_G(fx, fy) &\leq k[G(gx, fx, fx) + G(gx, fx, fx)] \\ &= kd_G(gx, fx). \end{aligned}$$

Since $k < 1$, the existence and uniqueness of a common fixed point follows from [9, Corollary 2.7].

(3) If $u_{(f,g)}(x, y, y) = G(gy, fy, fy)$ and $u_{(f,g)}(y, x, x) = G(gy, fy, fy)$, then

$$\begin{aligned} d_G(fx, fy) &\leq k[G(gy, fy, fy) + G(gy, fy, fy)] \\ &= kd_G(gy, fy). \end{aligned}$$

Since $k < 1$, the existence and uniqueness of a common fixed point follows from [9, Corollary 2.7].

(4) If $u_{(f,g)}(x,y,y) = G(gy,fy,fy)$ and $u_{(f,g)}(y,x,x) = G(gy,fy,fy)$, then

$$\begin{aligned} d_G(fx,fy) &\leq k[G(gy,fy,fy)+G(gx,fx,fx)] \\ &= \frac{k}{2}[d_G(gy,fy)+d_G(gx,fx)]. \end{aligned}$$

Since $k < 1$, the existence and uniqueness of a common fixed point follows from [9, Corollary 2.4].

However, if (X,G) is not symmetric, then (2.9) gives no information about the maps, as in this case, the contractive constant need not be less that 1. In this case, let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. From (2.9),

$$\begin{aligned} G(gx_n,gx_{n+1},gx_{n+1}) &= G(fx_{n-1},fx_n,fx_n) \\ &\leq ku_{(f,g)}(x_{n-1},x_n,x_n) \end{aligned}$$

where

$$u_{(f,g)}(x_{n-1},x_n,x_n) \in \{G(gx_{n-1},gx_n,gx_n), G(gx_n,gx_{n+1},gx_{n+1})\}.$$

If $u_{(f,g)}(x_{n-1},x_n,x_n) = G(gx_n,gx_{n+1},gx_{n+1})$, then

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq kG(gx_n,gx_{n+1},gx_{n+1}).$$

Remark 1.4(d), implies $gx_n = gx_{n+1}$ for each n, then since $gx_{n+1} = fx_n$, f and g have a coincidence point. Now if $u_{(f,g)}(x_{n-1},x_n,x_n) = G(gx_{n-1},gx_n,gx_n)$, then

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq kG(gx_{n-1},gx_n,gx_n),$$

and, continuing the above process, we obtain

$$G(gx_n,gx_{n+1},gx_{n+1}) \leq k^n G(gx_0,gx_1,gx_1).$$

As in the case of Theorem 2.3, it follows that $\{gx_n\}$ is a Cauchy sequence and hence converges to a point q in $g(X)$. Choose p so that $g(p) = q$. Suppose that $f(p) \neq g(q)$. Then

$$\begin{aligned} G(gx_n,fp,fp) &= G(fx_{n-1},fp,fp) \\ &\leq ku_{(f,g)}(x_{n-1},p,p), \end{aligned}$$

where

$$\begin{aligned} u_{(f,g)}(x_{n-1},p,p) &\in \{G(gx_{n-1},fx_{n-1},fx_{n-1}), G(gp,fp,fp), G(gp,fp,fp)\} \\ &= \{G(gx_{n-1},fx_{n-1},fx_{n-1}), G(gp,fp,fp)\}. \end{aligned}$$

If $u_{(f,g)}(x_{n-1}, p, p) = G(gx_{n-1}, fx_{n-1}, fx_{n-1})$, then

$$G(gx_n, fp, fp) \leq kG(gx_{n-1}, fx_{n-1}, fx_{n-1}),$$

which, on taking the limit as $n \to \infty$, implies that

$$G(gp, fp, fp) \leq kG(gp, fp, fp),$$

by using Remark 1.4(d), we get $gp = fp$. Also if $u_{(f,g)}(x_{n-1}, p, p) = G(gp, fp, fp)$, then again on taking the limit as $n \to \infty$, implies that

$$G(gp, fp, fp) \leq kG(gp, fp, fp),$$

by same above argument, we have $f(p) = g(p)$. Now we show that f and g have a unique point of coincidence. Assume that there exists another point q in X such that $fq = gq$. Now for

$$G(gq, gp, gp) = G(fq, fp, fp) \leq ku_{(f,g)}(q, p, p),$$

where

$$u_{(f,g)}(q, p, p) \in \{G(gq, fq, fq), G(gp, fp, fp), G(gp, fp, fp)\} = \{0\}.$$

