Nonlinear Functional Analysis and Applications
Vol. 15, No. 2 (2010), pp. 247-260

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

SOME GENERALIZATIONS OF POLYNOMIAL INEQUALITIES

A. Liman[1], W. M. Shah[2] and Shamim Ahmad Bhat[3]

[1]Department of Mathematics, National Institute of Technology, Kashmir, 190006, India
e-mail: abliman22@yahoo.co.in

[2]P.G.Department of Mathematics, Kashmir University, 190006, India
e-mail: wmshah@rediffmail.com

[3]Vinayaka Mission Research Foundation, Deemed University, Tamilnadu, 636308, India
e-mail: bhatshamim@gmail.com

Abstract. In this paper, we investigate the dependence of $\left\{ \int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}$ on $\left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}$ for each real or complex number β with $\beta \leq 1,\ R > r \geq 1$ and $q > 0$ and present compact generalizations of some well-known polynomial inequalities.

1. Introduction

Let P_n be the class of polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. Then

$$\left\{ \int_0^{2\pi} |P^{'}(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \leq n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}},\ q \geq 1 \tag{1}$$

and

$$\left\{ \int_0^{2\pi} |P(Re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \leq R^n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}},\ R \geq 1,\ q > 0. \tag{2}$$

[0]Received February 20, 2009. Revised October 18, 2009.
[0]200 Mathematics subject classification: 26D05, 30D15, 30A10, 26A48.
[0]Keywords: Polynomials, growth, integral mean estimates, zeros.

Inequality (1) is due to Zygmund [17], whereas inequality (2) is a simple consequence of a result due to Hardy [7]. Arestove [2] verified that (1) remains true for $0 < q < 1$ as well. If we make $q \to \infty$ in inequalities (1) and (2) and note that

$$\lim_{q\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} = \max_{|z|=1}|P(z)|,$$

we get

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| \tag{3}$$

and

$$\max_{|z|=R>1}|P(z)| \leq R^n \max_{|z|=1}|P(z)|. \tag{4}$$

Inequality (3) is an immediate consequence of a famous results due to Bernstein on the derivative of a trigonometric polynomial (for reference see [9, p.531], [10], [16]), whereas inequality (4) is a simple deduction from the maximum modulus principle (see [9, p. 346], [11] or [13, p. 158 problem 269]).

Inequalities (1) and (2) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In such a case, we have

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|P^{'}(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq n\ A_q\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}},\ q \geq 1 \tag{5}$$

where

$$A_q = \left\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\alpha}|^q d\alpha\right\}^{\frac{-1}{q}},$$

and

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq B_q\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}},\ q \geq 1, \tag{6}$$

where

$$B_q = \left\{\frac{1}{2\pi}\int_0^{2\pi}|1+R^n e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}} / \left\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}.$$

Inequality (5) is due to deBrujin [6, Theorem 13], whereas inequality (6) was proved by Boas and Rahman [5]. Both these inequalities were latter extended by Rahman and Schmeisser [14] for $0 < q < 1$ as well.

Aziz and Rather [3] investigated the dependence of

$$\left\{\int_0^{2\pi} |P(Re^{i\theta}) - \beta P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}$$

on $\left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}$ for $q \geq 0$ and proved the following compact generalization of (1) and (2).

Theorem A. If $P \in P_n$, then for every real or complex number β with $|\beta| \leq 1,\ R \geq 1$ and $q > 0$,

$$\left\{\int_0^{2\pi} |P(Re^{i\theta}) - \beta P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq |R^n - \beta| \left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{7}$$

In this paper, we first prove the following more general result which also yields a compact generalization of inequalities (1) and (2).

Theorem 1. If $P \in P_n$, then for every real or complex number β with $|\beta| \leq 1,\ R \geq r \geq 1$ and $q > 0$,

$$\left\{\int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq |R^n - \beta r^n| \left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{8}$$

The result is best possible and equality in (8) holds for $P(z) = \alpha z^n,\ \alpha \neq 0$.

