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STABILITY OF FUNCTIONAL EQUATIONS OF SEVERAL VARIABLES DERIVING FROM ADDITIVE AND QUADRATIC MAPS

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Abstract. In this paper, we establish the Hyers–Ulam–Rassias stability of the system of functional equations with n variables which are additive on the first k variables, and are quadratic on other $n-k$ variables.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [25] in 1940, concerning the stability of group homomorphisms. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [8] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

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for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in t for each fixed $x \in E$, then T is linear. In 1978, Th. M. Rassias [16] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded (see also [5], [6], [9-12] and [16-23]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{1.1}$$

is related to symmetric bi-additive function [1,2,13,15]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [1,15]). A Hyers–Ulam–Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \longrightarrow B$, where A is normed space and B Banach space (see [24]). Cholewa [3] noticed that the Theorem of Skof is still true if relevant domain A is replaced an abelian group. In the paper [4] , Czerwik proved the Hyers–Ulam–Rassias stability of the equation (1.1). Grabiec [7] has generalized these result mentioned above.

Let $n > 1$ and let $X_1, X_2, ..., X_n$ be a normed spaces and Y be a Banach space. In this paper we investigate the Hyers-Ulam-Rassias stability of the system of functional equations

$$\begin{cases} f(x_1, x_2, \cdots, x_{i-1}, a+b, x_{i+1}, \cdots, x_n) \\ = f(x_1, x_2, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) \\ \quad + f(x_1, x_2, \cdots, x_{i-1}, b, x_{i+1}, \cdots, x_n), \\ f(x_1, x_2, \cdots, x_{j-1}, a+b, x_{j+1}, \cdots, x_n) \\ \quad + f(x_1, x_2, \cdots, x_{j-1}, a-b, x_{j+1}, \cdots, x_n) \\ = 2f(x_1, x_2, \cdots, x_{j-1}, a, x_{j+1}, \cdots, x_n) \\ \quad + 2f(x_1, x_2, \cdots, x_{j-1}, b, x_{j+1}, \cdots, x_n) \end{cases} \tag{1.2}$$

where $1 \leq i \leq k < n,\ k+1 < j \leq n$, and f is a mapping from $X_1 \times X_2 \times \cdots \times X_n$ into Y.

It is easy to see that the functions $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_k x_{k+1}^2 x_{k+2}^2 \cdots x_n^2$$

satisfies (1.3). As another example, let A be a normed algebra and let X be a Banach $A-$module. Define $f : A^{n-1} \times X \to A$, $g : X \times A^{n-1} \to X$ by

$$f(a_1, a_2, \cdots a_{n-1}, x) = a_1 a_2 \cdots a_k a_{k+1}^2 a_{k+2}^2 \cdots a_{n-1}^2 \|x\|^2,$$

and

$$g(x, a_1, a_2, \cdots a_{n-1}) = x a_1 a_2 \cdots a_{k-1} \|a_k\|^2 \|a_{k+1}\|^2 \cdots \|a_{n-1}\|^2,$$

respectively. It is easy to see that f, g satisfying (1.2).

2. Main results

From now on, $X_1, X_2, \cdots, X_n$ will be normed spaces and Y will be a Banach space. Let $n > 1$ and let $1 \leq k < n, 1 \leq r \leq n$. Let $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ be a mapping. For each $(x_1, x_2, \cdots x_{i-1}, a, b, x_{i+1}, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_{i-1} \times X_i \times X_i \times X_{i+1} \times \cdots \times X_n$, we denote $D_f^r(x_1, x_2, \cdots x_{r-1}, a, b, x_{r+1}, \cdots, x_n)$ by

$$\begin{cases} \|f(x_1, x_2, \cdots, x_{r-1}, a+b, x_{r+1}, \cdots, x_n) \\ -f(x_1, x_2, \cdots, x_{r-1}, a, x_{r+1}, \cdots, x_n) \\ -f(x_1, x_2, \cdots, x_{r-1}, b, x_{r+1}, \cdots, x_n)\|, & \text{if } 1 \leq r \leq k; \\ \|f(x_1, x_2, \cdots, x_{r-1}, a+b, x_{r+1}, \cdots, x_n) \\ +f(x_1, x_2, \cdots, x_{r-1}, a-b, x_{r+1}, \cdots, x_n) \\ -2f(x_1, x_2, \cdots, x_{r-1}, a, x_{r+1}, \cdots, x_n) \\ -2f(x_1, x_2, \cdots, x_{r-1}, b, x_{r+1}, \cdots, x_n)\|, & \text{if } k < r \leq n; \end{cases}$$

