Nonlinear Functional Analysis and Applications
Vol. 23, No. 4 (2018), pp. 613-627
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

MEAN CONVERGENCE THEOREMS FOR ASYMPTOTICALLY DEMICONTRACTIVE MAPPINGS IN THE INTERMEDIATE SENSE

Godwin Amechi Okeke[1], Johnson O. Olaleru[2] and Jong Kyu Kim[3]

[1]Department of Mathematics, School of Physical Sciences,
Federal University of Technology,
Owerri, P.M.B. 1526 Owerri, Imo State, Nigeria
e-mail: `gaokeke1@yahoo.co.uk`

[2]Department of Mathematics, Faculty of Science, University of Lagos,
Akoka, Lagos, Nigeria
e-mail: `jolaleru@unilag.edu.ng`

[3]Department of mathematics Education, Kyungnam University,
Changwon, Gyeongnam, 51767, Korea
e-mail: `jongkyuk@kyungnam.ac.kr`

Abstract. Recently, Olaleru and Okeke [19] introduced the class of asymptotically demicontractive mappings in the intermediate sense as a generalization of the class of asymptotically demicontractive mappings. The authors proved some convergence theorems for this class of nonlinear mappings in Hilbert spaces (see, [19]). The purpose of this paper is to continue the study of this class of nonlinear mappings. We prove some fixed point theorems for the class of asymptotically demicontractive mappings in the intermediate sense. We also prove some mean convergence theorems for this class of mappings in Hilbert spaces.

1. Introduction

We assume that H is a real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The symbols $\to$ and $\rightharpoonup$ denotes strong convergence and weak convergence, respectively. $\omega_w(x_n) = \{x : \exists\, x_{n_i} \rightharpoonup x\}$

[0]Received June 22, 2018. Revised August 24, 2018.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Demiclosedness Principle, asymptotically demicontractive mappings in the intermediate sense, mean convergence theorem.
[0]Corresponding author: J. K. Kim(`jongkyuk@kyungnam.ac.kr`).

denotes the weak ω-limit set of $\{x_n\}$. Let C be a nonempty closed convex subset of H and $T : C \to C$ be a mapping. We denote the fixed point set of T by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$.

First, we give the definitions of several well-known classes of mappings.

Definition 1.1. Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be:

(1) *k-strictly asymptotically pseudocontractive* [21] if there exists a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = 1$ such that

$$\|T^n x - T^n y\|^2 \le a_n^2 \|x - y\| + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.1}$$

for some $k \in [0, 1)$ and for all $x, y \in C$ and $n \in \mathbb{N}$.

(2) a *demicontractive mapping* [6] if $F(T) \neq \emptyset$ and there exists $k \in (0, 1)$ such that

$$\|Tx - p\|^2 \le \|x - p\|^2 + k\|x - Tx\|^2, \tag{1.2}$$

for each $x \in C$ and $p \in F(T)$.

(3) *asymptotically demicontractive* [21] if $F(T) \neq \emptyset$ and there exists a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = 1$ such that

$$\|T^n x - p\|^2 \le a_n^2 \|x - p\|^2 + k\|x - T^n x\|^2, \tag{1.3}$$

for some $k \in [0, 1)$ and for each $x \in C$ and $p \in F(T)$.

(4) *asymptotically nonexpansive in the intermediate sense* [3] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

(5) an *asymptotically k-strict pseudocontractive mapping in the intermediate sense* with sequence $\{\gamma_n\}$ [24] if there exists a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\begin{aligned}\limsup_{n\to\infty} \sup_{x,y\in C} \quad & (\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 \\ & -k\|x - T^n x - (y - T^n y)\|^2) \le 0.\end{aligned} \tag{1.4}$$

Put

$$\begin{aligned}c_n := \max \quad & \{0, \sup_{x,y\in C}(\|T^n x - T^n y\|^2 - (1 + \gamma_n)\|x - y\|^2 \\ & -k\|(x - T^n x - (y - T^n y)\|^2)\}.\end{aligned} \tag{1.5}$$

Then $c_n \ge 0$ for all $n \in \mathbb{N}$, $c_n \to 0$ as $n \to \infty$ and (1.4) reduces to the relation

$$\begin{aligned}\|T^n x - T^n y\|^2 \le & (1 + \gamma_n)\|x - y\|^2 \\ & +k\|x - T^n x - (y - T^n y)\|^2 + c_n,\end{aligned} \tag{1.6}$$

for all $n \in \mathbb{N}$, $x, y \in C$.