Remark. For $r = 1$, Theorem 1 reduces to Theorem A, for $\beta = 0$, it reduces to inequality (2) and for $\beta = 1$, we get the following:

Corollary 1. If $P \in P_n$, then for $R \geq r \geq 1$ and $q > 0$,

$$\left\{\int_0^{2\pi} |P(Re^{i\theta}) - P(re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq |R^n - r^n| \left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{9}$$

For $r = 1$, Corollary 1 reduces to a result proved by Aziz and Rather [3, inequality (8)].

If we divide the two sides of (8) by $R - r$ and make $R \to r$, we get the following:

Corollary 2. If $P \in P_n$, then for $r \geq 1$ and $q > 0$,

$$\int_0^{2\pi} |P'(re^{i\theta})|^q d\theta \leq nr^{n-1} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta.$$

By taking $r = 1$ in Corollary 2, we get Zygmund's inequality for every $q > 0$.

If we let $q \to \infty$ in (8), we immediately get the following result which is a compact generalization of inequalities (3) and (4).

Corollary 3. If $P \in P_n$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$\max_{|z|=1} |P(Rz) - \beta P(rz)| \leq |R^n - \beta r^n| \max_{|z|=1} |P(z)|. \tag{10}$$

The result is sharp and equality in (10) holds for $P(z) = \alpha z^n, \ \alpha \neq 0$.

For polynomials $P \in P_n$ having no zeros in $|z| < 1$, we next prove the following interesting result which among other things includes deBruijn's theorem (inequality (5)) and a result of Boas and Rahman (inequality (6)) as special cases.

Theorem 2. If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1, \ R \geq r \geq 1$ and $q > 0$,

$$\left\{ \int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}$$

$$\leq \frac{\left\{ \int_0^{2\pi} |(R^n - \beta r^n)e^{i\alpha} + (1-\beta)|^q d\alpha \right\}^{\frac{1}{q}}}{\left\{ \int_0^{2\pi} |1 + e^{i\alpha}|^q d\alpha \right\}^{\frac{1}{q}}} \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}. \tag{11}$$

For $\beta = 0$, Theorem 2 reduces to a result due to Boas and Rahman (inequality (6)) for each $q > 1$. A variety of interesting results can be easily deduced from Theorem 2. For example the following Corollary which is an improvement of inequality (8) for polynomials $P \in P_n$ having no zeros in $|z| < 1$, immediately follows from Theorem 2 by taking $\beta = 1$.

Corollary 4. If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $R \geq r \geq 1$ and $q > 0$, we have

$$\left\{ \int_0^{2\pi} |P(Re^{i\theta}) - P(re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \leq \frac{R^n - r^n}{\left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\alpha}|^q d\alpha \right\}^{\frac{1}{q}}} \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}. \tag{12}$$

For $r = 1$, Theorem 2 reduces to a result earlier proved by Aziz and Rather [3, Theorem 4].

Again, by making $q \to \infty$ in (11), we immediately get the following:

Corollary 5. If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\max_{|z|=1} |P(Rz) - \beta P(rz)| \leq \frac{|(R^n - \beta r^n) + (1-\beta)|}{2} \max_{|z|=1} |P(z)|. \tag{13}$$

The result is sharp and equality in (13) holds for $P(z) = z^n + 1$.

Taking $\beta = 1$ and dividing the two sides of inequality (13) by $R - r$ and letting $R \to r$, we get

$$\max_{|z|=1} |P'(rz)| \leq \frac{nr^{n-1}}{2} \max_{|z|=1} |P(z)|, \ r \geq 1. \tag{14}$$

For $r = 1$, inequality (14) was conjectured by Erdös and later verified by Lax [8]. Also, if we take $\beta = 0$ in (13), we get a result proved by Ankeny and Rivilin [1].

2. Lemmas

For the proofs of these theorems, we need the following lemmas:

The first lemma is based on a result of Arestov, which we shall describe first.

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n) \in C^{n+1}$ and $P(z) = \sum\limits_{j=0}^{n} a_j z^j$, we define

$$\Lambda_\gamma P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties :

(i) $P(z)$ has all its zeros in $\{z \in C : |z| \leq 1\}$,

(ii) $P(z)$ has all its zeros in $\{z \in C : |z| \geq 1\}$.