Let $\phi_r, \psi_r : X_1 \times X_2 \times \cdots \times X_{r-1} \times X_r \times X_r \times X_{r+1} \times \cdots \times X_n \to [0, \infty)$ be mappings. Then for every $m \in \mathbb{N}, (x_1, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$, we define

$$\begin{aligned} \phi_r^m((x_1, x_2 \cdots, x_n)) :=& \chi_{\{1,2,\cdots,k\}}(r) \phi_r(2^m x_1, 2^m x_2, \cdots, 2^m x_{r-1}, 2^{m-1} x_r, \\ & 2^{m-1} x_r, 2^{m-1} x_{r+1}, \cdots, 2^{m-1} x_k, 2^m x_{k+1}, \cdots, 2^m x_n), \end{aligned}$$

$$\begin{aligned} \psi_r^m((x_1, x_2, \cdots, x_n)) :=& \chi_{\{k+1,k+2,\cdots,n\}}(r) \psi_r(2^{m-1} x_1, 2^{m-1} x_2, \cdots, 2^{m-1} x_r, \\ & 2^{m-1} x_r, 2^m x_{r+1}, \cdots, 2^m x_n). \end{aligned}$$

Theorem 2.1. *Let $\phi_r, \psi_r : X_1 \times X_2 \times \cdots \times X_{r-1} \times X_r \times X_r \times X_{r+1} \times \cdots \times X_n \to [0, \infty)$ be mappings for all $r \in \{1, 2, \cdots, n\}$. Let*

$$\sum_{m=0}^{\infty} \frac{\phi_r^m((x_1, x_2, \cdots, x_n)) + \psi_r^m((x_1, x_2, \cdots, x_n))}{(2^k 4^{n-k})^m} < \infty, \tag{2.1}$$

for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$, and

$$\lim_m \chi_{\{1,2,\cdots,k\}}(r)\frac{\phi_r(2^m x_1, 2^m x_2, \cdots, 2^m x_{r-1}, 2^m x_r, 2^m x_r, 2^m x_{r+1}, \cdots, 2^m x_n)}{(2^k 4^{n-k})^m}$$
$$= 0,$$

and

$$\lim_m \chi_{\{k+1,k+2,\dots,n\}}(r)\frac{\psi_r(2^m x_1, 2^m x_2, \dots, 2^m x_{r-1}, 2^m x_r, 2^m x_r, 2^m x_{r+1}, \dots, 2^m x_n)}{(2^k 4^{n-k})^m}$$
$$= 0, \tag{2.2}$$

for all $(x_1, x_2, \cdots, x_{r-1}, x_r, x_r, x_{r+1}, \cdots, x_n, x_n) \in X_1 \times X_2 \times \cdots \times X_{r-1} \times X_r \times X_r \times X_{r+1} \times \cdots \times X_n$. *Suppose* $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ *is a mapping such that*

$$D_f^r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n)$$
$$\leq \phi_r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n),$$

if $1 \leq r \leq k$*; and*

$$D_f^r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n)$$
$$\leq \psi_r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n), \tag{2.3}$$

if $k < r \leq n$*; for all* $(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_{r-1} \times X_r \times X_r \times X_{r+1} \times \cdots \times X_n, r \in \{1, 2, \cdots, n\}$. *Then there exists a unique mapping* $T : X_1 \times X_2 \times \cdots \times X_n \to Y$ *satisfying (1.2) and*

$$\|T(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)\| \leq \sum_{m=0}^{\infty} \frac{1}{(2^k 4^{n-k})^m}[\sum_{r=1}^{k} 2^{r-1}\phi_r^m$$
$$\times((x_1, x_2, \cdots, x_n)) + \sum_{r=k+1}^{n} 4^{r-1}2^k \psi_r^m((x_1, x_2, \cdots, x_n))], \tag{2.4}$$

for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$.