(6) an *asymptotically pseudocontractive mapping in the intermediate sense* ([22]) if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2\big) \le 0, \tag{1.7}$$

where $\{k_n\}$ is a sequence in $[1,\infty)$ such that $k_n \to 1$ as $n\to\infty$.
Put

$$\nu_n = \max\big\{0, \sup_{x,y\in C} \big(\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2\big)\big\}. \tag{1.8}$$

It follows that $\nu_n \to 0$ as $n\to\infty$. Hence, (1.7) is reduced to the following:

$$\langle T^n x - T^n y, x-y\rangle \le k_n\|x-y\|^2 + \nu_n, \tag{1.9}$$

for all $n\ge 1,\ x,y\in C$. In real Hilbert spaces, it is easy to check that (1.9) is equivalent to:

$$\begin{aligned}\|T^n x - T^n y\|^2 \le{}& (2k_n-1)\|x-y\|^2\\ &+\|(I-T^n)x-(I-T^n)y\|^2 + 2\nu_n,\end{aligned} \tag{1.10}$$

for all $n\ge 1,\ x,y\in C$.

(7) an *asymptotically demicontractive mapping in the intermediate sense* ([19]) if there exists a constant $k\in(0,1)$ such that for all $(x,p)\in C\times F(T)$,

$$\begin{aligned}\limsup\nolimits_{n\to\infty} \sup\nolimits_{(x,p)\in C\times F(T)} \quad &(\|T^n x - p\|^2 - a_n^2\|x-p\|^2\\ &-k\|x-T^n x\|^2) \le 0,\end{aligned} \tag{1.11}$$

where $\{a_n\}$ is a sequence in $[1,\infty)$ such that $a_n\to 1$ as $n\to\infty$. Observe that if we put

$$\begin{aligned}\nu_n = \max\big\{0, \sup\nolimits_{(x,p)\in C\times F(T)} \quad &(\|T^n x - p\|^2 - a_n^2\|x-p\|^2\\ &-k\|x-T^n x\|^2)\big\},\end{aligned} \tag{1.12}$$

then we get that $\nu_n \longrightarrow 0$ as $n\to\infty$ and (1.11) is reduced to the following:

$$\|T^n x - p\|^2 \le a_n^2\|x-p\|^2 + k\|x-T^n x\|^2 + \nu_n. \tag{1.13}$$

The classes of k-strictly asymptotically pseudocontractive mappings and asymptotically demicontractive mappings were introduced in 1996 by Liu [21]. If $k=0$ in (1.1) then T is called *asymptotically nonexpansive mapping*. If $k=0$, $a_n^2=1$ and $y\in F(T)$ in (1.1) then T is called *quasi-nonexpansive*. Observe that every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive, and every quasi-nonexpansive mapping is demicontractive mapping. Furthermore, every k-strictly asymptotically pseudocontractive mapping with a nonempty fixed point set $F(T)$ is asymptotically demicontractive.

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck *et al.* [3] and the iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings have been studied by several well-known mathematicians (see, e.g. [1], [3], [4], [8], [14]). Yang *et al.* [29] proved a demiclosedness principle for the class of asymptotically nonexpansive mapping in the intermediate sense.

Sahu *et al.* [24] proved that the modified Mann iteration process converges weakly to a fixed point of an asymptotically k-strict pseudocontractive mapping T in the intermediate sense which is not necessarily Lipschitzian. They established that if C is a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ is a continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero. Hu and Cai [7] introduced a modified explicit iterative process for finding a common element of the solutions of an equilibrium problem and the set of common fixed points of a finite family of asymptotically k-strictly pseudocontractive mappings in the intermediate sense in the framework of Hilbert spaces. Zhou [31] established a demiclosedness principle for asymptotically pseudocontractions in Hilbert spaces.