The result of Arestov [2, Theorem 4] may now be stated as follows:

Lemma 1. For polynomials $P(z)$ of degree at most n and each admissible operator Λ_γ,

$$\left\{ \int_0^{2\pi} \left| \Lambda_\gamma P(e^{i\theta}) \right|^q d\theta \right\}^{\frac{1}{q}} \leq C(\gamma, n) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{\frac{1}{q}}, \ 0 < q < \infty,$$

where $C(\gamma, n) = \max\left(|\gamma_0|, \ |\gamma_n|\right)$.

Lemma 2. If $P \in P_n$ and $P(z)$ has all zeros in $|z| \leq 1$, then for $R > r \geq 1$,

$$|P(Rz)| > |P(rz)| \ for \ |z| \geq 1. \tag{15}$$

The above Lemma is a special case of a result due to Aziz and Zargar [4, Lemma 3].

Lemma 3. If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$,

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)|, \ for \ |z| \geq 1 \ and \ R \geq r \geq 1, \quad (16)$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Proof. For $R = r > 1$, the result follows by observing that $|P(z)| \leq |Q(z)|$ for $|z| \geq 1$. Henceforth, we assume that $R > r \geq 1$. Since the polynomial $P(z)$ has all its zeros in $|z| \geq 1$, therefore for every real or complex number α with $|\alpha| > 1$, the polynomial $F(z) = P(z) - \alpha Q(z)$ where $Q(z) = z^n \overline{P(1/\bar{z})}$ has all its zeros in $|z| \leq 1$. Applying Lemma 2 to the polynomial $F(z)$, we get

$$|F(rz)| < |F(Rz)| \ for \ |z| = 1 \ and \ R > r \geq 1.$$

Using Rouche's theorem and noting that all the zeros of $F(Rz)$ lie in $|z| \leq (1/R) < 1$, we conclude that for every real or complex number β with $|\beta| \leq 1$, the polynomial $G(z) = F(Rz) - \beta F(rz)$ has all its zeros in $|z| < 1$. Replacing $F(z)$ by $P(z) - \alpha Q(z)$, it follows that all the zeros of the polynomial

$$G(z) = (P(Rz) - \beta P(rz)) - \alpha(Q(Rz) - \beta Q(rz)) \quad (17)$$

lie in $|z| < 1$ for every α, β with $|\alpha| > 1, \ |\beta| \leq 1$ and $R > r \geq 1$. This implies

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| \ for \ |z| \geq 1 \ and \ R > r \geq 1. \quad (18)$$

If inequality (18) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|P(Rz_0) - \beta P(rz_0)| > |Q(Rz_0) - \beta Q(rz_0)|.$$

Since all the zeros of $Q(rz)$ lie in $|z| \leq 1$, it follows that all the zeros of $Q(Rz) - \beta Q(rz)$ lie in $|z| < 1$ for every real or complex number β with $|\beta| \leq 1$ and $R > 1$. Hence $Q(Rz_0) - \beta Q(rz_0) \neq 0$ with $|z_0| \geq 1$. We choose

$$\alpha = P(Rz_0) - \beta P(rz_0)/Q(Rz_0) - \beta Q(rz_0), \quad (19)$$

so that α is well-defined real or complex number with $|\alpha| > 1$ and with this choice of α, from (17) we get

$$G(z_0) = 0 \ with \ |z_0| \geq 1.$$

This is clearly a contradiction to the fact that all the zeros of $G(z)$ lie in $|z| < 1$. Thus for every β with $|\beta| \leq 1$,

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| \ for \ |z| \geq 1 \ and \ R > r \geq 1.$$

This proves Lemma 3. □

Next, we use Lemma 1 to prove the following result:

Lemma 4. If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$, $R \geq r \geq 1$, $q > 0$ and α real

$$\int_0^{2\pi} \left| \left(P(Re^{i\theta}) - \beta P(re^{i\theta}) \right) + e^{i\alpha} \left(R^n P(e^{i\theta}/R) - \bar{\beta} r^n P(e^{i\theta}/r) \right) \right|^q d\theta$$

$$\leq \left| (R^n - \bar{\beta} r^n) e^{i\alpha} + (1-\beta) \right|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{20}$$

Proof. If $Q(z) = z^n \overline{P(1/\bar{z})}$, where $R \geq r \geq 1$, then we have by Lemma 3