Proof. Let $r \in \{1, 2, \cdots, k\}$. Replacing a, b by x_r in (2.3) to get

$$\|f(x_1, x_2, \cdots, x_{r-1}, 2x_r, x_{r+1}, \cdots, x_n) - 2f(x_1, x_2, \cdots, x_n)\|$$
$$\leq \phi_r(x_1, x_2, \cdots, x_n). \tag{2.5}$$

By (2.5), we conclude that

$$\begin{aligned}
&\|f(2x_1, 2x_2, \cdots, 2x_k, x_{k+1}, \cdots, x_n) - 2^k f(x_1, x_2, \cdots, x_n)\| \\
&\le \|f(2x_1, 2x_2, \cdots, 2x_k, x_{k+1}, \cdots, x_n) \\
&\quad - 2f(2x_1, 2x_2, \cdots, 2x_{k-1}, x_k, x_{k+1}, \cdots, x_n)\| \\
&\quad + \|2f(2x_1, 2x_2, \cdots, 2x_{k-1}, x_k, x_{k+1}, \cdots, x_n) \\
&\quad - 2^2 f(2x_1, 2x_2, \cdots, 2x_{k-2}, x_{k-1}, x_k, \cdots, x_n)\| \\
&\quad + \|2^2 f(2x_1, 2x_2, \cdots, 2x_{k-2}, x_{k-1}, x_k, \cdots, x_n) \\
&\quad - 2^3 f(2x_1, 2x_2, \cdots, 2x_{k-3}, x_{k-2}, x_{k-1}, \cdots, x_n)\| \\
&\quad + \cdots \\
&\quad + \|2^{k-1} f(2x_1, x_2, x_3, \cdots, x_n) - 2^k f(x_1, x_2, \cdots, x_n)\| \\
&\le \sum_{r=1}^{k} 2^{k-r} \phi_r(2x_1, 2x_2, \cdots, 2x_{r-1}, x_r, x_{r+1}, \cdots, x_n).
\end{aligned} \tag{2.6}$$

Now let $r \in \{k+1, k+2, \cdots, n\}$. Replacing a, b by x_r in (2.3) to obtain

$$\begin{aligned}
&\|f(x_1, x_2, \cdots, x_{r-1}, 2x_r, x_{r+1}, \cdots, x_n) - 4f(x_1, x_2, \cdots, x_n)\| \\
&\qquad \le \psi_r(x_1, x_2, \cdots, x_n).
\end{aligned} \tag{2.7}$$

It follows from (2.7) that

$$\begin{aligned}
&\|f(x_1, x_2, \cdots, x_k, 2x_{k+1}, 2x_{k+2}, \cdots, 2x_n) - 4^{n-k} f(x_1, x_2, \cdots, x_n)\| \\
&\quad \le \|f(x_1, x_2, \cdots, x_k, 2x_{k+1}, 2x_{k+2}, \cdots, 2x_n) \\
&\qquad - 4f(x_1, x_2, \cdots, x_{k+1}, 2x_{k+2}, 2x_{k+3}, \cdots, 2x_n)\| \\
&\qquad + \|4f(x_1, x_2, \cdots, x_{k+1}, 2x_{k+2}, 2x_{k+3}, , 2x_n) \\
&\qquad - 4^2 f(x_1, x_2, \cdots, x_{k+2}, 2x_{k+3}, 2x_{k+4}, \cdots, 2x_n)\| \\
&\qquad + \cdots \\
&\qquad + \|4^{n-k-1} f(x_1, x_2, \cdots, x_{n-1}, 2x_n) - 4^{n-k} f(x_1, x_2, \cdots, x_n)\| \\
&\quad \le \sum_{r=1}^{n-k} 4^{r-1} \psi_{k+r}(x_1, x_2, \cdots, x_{k+r}, 2x_{k+r+1}, 2x_{k+r+2}, \cdots, 2x_n).
\end{aligned} \tag{2.8}$$