Qin *et al.* [22] introduced the class of asymptotically pseudocontractive mappings in the intermediate sense in 2010. They obtained some weak convergence theorems for this class of nonlinear mappings. They also established a strong convergence theorem without any compact assumption by considering the so-called hybrid projection method. They established that if $T : C \to C$ is a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in the intermediate sense, then $F(T)$ is a closed convex subset of C. Several well-known mathematicians have continued the study of this class of nonlinear mappings (see, e.g. [5], [17], [30]) among others.

Several authors proved some related results of certain classes of nonlinear mappings (see, e.g., [9], [10], [11], [12] [14], [15], [16], [17], [18], [23], [28]).

Olaleru and Okeke [19] introduced the class of asymptotically demicontractive mappings in the intermediate sense as a generalization of the class of asymptotically demicontractive mappings and they study the weak and strong convergence theorems for the iterative sequences defined by this class of mappings..

We now give an example of this class of nonlinear mappings.

Example 1.2. Let $X = \mathbb{R}$ (the set of real numbers with the usual metric). Define $T : X \to 2^X$ by

$$Tx = \begin{cases} [-kx, -\frac{1}{2}x], \ x \in [0, \infty), \\ [-\frac{1}{2}x, -kx], \ x \in (-\infty, 0], \ k \in (0, 1). \end{cases}$$

Then, we know that $F(T) = \{0\}$ and T is an asymptotically demicontractive mapping in the intermediate sense which is not asymptotically demicontractive mapping.

Definition 1.3. The map $T : C \to C$ is said to be *uniformly L-Lipschitzian* if

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.14}$$

for some constant $L > 0$, for all $n \in \mathbb{N}$ and $x, y \in C$.

Olaleru and Okeke [19] proved the following theorem.

Theorem 1.4. ([19], Theorem 2.1) *Let C be a nonempty bounded closed convex subset of H, and $T : C \to C$ be a completely continuous, uniformly L-Lipschitzian and asymptotically demicontractive mapping in the intermediate sense with sequence $\{\nu_n\}$ as defined in (1.13). Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \beta_n T^n x_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \ n \geq 1. \end{cases}$$

where α_n, $\beta_n \in [0, 1]$. Assume that the following conditions are satisfied:

(i) *for all $n \in \mathbb{N}$, $a_n \in [1, +\infty)$ and* $\sum_{n=1}^{\infty}(a_n^2 - 1) < +\infty$,
(ii) $\sum_{n=0}^{\infty} \nu_n < +\infty$,
(iii) *for all $n \in \mathbb{N}$ and for some $\epsilon > 0$, $k \in [0, 1)$, $\epsilon \leq k \leq \alpha_n \leq \beta_n \leq b$ and $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1)$.*

Then $\{x_n\}$ converges strongly to a fixed point of T.

2. Preliminaries

Let C be a closed convex subset of a Hilbert space H. For every $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that for all $y \in C$,

$$\|x - P_C x\| \leq \|x - y\|.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies the following inequality

$$\|P_C x - P_C y\| \leq \langle P_C x - P_C y, x - y\rangle,$$

for every $x, y \in H$ and

$$\langle y - P_C x, x - P_C x\rangle \leq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$. We know that for $x \in H$ and $z \in C$, $z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$, for all $y \in C$ (see [26]).

Definition 2.1. A point $p \in C$ is said to be an *asymptotic fixed point of* T if there exists a weakly convergent sequence $\{x_n\} \subset C$ to p such that $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to 0. We denote the set of asymptotic fixed points of T by $\widehat{F(T)}$.

The following lemma will play important roles in the proofs of main theorems.

Lemma 2.2. ([27]) *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality:*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n,$$

for all $n \in \mathbb{N}$. If $\beta_n \geq 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 2.3. ([1]) *Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If $\omega_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.*

Lemma 2.4. *Let H be a real Hilbert space. Then, for all $x, y \in H$, we have the followings:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$.

(b) *For all $t \in [0, 1]$, $\|(1-t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x - y\|^2$.*

(c) *If $\{x_n\} \subset H$ is a weakly convergent sequence to x, then we have*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2,$$

for all $y \in H$.

Lemma 2.5. ([13]) *Let C be a closed convex subset of a real Hilbert space H. Then for $x, y, z \in H$ and $a \in \mathbb{R}$, the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is closed and convex subset of C.