Hence $G(gq, gp, gp) \leq 0$ implies that $-G(gp, gp, gq) \in P$. But $G(gp, gp, gq) \geq 0$, therefore $G(gp, gp, gq) = 0$, and hence $gq = gp$. From Lemma 1.8, f and g have a unique common fixed point. The proof using (2.8) is similar. □

Corollary 2.6. Let X be a $G-$cone metric space and $f, g : X \to X$ be two mappings such that for some $m \in \mathbb{N}$, satisfies

$$G(f^m x, f^m y, f^m z) \leq hu_{(f,g)}(x, y, z), \tag{2.11}$$

where

$$u_{(f,g)}(x, y, z) \in \{G(g^m x, g^m x, g^m x), G(g^m y, f^m y, f^m y), G(g^m z, f^m z, f^m z)\},$$

for all $x, y, z \in X$, where $0 \leq h < \frac{1}{2}$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Theorem 2.7. Let X be a $G-$ cone metric space. Suppose that the mappings $f, g : X \longrightarrow X$ satisfy one of the following conditions

$$G(fx, fy, fy) \leq a\{G(gx, fy, fy) + G(gy, fx, fx)\}, \tag{2.12}$$

or

$$G(fx, fy, fy) \leq a\{G(gx, gx, fy) + G(gy, gy, fx)\}, \tag{2.13}$$

for all $x, y \in X$, where $0 \leq a < \frac{1}{2}$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. If X is symmetric, then from (2.12), we have

$$G(fy, fx, fx) \leq a\{G(gy, fx, fx) + G(gx, fy, fy)\}. \tag{2.14}$$

Adding (2.12) and (2.14), we have

$$d_G(fx, fy) \leq a\{d_G(gx, fy) + d_G(fx, gy)\}. \tag{2.15}$$

Since $a < \frac{1}{2}$, the existence and uniqueness of a common fixed point follows from [9, Corollary 2.5]. However, if X is not symmetric, then (2.12) gives no information about the maps, as in this case, the contractive constant need not be less that 1. In this case, let x_0 be an arbitrary point in X. Choose $\{x_n\}$ as in Theorem 2.3. Then

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq a\{G(gx_{n-1}, fx_n, fx_n) + G(gx_n, fx_{n-1}, fx_{n-1})\}, \\ &= aG(gx_{n-1}, gx_{n+1}, gx_{n+1}) \\ &\leq aG(gx_{n-1}, gx_n, gx_n) + aG(gx_n, gx_{n+1}, gx_{n+1}), \end{aligned}$$

it gives,

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq kG(gx_{n-1}, gx_n, gx_n),$$

where $k = \frac{a}{1-a}, 0 \leq k < 1$. Continuing the above process we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq k^n G(gx_0, gx_1, gx_1).$$

Using the same argument as that of Theorem 2.3 yields the result. □

Example 2.8. Let $X = [0, 1]$ and

$$G(x, y, z) = (|x - y| + |y - z| + |z - x|, \alpha(|x - y| + |y - z| + |z - x|)),$$

for $\alpha \geq 0$, be a $G-$cone metric on X. Define $f, g : X \to X$ by $f(x) = \frac{x}{8}$, and $g(x) = \frac{x}{2}$.

Note that

$$G(gx, fy, fy) = \frac{1}{2}(\left|x - \frac{y}{4}\right|, \alpha\left|x - \frac{y}{4}\right|),$$

and

$$G(gy, fx, fx) = \frac{1}{2}(\left|y - \frac{x}{4}\right|, \alpha\left|y - \frac{x}{4}\right|).$$

Now

$$\begin{aligned} G(fx,fy,fy) &= \frac{1}{8}(|x-y|\,,\alpha\,|x-y|) \\ &\le \frac{1}{8}(\left|x-\frac{y}{4}\right| + \left|y-\frac{x}{4}\right|, \alpha(\left|x-\frac{y}{4}\right| + \left|y-\frac{x}{4}\right|)) \\ &= \frac{1}{4}(\frac{1}{2}\left|x-\frac{y}{4}\right| + \frac{1}{2}\left|y-\frac{x}{4}\right|, \alpha(\frac{1}{2}\left|x-\frac{y}{4}\right| + \frac{1}{2}\left|y-\frac{x}{4}\right|)) \\ &= \frac{1}{4}\left\{G(gx,fy,fy)+G(gy,fx,fx)\right\}. \end{aligned}$$

Therefore

$$G(fx,fy,fy) \le a\left\{G(gx,fy,fy)+G(gy,fx,fx)\right\},$$

is satisfied for all $x, y \in X$, where $a = \frac{1}{4} < \frac{1}{2}$. Also, range of g contains the range of f, $g(X)$ is a complete subset of X and f and g are weakly compatible. Therefore, f and g satisfy all conditions of Theorem 2.7. Here, 0 is a unique common fixed point of f and g.

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