Now, combine (2.6) and (2.8) by use of the triangle inequality to get

$$\begin{aligned}
&\|f(2x_1,2x_2,\cdots,2x_n)-2^k4^{n-k}f(x_1,x_2,\cdots,x_n)\| \\
&\quad\leq \|f(2x_1,2x_2,\cdots,2x_n)-2^kf(x_1,x_2,\cdots,x_k,2x_{k+1},2x_{k+2},\cdots,2x_n)\| \\
&\qquad+\|2^kf(x_1,x_2,\cdots,x_k,2x_{k+1},2x_{k+2},\cdots,2x_n)-2^k4^{n-k}f(x_1,x_2,\cdots,x_n)\| \\
&\quad\leq \sum_{r=1}^{k}2^{k-r}\phi_r(2x_1,2x_2,\cdots,2x_{r-1},x_r,x_{r+1},\cdots,x_k,2x_{k+1},2x_{k+2},\cdots,2x_n) \\
&\qquad+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_{k+r}(x_1,x_2,\cdots,x_{k+r},2x_{k+r+1},2x_{k+r+2},\cdots,2x_n) \\
&\quad=\sum_{r=1}^{k}2^{k-r}\phi_r^1(x_1,x_2,\cdots,x_n)+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_r^1(x_1,x_2,\cdots,x_n). \qquad (2.9)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|\frac{1}{2^k4^{n-k}}f(2x_1,2x_2,\cdots,2x_n)-f(x_1,x_2,\cdots,x_n)\| \qquad (2.10)\\
&\qquad\leq \frac{1}{2^k4^{n-k}}[\sum_{r=1}^{k}2^{k-r}\phi_r^1(x_1,x_2,\cdots,x_n)+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_r^1(x_1,x_2,\cdots,x_n)]
\end{aligned}$$

for all $(x_1,x_2,\cdots,x_n)\in X_1\times X_2\times\cdots\times X_n$.

Divide both sides of (2.10) by 2^k4^{n-k}, and replacing $x_1,x_2,\cdots,x_n$ by $2x_1,2x_2,\cdots,2x_n$, respectively, we obtain

$$\begin{aligned}
&\|\frac{1}{(2^k4^{n-k})^2}f(2^2x_1,2^2x_2,\cdots,2^2x_n)-\frac{1}{2^k4^{n-k}}f(2x_1,2x_2,\cdots,2x_n)\| \\
&\qquad\leq \frac{1}{(2^k4^{n-k})2}[\sum_{r=1}^{k}2^{k-r}\phi_r^1(2x_1,2x_2,\cdots,2x_n) \\
&\qquad\quad+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_r^1(2x_1,2x_2,\cdots,2x_n)] \qquad (2.11)
\end{aligned}$$

Combine (2.10) and (2.11) by use of the triangle inequality to get

$$\|\frac{1}{(2^k4^{n-k})^2}f(2^2x_1,2^2x_2,\cdots,2^2x_n)-f(x_1,x_2,\cdots,x_n)\| \tag{2.12}$$
$$\leq \frac{1}{(2^k4^{n-k})2}[\sum_{r=1}^{k}2^{k-r}\phi_r^2(x_1,x_2,\cdots,x_n)+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_r^2(x_1,x_2,\cdots,x_n)]$$
$$+\frac{1}{2^k4^{n-k}}[\sum_{r=1}^{k}2^{k-r}\phi_r^1(x_1,x_2,\cdots,x_n)+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_r^1(x_1,x_2,\cdots,x_n)]$$

Now, proceed in this way to prove by induction on m that

$$\|\frac{1}{(2^k4^{n-k})^m}f(2^mx_1,2^mx_2,\cdots,2^mx_n)-f(x_1,x_2,\cdots,x_n)\|$$
$$\leq \sum_{s=1}^{m}\frac{1}{(2^k4^{n-k})^s}[\sum_{r=1}^{k}2^{k-r}\phi_r^s(x_1,x_2,\cdots,x_n)$$
$$+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_r^s(x_1,x_2,\cdots,x_n)] \tag{2.13}$$

for all $(x_1,x_2,\cdots,x_n)\in X_1\times X_2\times\cdots\times X_n$.