Lemma 2.6. ([2]) *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Hilbert space H and $\{\eta_n\}$ be a sequence of real numbers. Suppose that $\{x_n\}$ is bounded and both $\{y_n\}$ and $\{\eta_n\}$ are convergent. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \eta_k \langle x_{k+1} - x_k, y_k\rangle = 0.$$

Lemma 2.7. ([2]) *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ be a mapping. Let $\{x_n\}$ be a sequence in C, $\{\xi_n\}$ be a sequence of real numbers, $\{z_n\}$ be a sequence in C defined by $z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$ for $n \in \mathbb{N}$, and z be a weak cluster point of $\{z_n\}$. Suppose that*

$$\xi_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \xi_k = 0$. Then z is a fixed point of T.

Lemma 2.8. ([25]) *Let C be a nonempty closed convex subset of a Hilbert space H, P be the metric projection of H onto C, and $\{x_n\}$ be a sequence in H such that*

$$\|x_{n+1} - u\| \leq \|x_n - u\|$$

for all $u \in C$ and $n \in \mathbb{N}$. Then $\{Px_n\}$ converges strongly to some point in C.

3. Demiclosedness Principle and Fixed Point

In this section, we prove some fixed point theorems for the class of asymptotically demicontractive mappings in the intermediate sense in Hilbert spaces.

Lemma 3.1. *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ be an asymptotically demicontractive mapping in the intermediate sense with sequence $\{\nu_n\}$. Then for $p \in F(T)$,*

$$\|T^n x - p\| \leq \frac{\sqrt{\nu_n} + (a_n + \sqrt{k})\|x - p\|}{1 - \sqrt{k}}.$$

Proof. For $p \in F(T)$, we have

$$\begin{aligned} ||T^n x - p||^2 &\leq a_n^2||x - p||^2 + k||x - T^n x||^2 + \nu_n \\ &= (a_n\|x - p\|)^2 + (\sqrt{k}\|x - T^n x\|)^2 + (\sqrt{\nu_n})^2 \\ &\leq (a_n\|x - p\| + \sqrt{k}\|x - T^n x\| + \sqrt{\nu_n})^2. \end{aligned} \tag{3.1}$$

This implies that

$$\begin{aligned} \|T^n x - p\| &\leq a_n\|x - p\| + \sqrt{k}\|x - T^n x\| + \sqrt{\nu_n} \\ &\leq a_n\|x - p\| + \sqrt{k}(\|x - p\| + \|p - T^n x\|) + \sqrt{\nu_n}. \end{aligned} \tag{3.2}$$

Hence, we have

$$\|T^n x - p\| \leq \frac{\sqrt{\nu_n} + (a_n + \sqrt{k})\|x - p\|}{1 - \sqrt{k}}. \tag{3.3}$$

This completes the proof. □

Remark 3.2. It is clear from Lemma 3.1 that T is not necessarily Lipschitzian. Moreover, if $\nu_n = 0$ for each $n \in \mathbb{N}$ in (3.3), then we obtain the inequality that was established by Osilike ([20], Remark 1) for the class of asymptotically demicontractive mappings.

Lemma 3.3. *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ be a uniformly continuous asymptotically demicontractive mapping in the intermediate sense with sequence $\{\nu_n\}$ defined in* (1.13)*. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then, we have*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Proof. Since T is an asymptotically demicontractive mapping in the intermediate sense, it was established from Lemma 3.1 that

$$\|T^{n+1}x_n - T^{n+1}x_{n+1}\| \leq \frac{\sqrt{\nu_n} + (a_n + \sqrt{k})\|x_n - x_{n+1}\|}{1 - \sqrt{k}}. \tag{3.4}$$

Since $\|x_n - x_{n+1}\| \to 0$, this implies that $\|T^{n+1}x_n - T^{n+1}x_{n+1}\| \to 0$. Clearly,

$$\begin{aligned} \|x_n - Tx_n\| \quad \leq \quad & \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ & + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\|. \end{aligned} \tag{3.5}$$