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)|, \ for \ |z| \geq 1$$

and

$$|P(Rz) - \beta P(rz)| = \left| R^n P(z/R) - \bar{\beta} r^n P(z/r) \right|, \ for \ |z| = 1.$$

Now if $H(z) = Q(Rz) - \beta Q(rz)$, then $H(z)$ has all its zeros in $|z| < 1$ for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$. Therefore, it follows that the polynomial $z^n \overline{H(1/\bar{z})} = R^n P(z/R) - \bar{\beta} r^n P(z/r)$ has all its zeros in $|z| > 1$. Hence $G(z) = \frac{P(Rz) - \beta P(rz)}{R^n P(z/R) - \bar{\beta} r^n P(z/r)}$ is analytic in $|z| \leq 1$ and $|G(z)| \leq 1$ for $|z| = 1$. Since $G(z)$ is not constant, it follows by maximum modulus principle that $|G(z)| < 1$ for $|z| < 1$.
Equivalently,

$$|P(Rz) - \beta P(rz)| < \left| R^n P(z/R) - \bar{\beta} r^n P(z/r) \right|, \ for \ |z| < 1. \tag{21}$$

By Rouche's theorem

$$\Lambda_\gamma P(z) = \left(P(Re^{i\theta}) - \beta P(re^{i\theta}) \right) + e^{i\alpha} \left(R^n P(z/R) - \bar{\beta} r^n P(z/r) \right)$$

$$= \left((R^n - \beta r^n) + e^{i\alpha}(1 - \bar{\beta}) \right) a_n z^n + \cdots + \left((1-\beta) + e^{i\alpha}(R^n - \bar{\beta} r^n) \right) a_0$$

does not vanish in $|z| < 1$, for every real or complex number β with $|\beta| \leq 1$, $R \geq r \geq 1$ and α real. Therefore Λ_γ is an admissible operator and hence by Lemma 1, we have for $q > 0$,

$$\left\{ \int_0^{2\pi} \left| \Lambda_\gamma P(e^{i\theta}) \right|^q d\theta \right\}^{\frac{1}{q}} \leq C(\gamma, n) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{\frac{1}{q}}.$$

where $C(\gamma,n)=\max\left(|\gamma_0|,\ |\gamma_n|\right)$.
This implies

$$\int_0^{2\pi}\left|\left(P(Re^{i\theta})-\beta P(re^{i\theta})\right)+e^{i\alpha}\left(R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)\right)\right|^q d\theta$$

$$\leq\left|(R^n-\bar{\beta}r^n)e^{i\alpha}+(1-\beta)\right|^q\int_0^{2\pi}|P(e^{i\theta})|^q d\theta,$$

which is inequality (20) and this proves Lemma 4. □

Lemma 5. If $P\in P_n$ then for every real or complex number β with $|\beta|\leq 1$, $R\geq r\geq 1$, $q>0$ and α real

$$\int_0^{2\pi}\left|\left(P(Re^{i\theta})-\beta P(re^{i\theta})\right)+e^{i\alpha}\left(R^nP(e^{i\theta}/R)-\bar{\beta}r^nP(e^{i\theta}/r)\right)\right|^q d\theta$$

$$\leq\left|(R^n-\bar{\beta}r^n)e^{i\alpha}+(1-\beta)\right|^q\int_0^{2\pi}|P(e^{i\theta})|^q d\theta. \tag{22}$$

Proof. The result is trivial for $R=r=1$. Henceforth, we assume $R>r\geq 1$. Since $P(z)$ is a polynomial of degree at most n, we can write

$$P(z)=P_1(z)P_2(z)=\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(z-z_j),\ k\geq 0$$

where all the zeros of $P_1(z)$ lie in $|z|\geq 1$ and all the zeros of $P_2(z)$ lie in $|z|<1$. First we suppose that $P_1(z)$ has no zero on $|z|=1$ so that all the zeros of $P_1(z)$ lie in $|z|>1$. Let $Q_2(z)=z^{n-k}\overline{P_2(1/\bar{z})}$. Then all the zeros of $Q_2(z)$ lie in $|z|>1$ and $|Q_2(z)|=|P_2(z)|$ for $|z|=1$. Now consider the polynomial

$$F(z)=P_1(z)Q_2(z)=\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(1-z\bar{z}_j).$$