In order to show that functions

$$T_m(x_1,x_2,\cdots,x_n)=\frac{1}{(2^k4^{n-k})^m}f(2^mx_1,2^mx_2,\cdots,2^mx_n)$$

form a convergent sequence, we used Cauchy convergence criterion. Indeed, replace x_i by $2^lx_i(1\leq i\leq n)$ in (2.13) and result divide by $(2^k4^{n-k})^l$, where l is an arbitrary positive integer, we find that

$$\|\frac{1}{(2^k4^{n-k})^{m+l}}f(2^{m+l}x_1,2^{m+l}x_2,\cdots,2^{m+l}x_n)-f(2^lx_1,2^lx_2,\cdots,2^lx_n)\|$$
$$\leq \sum_{s=l}^{m+l}\frac{1}{(2^k4^{n-k})^s}[\sum_{r=1}^{k}\phi_r^s((2^lx_1,2^lx_2,\cdots,2^lx_n))$$
$$+2^k\sum_{r=1}^{n-k}4^{r-1}\psi_r^s((2^lx_1,2^lx_2,\cdots,2^lx_n))].$$

It follows from (2.1) that $\{T_m(x_1,x_2,\cdots,x_n)\}$ is a Cauchy sequence in Y. Since Y is complete, then $T(x_1,x_2,\cdots,x_n):=\lim_m T_m(x_1,x_2,\cdots,x_n)$ exists for all $(x_1,x_2,\cdots,x_n)\in X_1\times X_2\times\cdots\times X_n$. It follows from (2.2) and (2.3) that

$$\begin{aligned}
&D_T^r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n)\\
&= \lim_m \frac{1}{(2^k 4^{n-k})^m} D_f^r(2^m x_1, 2^m x_2, \cdots, 2^m x_{i_r-1}, 2^m a, 2^m b, 2^m x_{i_r+1}, \cdots, 2^m x_n)\\
&\le \lim_m \frac{1}{(2^k 4^{n-k})^m} \phi_r(2^m x_1, 2^m x_2, \cdots, 2^m x_{i_r-1}, 2^m a, 2^m b, 2^m x_{i_r+1}, \cdots, 2^m x_n)\\
&= 0
\end{aligned}$$

for all $r \in \{1, 2, \cdots, k\}$. Similarly, for every $r \in \{k+1, k+2, \cdots, n\}$, it follows from (2.2) and (2.3) that

$$\begin{aligned}
&D_T^r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n)\\
&= \lim_m \frac{1}{(2^k 4^{n-k})^m} D_f^r(2^m x_1, 2^m x_2, \cdots, 2^m x_{r-1}, 2^m a, 2^m b, 2^m x_{r+1}, \cdots, 2^m x_n)\\
&\le \lim_m \frac{1}{(2^k 4^{n-k})^m} \psi^r(2^m x_1, 2^m x_2, \cdots, 2^m x_{r-1}, 2^m a, 2^m b, 2^m x_{r+1}, \cdots, 2^m x_n)\\
&= 0.
\end{aligned}$$

This means that T satisfies (1.2). It remains to show that T is unique. Suppose that there exists another mapping $T' : X_1 \times X_2 \times ... \times X_n \to Y$ which satisfies (1.2) and (2.4). Since

$$(2^k 4^{n-k})^l T(x_1, x_2, \cdots, x_n) = T(2^l x_1, 2^l x_2, \cdots, 2^l x_n),$$
$$(2^k 4^{n-k})^l T'(x_1, x_2, \cdots, x_n) = T'(2^l x_1, 2^l x_2, \cdots, 2^l x_n)$$

for all $l \in \mathbb{N}, (x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$, we conclude that

$$\begin{aligned}
&\|T((x_1, x_2, \cdots, x_n)) - T'((x_1, x_2, \cdots, x_n))\|\\
&\quad = \frac{1}{(2^k 4^{n-k})^l} \|T((2^l x_1, 2^l x_2, \cdots, 2^l x_n)) - T'((2^l x_1, 2^l x_2, \cdots, 2^l x_n))\|\\
&\quad \le \frac{1}{(2^k 4^{n-k})^l} \|T((2^l x_1, 2^l x_2, \cdots, 2^l x_n)) - f((2^l x_1, 2^l x_2, \cdots, 2^l x_n))\|\\
&\qquad + \frac{1}{(2^k 4^{n-k})^l} \|f((2^l x_1, 2^l x_2, \cdots, 2^l x_n)) - T'((2^l x_1, 2^l x_2, \cdots, 2^l x_n))\|\\
&\quad \le \frac{2}{(2^k 4^{n-k})^l} \sum_{m=0}^{\infty} \frac{1}{(2^k 4^{n-k})^m} [\sum_{r=1}^{k} 2^{r-1} \phi_r^m((2^l x_1, 2^l x_2, \cdots, 2^l x_n))\\
&\qquad + \sum_{r=k+1}^{n} 4^{r-1} 2^k \psi_r^m((2^l x_1, 2^l x_2, \cdots, 2^l x_n))]
\end{aligned}$$