Since T is uniformly continuous, we have

$$\|Tx_n - T^{n+1}x_n\| \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Since $||x_n - T^n x_n|| \to 0$ and $||x_n - x_{n+1}|| \to 0$, it follows from (3.5) and (3.6) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Theorem 3.4. (Demiclosedness principle) *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically demicontractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m,n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.*

Proof. Assume that T is a continuous asymptotically demicontractive mapping in the intermediate sense with sequence $\{\nu_n\}$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and

$$\limsup_{m,n\to\infty} \|x_n - T^m x_n\| = 0. \tag{3.7}$$

Using Lemma 3.1, we have

$$\begin{aligned} \|T^m x_n - T^m x\| \quad & \leq \quad \tfrac{\sqrt{\nu_m} + (a_n + \sqrt{k})\|x_n - x\|}{1 - \sqrt{k}} \\ & \leq \quad K^*, \end{aligned} \tag{3.8}$$

for all $m, n \in \mathbb{N}$ and for some $K^* > 0$. We define

$$f(x) := \limsup_{n\to\infty} \|x_n - x\|^2, \text{ for all } x \in F(T). \tag{3.9}$$

Since $x_n \rightharpoonup x$, it follows from Lemma 2.4 (c) that

$$f(y) = f(x) + \|x-y\|^2, \text{ for all } y \in H. \tag{3.10}$$

Since T is asymptotically demicontractive mapping in the intermediate sense, by relation (1.13), (3.8) and the fact that $\lim_{n\to\infty} a_n = 1$, we have

$$\begin{aligned} f(T^m x) &= \limsup_{n\to\infty} \|x_n - T^m x\|^2 \\ &\le \limsup_{n\to\infty} (\|x_n - T^m x_n\| + \|T^m x_n - T^m x\|)^2 \\ &\le \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + \|T^m x_n - T^m x\|^2 \\ &\quad +2\|x_n - T^m x_n\|\|T^m x_n - T^m x\|) \\ &\le \limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + \|T^m x_n - T^m x\|^2 \\ &\quad +2K^*\|x_n - T^m x_n\|) \\ &\le \limsup_{n\to\infty} \|T^m x_n - T^m x\|^2 \\ &\quad +\limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2K^*\|x_n - T^m x_n\|) \\ &\le \limsup_{n\to\infty} (a_n^2\|x_n - x\|^2 + k\|x_n - T^m x_n\|^2 + \nu_m) \\ &\quad +\limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2K^*\|x_n - T^m x_n\|) \\ &= \limsup_{n\to\infty} (\|x_n - x\|^2 + k\|x_n - T^m x_n\|^2 + \nu_m) \\ &\quad +\limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2K^*\|x_n - T^m x_n\|) \\ &\le f(x) + k\limsup_{n\to\infty} \|x_n - T^m x_n\|^2 + \nu_m \\ &\quad +\limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2K^*\|x_n - T^m x_n\|), \end{aligned} \tag{3.11}$$

for each $m \in \mathbb{N}$. Using (3.10), we have

$$\begin{aligned} f(x) + \|x - T^m x\|^2 &= f(T^m x) \\ &\le f(x) + k\limsup_{n\to\infty} \|x_n - T^m x_n\|^2 + \nu_m \\ &\quad +\limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2K^*\|x_n - T^m x_n\|). \end{aligned} \tag{3.12}$$

This implies that

$$\begin{aligned} \|x - T^m x\|^2 &\le k\limsup_{n\to\infty} \|x_n - T^m x_n\|^2 + \nu_m \\ &\quad +\limsup_{n\to\infty} (\|x_n - T^m x_n\|^2 + 2K^*\|x_n - T^m x_n\|). \end{aligned} \tag{3.13}$$

Since $\limsup_{m,n\to\infty} \|x_n - T^m x_n\| = 0$, it follows from (3.13) that

$$\limsup_{m\to\infty} \|x - T^m x\|^2 \le \limsup_{m\to\infty} \nu_m = 0. \tag{3.14}$$

This means that $T^m x \longrightarrow x$ as $m \to \infty$. By using the continuity of T, we have $(I - T)x = 0$. This completes the proof. □

Proposition 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a continuous asymptotically demicontractive mapping in the intermediate sense. Then $F(T)$ is closed and convex.*

Proof. Let $\{x_n\}_{n=1}^{\infty} \subseteq F(T)$ be a sequence which converges to p. Then we have to show that $p \in F(T)$. Clearly,

$$\begin{array}{rcl} \|T^n p - p\| & \leq & \|T^n p - T^n x_n\| + \|x_n - p\| \\ & \leq & 2\|x_n - p\| \\ & \to & 0, \text{ as } n \to \infty. \end{array} \tag{3.15}$$

Using the continuity of T, it follows that $p = Tp$. Hence $p \in F(T)$.