Then all the zeros of $F(z)$ lie in $|z|>1$ and for $|z|=1$,

$$|F(z)|=|P_1(z)||Q_2(z)|=|P_1(z)||P_2(z)|=|P(z)|. \tag{23}$$

Since $P(z)/F(z)$ is not a constant by the maximum modulus principle

$$|P(z)|<|F(z)|\ for\ |z|\leq 1.$$

Using Rouche's theorem, it follows that the polynomial $G(z)=P(z)+\lambda F(z)$ does not vanish in $|z|\leq t,\ t>1$ and for every real or complex number λ with

$|\lambda| > 1$. Hence for every $t > 1$, all the zeros of $G(tz)$ lie in $|z| \geq 1$. Applying (21) to the polynomial $G(tz)$, we get

$$\left|G(Rtz) - \beta G(rtz)\right| < \left|R^n G(tz/R) - \bar{\beta} r^n G(tz/r)\right| \; for \; |z| < 1.$$

Taking $z = e^{i\theta}/t, \; 0 \leq \theta < 2\pi$, then $|z| < 1/t < 1$ and we get

$$\left|G(Re^{i\theta}) - \beta G(re^{i\theta})\right| < \left|R^n G(e^{i\theta}/R) - \bar{\beta} r^n G(e^{i\theta}/r)\right|,$$

for each $\theta, \; 0 \leq \theta < 2\pi, \; R > r \geq 1$ and β with $|\beta| \leq 1$. This implies

$$\left|G(Rz) - \beta G(rz)\right| < \left|R^n G(z/R) - \bar{\beta} r^n G(z/r)\right|, \; for \; |z| = 1.$$

Since $R^n G(z/R) - \bar{\beta} r^n G(z/r)$ does not vanish in $|z| \leq 1$, therefore an application of Rouche's theorem shows that the polynomial

$$T(z) = \left(G(Rz) - \beta G(rz)\right) + e^{i\alpha}\left(R^n G(z/R) - \bar{\beta} r^n G(z/r)\right)$$

does not vanish in $|z| \leq 1$, for every real or complex number β with $|\beta| \leq 1$, $R \geq r \geq 1$ and α real. Replacing $G(z)$ by $P(z) + \lambda F(z)$, it follows that the polynomial

$$T(z) = \left(P(Rz) - \beta P(rz)\right) + e^{i\alpha}\left(R^n P(z/R) - \bar{\beta} r^n P(z/r)\right)$$

$$+\lambda\left[\left(F(Rz) - \beta F(rz)\right) + e^{i\alpha}\left(R^n F(z/R) - \bar{\beta} r^n F(z/r)\right)\right],$$

does not vanish in $|z| \leq 1$, for every $\beta, \; \lambda$ with $|\beta| \leq 1, \; |\lambda| > 1$. This implies by the similar argument as in the proof of Lemma 3 that

$$\left|\left(P(Rz) - \beta P(rz)\right) + e^{i\alpha}\left(R^n P(z/R) - \bar{\beta} r^n P(z/r)\right)\right|$$

$$\leq \left|\left(F(Rz) - \beta F(rz)\right) + e^{i\alpha}\left(R^n F(z/R) - \bar{\beta} r^n F(z/r)\right)\right|, \tag{24}$$

for $|z| \leq 1$, which in particular gives for $R > r \geq 1, \; |\beta| \leq 1$ and $|z| = 1$

$$\left|\left(P(Rz) - \beta P(rz)\right) + e^{i\alpha}\left(R^n P(z/R) - \bar{\beta} r^n P(z/r)\right)\right|$$

$$\leq \left|\left(F(Rz) - \beta F(rz)\right) + e^{i\alpha}\left(R^n F(z/R) - \bar{\beta} r^n F(z/r)\right)\right|.$$

Hence for each $q>0$ and $0\le\theta<2\pi$, we obtain

$$\int_0^{2\pi}\left|\Big(P(Rz)-\beta P(rz)\Big)+e^{i\alpha}\Big(R^nP(z/R)-\bar{\beta}r^nP(z/r)\Big)\right|^q d\theta$$

$$\le\int_0^{2\pi}\left|\Big(F(Rz)-\beta F(rz)\Big)+e^{i\alpha}\Big(R^nF(z/R)-\bar{\beta}r^nF(z/r)\Big)\right|^q d\theta.$$