By letting $l \to \infty$ in this inequality, it follows that

$$T(x_1, x_2, \cdots, x_n) = T'(x_1, x_2, \cdots, x_n)$$

for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$, which gives the conclusion. $\square$

We are going to investigate the Hyers-Ulam-Rassias stability problem for system of functional equations (1.2).

Corollary 2.2. *Let $\epsilon > 0, p < 2n-k+1$. Suppose $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ is a mapping such that*

$$D_f^r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n) \le \epsilon[(\sum_{i=1, i\neq r}^{n} \|x_i\|^p) + \|a\|^p + \|b\|^p]$$

for all $(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_{r-1} \times X_r \times X_r \times X_{r+1} \times \cdots \times X_n, r \in \{1, 2, \cdots, n-1\}$. Then there exists a unique mapping $T : X_1 \times X_2 \times \cdots \times X_n \to Y$ satisfying (1.2) and

$$\begin{aligned}
&\|T(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)\| \\
&\quad \le \epsilon \sum_{m=0}^{\infty} 2^{(p-2n+k)m} \{[\sum_{r=1}^{k} 2^{r-1} (\sum_{j=1}^{r-1} \|x_j\|^p + 2^{-p}\|x_r\|^p + 2^{-p} \sum_{s=r}^{k} \|x_s\|^p) \\
&\qquad + (2^{k-1} \sum_{i=k+1}^{n} \|x_i\|^p)] + \sum_{r=k+1}^{n} [2^{-p} \sum_{i=1}^{r} \|x_i\|^p + 2^{-p}\|x_r\|^p + \sum_{j=r+1}^{n} \|x_j\|^p]\}
\end{aligned}$$

for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$.

Proof. Put

$$\begin{aligned}
&\phi_r(x_1, x_2, \cdots, x_{r-1}, x_r, x_r, x_{r+1}, \cdots, x_n) \\
&= \phi_r(x_1, x_2, \cdots, x_{r-1}, x_r, x_r, x_{r+1}, \cdots, x_n) \\
&= (\sum_{i=1}^{n} \|x_i\|^p) + \|x_r\|^p
\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X_1 \times X_2 \times \cdots \times X_{r-1} \times X_r \times X_r \times X_{r+1} \times \cdots \times X_n$. Then for every $r \in \{1, 2, \cdots, k\}$, we have

$$\phi_r^m(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{r-1} \|2^m x_i\|^p + \|2^{m-1} x_r\|^p + \sum_{j=r}^{k} \|2^{m-1} x_j\|^p + \sum_{s=r+1}^{n} \|2^m x_s\|^p$$

and for every $r \in \{k+1, k+2, \cdots, n\}$, we have

$$\psi_r^m(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{r} \|2^{m-1}x_i\|^p + \|2^{m-1}x_r\|^p + \sum_{s=r+1}^{n} \|2^m x_s\|^p.$$

The conclusion follows from theorem 2.1. □

By Corollary 2.2, we solve the following Hyers-Ulam stability problem for system of functional equations (1.2).

Corollary 2.3. *Let $\epsilon > 0$. Suppose $f : X_1 \times X_2 \times \cdots \times X_n \to Y$ is a mapping such that*

$$D_f^r(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n) \leq \epsilon$$

for all $(x_1, x_2, \cdots, x_{r-1}, a, b, x_{r+1}, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_{r-1} \times X_r \times X_r \times X_{r+1} \times \cdots \times X_n, r \in \{1, 2, \cdots, n-1\}$. Then there exists a unique mapping $T : X_1 \times X_2 \times \cdots \times X_n \to Y$ satisfying (1.2) and

$$\|T(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)\| \leq \frac{2^{2n} - 2^k}{2^{2n+k}}\epsilon$$

for all $(x_1, x_2, \cdots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$.

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