Since $F(z)$ does not vanish in $|z|<1$, therefore using Lemma 4 and (23), it follows that for every β with $|\beta|\le 1$, $R\ge r\ge 1$, $q>0$ and α real,

$$\int_0^{2\pi}\left|\Big(P(Rz)-\beta P(rz)\Big)+e^{i\alpha}\Big(R^nP(z/R)-\bar{\beta}r^nP(z/r)\Big)\right|^q d\theta$$

$$\le\left|(R^n-\bar{\beta}r^n)e^{i\alpha}+(1-\beta)\right|^q\int_0^{2\pi}|F(e^{i\theta})|^q d\theta$$

$$=\left|(R^n-\bar{\beta}r^n)e^{i\alpha}+(1-\beta)\right|^q\int_0^{2\pi}|P(e^{i\theta})|^q d\theta. \tag{25}$$

Now if $P_1(z)$ has a zero on $|z|=1$, then the polynomial $P^*(z)=P_1(uz)P_2(z)$ where $u<1$, does not vanish in $|z|<1$. Therefore, applying (25), we get for every β with $|\beta|\le 1$, $R\ge r\ge 1$, $q>0$ and α real,

$$\int_0^{2\pi}\left|\Big(P^*(Rz)-\beta P(rz)\Big)+e^{i\alpha}\Big(R^nP^*(z/R)-\bar{\beta}r^nP^*(z/r)\Big)\right|^q d\theta$$

$$\le\left|(R^n-\bar{\beta}r^n)e^{i\alpha}+(1-\beta)\right|^q\int_0^{2\pi}|P^*(e^{i\theta})|^q d\theta. \tag{26}$$

Letting $u\to 1$ in (26) so that $P^*\to P$ and using continuity, the desired result follows immediately and this proves Lemma 5. □

3. Proofs of the theorems

Proof of Theorem 1. The result is trivial for $R=r=1$. Henceforth, we assume $R>r\ge 1$. Since $P(z)$ is a polynomial of degree atmost n, we can write

$$P(z)=P_1(z)P_2(z)=\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(z-z_j),\ k\ge 0$$

where all the zeros of $P_1(z)$ lie in $|z| \leq 1$ and all the zeros of $P_2(z)$ lie in $|z| > 1$. First we suppose that $P_1(z)$ has no zero on $|z| = 1$ so that all the zeros of $P_1(z)$ lie in $|z| < 1$. Let $Q_2(z) = z^{n-k}\overline{P_2(1/\bar{z})}$. Then all the zeros of $Q_2(z)$ lie in $|z| < 1$ and $|Q_2(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$f(z) = P_1(z)Q_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(1 - \bar{z}_j),\ k \geq 0,$$

then all the zeros of $f(z) = \sum_{j=0}^{n} b_j z^j$ lie in $|z| < 1$ and for $|z| = 1$,

$$|f(z)| = |P_1(z)||Q_2(z)| = |P_1(z)||P_2(z)| = |P(z)|.$$

Since $P(z)/f(z)$ is not a constant by the maximum modulus principle

$$|P(z)| < |f(z)| \ for\ |z| > 1.$$

Using Rouche's theorem, it follows that the polynomial $g(z) = P(z) + \lambda f(z)$ has all its zeros in $|z| < 1$ and for every real or complex number λ with $|\lambda| > 1$. Hence by Lemma 1, we have

$$|g(rz)| < |g(Rz)| \ for\ |z| = 1\ and\ R > r \geq 1.$$

Since all the zeros of $g(Rz)$ lie in $|z| < (1/R) < 1$, we conclude that for every β with $|\beta| \leq 1$, all the zeros of $h(z) = g(Rz) - \beta g(rz) = (P(Rz) - \beta P(rz)) + \lambda(f(Rz) - \beta f(rz))$ lie in $|z| < 1$. This implies (as in the case of Lemma 5)

$$|P(Rz) - \beta P(rz)| \leq |f(Rz) - \beta f(rz)| \ for\ |z| \geq 1\ and\ R > r \geq 1.$$

This in particular gives for each $R > r \geq 1$ and $q > 0$,

$$\int_0^{2\pi} \left|P(Re^{i\theta}) - \beta P(re^{i\theta})\right|^q d\theta \leq \int_0^{2\pi} \left|f(Re^{i\theta}) - \beta f(re^{i\theta})\right|^q d\theta. \tag{27}$$

Again, since all the zeros of $f(z)$ lie in $|z| < 1$, therefore as before $f(Rz) - \beta f(rz)$ has all its zeros in $|z| < 1$ for every real or complex number β with $|\beta| \leq 1$ and hence the operator Λ_γ defined by

$$\Lambda_\gamma f(z) = P(Rz) - \beta f(rz) = (R^n - \beta r^n)b_n z^n + \cdots + (1 - \beta)b_0$$

is admissible. Thus by Lemma 1, for each $q > 0$, we have

$$\left\{\int_0^{2\pi} \left|f(Re^{i\theta}) - \beta f(re^{i\theta})\right|^q d\theta\right\}^{\frac{1}{q}} \leq \max\Big(|R^n - \beta r^n|, |1 - \beta|\Big)\left\{\int_0^{2\pi} \left|f(e^{i\theta})\right|^q d\theta\right\}^{\frac{1}{q}}. \tag{28}$$

Combining inequality (28) with (27) and noting that $|f(e^{i\theta})| = |P(e^{i\theta})|$, we get

$$\left\{ \int_0^{2\pi} \left| P(Re^{i\theta}) - \beta P(re^{i\theta}) \right|^q d\theta \right\}^{\frac{1}{q}} \leq |R^n - \beta r^n| \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{\frac{1}{q}}.$$

In case $P_1(z)$ has a zero on $|z| = 1$, the inequality (28) follows by using similar argument as in the case of Lemma 5. This completes the proof of Theorem 1.

Proof of Theorem 2. By hypothesis $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, therefore by Lemma 3, for every real or complex number β with $|\beta| \leq 1$, and $\theta,\ 0 < \theta \leq 2\pi$,

$$\left| P(Re^{i\theta}) - \beta P(re^{i\theta}) \right| \leq \left| R^n P(e^{i\theta}/R) - \bar{\beta} r^n P(e^{i\theta}/r) \right|, \ R \geq r \geq 1. \tag{29}$$

Also by Lemma 5

$$\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^q d\theta \leq \left| (R^n - \bar{\beta} r^n) e^{i\alpha} + (1 - \beta) \right|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \tag{30}$$

where

$$F(\theta) = P(Re^{i\theta}) - \beta P(re^{i\theta}) \ and \ G(\theta) = R^n P(e^{i\theta}/R) - \bar{\beta} r^n P(e^{i\theta}/r).$$

Integrating both sides of (30) with respect to α from 0 to 2π, we get for each $q > 0,\ R \geq r \geq 1$ and α real,

$$\int_0^{2\pi} \int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^q d\theta d\alpha$$

$$\leq \left\{ \int_0^{2\pi} \left| (R^n - \bar{\beta} r^n) e^{i\alpha} + (1 - \beta) \right|^q d\alpha \right\} \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}. \tag{31}$$

If $F(\theta) \neq 0$, then by (29), $\left| G(\theta)/F(\theta) \right| \geq 1$ and therefore, we have

$$\begin{aligned}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^q d\alpha &\le |F(\theta)|^q\int_0^{2\pi}\left|1+e^{i\alpha}\frac{G(\theta)}{F(\theta)}\right|^q d\alpha\\ &= |F(\theta)|^q\int_0^{2\pi}\left|\left|\frac{G(\theta)}{F(\theta)}\right|+e^{i\alpha}\right|^q d\alpha\\ &\le |F(\theta)|^q\int_0^{2\pi}\left|1+e^{i\alpha}\right|^q d\alpha\\ &= \left|P(Re^{i\theta})-\beta P(re^{i\theta})\right|^q\int_0^{2\pi}\left|1+e^{i\alpha}\right|^q d\alpha.\end{aligned}$$

For $F(\theta)=0$, this inequality is trivially true. Using this in (31), we conclude that for every real or complex number β with $|\beta|\le 1$, $R\ge 1$, $q>0$ and α real,

$$\left\{\int_0^{2\pi}|1+e^{i\alpha}|^q d\alpha\right\}\left\{\int_0^{2\pi}|P(Re^{i\theta})-\beta P(e^{i\theta})|^q d\theta\right\}$$
$$\le\left\{\int_0^{2\pi}\left|(R^n-\bar{\beta}r^n)e^{i\alpha}+(1-\beta)\right|^q d\alpha\right\}\left\{\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}. \tag{32}$$

And also, we have

$$\begin{aligned}&\left\{\int_0^{2\pi}\left|(R^n-\bar{\beta}r^n)e^{i\alpha}+(1-\beta)\right|^q d\alpha\right\}\\ =&\left\{\int_0^{2\pi}\left||(R^n-\bar{\beta}r^n)|e^{i\alpha}+|1-\beta|\right|^q d\alpha\right\}\\ =&\left\{\int_0^{2\pi}\left||(R^n-\beta r^n)|e^{i\alpha}+|1-\beta|\right|^q d\alpha\right\}\\ =&\left\{\int_0^{2\pi}\left|(R^n-\beta r^n)e^{i\alpha}+(1-\beta)\right|^q d\alpha\right\}.\end{aligned} \tag{33}$$

Combining (32) and (33), we get the desired result.

References

[1] N. C.Ankeny and T. J. Rivlin, *On a theorem of S.Bernstein,* Pacific J.Math., 5 (1955), 849-852.

[2] V. V. Arestov, *On inequalities for trigonometric polynomials and their derivatives,* Izv. Akad. Nauk SSSR Ser. Mat., 45 (1981), 3-22.

[3] A. Aziz and N. A. Rather, *Some compact generalizations of Zygmund type inequalities for polynomials,* Nonlinear Studies, 6 (2), (1999), 241-255.

[4] A. Aziz and B. A. Zargar, *Inequalities for a polynomial and its derivatives,* Math. Inequal. and Appl., 1(4), (1998), 543=550.

[5] R. P Boas, Jr. and Q. I. Rahman, *Lp inequalities for polynomials and entire functions,* Arch Rational Mech. Anal., II (1962), 34-39.

[6] N. G. de Bruijn, *Inequalities concerning polynomials in the complex domain*, Nederl. Akad. Wetensch. Proc. Ser. A, 50 (1947,) 1265-1272; Indag. Math. 9 (1947) 591-598.

[7] G. H. Hardy, *The mean value of the modulus of an analytic function*, Proc. London. Math. Soc., 14 (1915), 269-277.

[8] P. D. Lax, *Proof of a conjucture of P.Erdös on the derivative of a polynomial,* Bull. Amer. Math. Soc. (N.S), 50 (1944), 509-513.

[9] G. V. Milovanovic, D. S. Mitrinovic and Th. M. Rassias, *Topics in polynomials extremal properties,inequalities,zeros,* World Scientific Publishing Co., Singapore, 1994.

[10] G. V. Milovanovic and Th. M. Rassias, *New developments on Turan's extremal problems for polynomials,* Approx. Theory, Marcel and Dekker, Inc., New York, 1998, 433-447.

[11] G. V. Milovanovic and Th. M. Rassias, *Inequalities for polynomial zeros, Survey on classical inequalities,* Kluwer Academic Publishers, Dorderecht, 2000, 165-202.

[12] G. V. Milovanovic, D. S. Mitrinovic and Th. M. Rassias, *On some extremal problems for algebraic polynomials in Lr norm, Generalised functions and convergence,* World Scientific Publishing Co., New jersy, London, 1990, 343-354.

[13] G. Polya and G. Szego, *Problems and Theorems in Analysis,* Vol I. Springer Verlag, New York, 1972.

[14] Q. I. Rahman and G. Schmeisser, L^p *inequalities for polynomials,* J. Approx. Theory, 53 (1988), 26-32.

[15] Th. M. Rassias, H. M. Srivastava and A. Yanushauskas, *Topics in polynomials in one and several variables and their applications,* World Scientific Publishing Co., Singapore, New Jersy, London, 1993.

[16] A. C. Schaeffer, *Inequalities of A.Markoff and S.Bernstein for polynomials and related functions,* Bull. Amer. Math. Soc., 47 (1941), 565-579.

[17] A. Zygmund, *A remark on Canjugate series,* Proc London. Math., Soc., 34 (2) (1932), 392-400.