Next we show that $F(T)$ is convex. For $p_1, p_2 \in F(T)$ and $\lambda \in [0,1]$, we show that $\lambda p_1 + (1-\lambda)p_2 \in F(T)$. Let $z = \lambda p_1 + (1-\lambda)p_2$. Then $p_1 - z = (1-\lambda)(p_1 - p_2)$, $p_2 - z = \lambda(p_2 - p_1)$. Using relation (1.13) and Lemma 2.4 (b), we have

$$\begin{array}{rcl} \|z - T^n z\|^2 & = & \|\lambda p_1 + (1-\lambda)p_2 - T^n z\|^2 \\ & = & \|\lambda(p_1 - T^n z) + (1-\lambda)(p_2 - T^n z)\|^2 \\ & = & \lambda\|p_1 - T^n z\|^2 + (1-\lambda)\|p_2 - T^n z\|^2 \\ & & -\lambda(1-\lambda)\|p_1 - p_2\|^2 \\ & \leq & \lambda(a_n^2\|p_1 - z\|^2 + k\|z - T^n z\|^2 + \nu_n) \\ & & +(1-\lambda)(a_n^2\|p_2 - z\|^2 + k\|z - T^n z\|^2 + \nu_n) \\ & & -\lambda(1-\lambda)\|p_1 - p_2\|^2 \\ & \leq & \|p_1 - z\|^2 + (1-\lambda)\|p_2 - z\|^2 \\ & & -\lambda(1-\lambda)\|p_1 - p_2\|^2 \\ & = & 0. \end{array} \tag{3.16}$$

Hence, by continuity of T, we have $z = Tz$, which implies that $z \in F(T)$. The proof is completed. □

Proposition 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $T : C \to C$ is a continuous asymptotically demicontractive mapping in the intermediate sense. Then $F(T) = \widehat{F(T)}$.*

Proof. Let $p \in F(T)$. From Proposition 3.4, since $F(T)$ is closed, there exists a sequence $\{x_n\} \subseteq F(T) \subseteq C$ such that $x_n \to p$. Observe that $x_n - T^n x_n = 0$ for each $n \in \mathbb{N}$, hence by continuity of T, we have $x_n - Tx_n \to 0$. Hence, $p \in \widehat{F(T)}$. Conversely, let $p \in \widehat{F(T)}$. Then there exists a sequence $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. From Theorem 3.3, since $(I - T)$ is demiclosed at zero, we have $p \in F(T)$. This completes the proof. □

Motivated by the results of Aoyama and Kohsaka [2], we prove the following existence theorem of fixed point for the class of asymptotically demicontractive mappings in the intermediate sense.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T_n : C \to C$ be a continuous asymptotically demicontractive mapping in the intermediate sense for each $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be* *sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n^m x_n \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k,$$

for $m, n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent to T, that is,

$$Tx = \lim_{n\to\infty} T_n x$$

for $x \in C$. Then we have the followings:

(a) *The mapping T is asymptotically demicontractive in the intermediate sense and $\cap_{n=1}^{\infty} F(T_n) \subset F(T)$.*
(b) *If $\{x_n\}$ is bounded, then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T.*

Proof. (a) Let $x, y \in C$. Since T_n is an asymptotically demicontractive mapping in the intermediate sense for each $n \in \mathbb{N}$, we have

$$\|T_n^m x - T_n^m p\|^2 \le a_m^2 \|x - p\|^2 + k\|x - T_n^m x\|^2 + \nu_m, \tag{3.17}$$

for every $m, n \in \mathbb{N}$, $p \in F(T)$. Taking the limit $n \to \infty$, we obtain

$$\|T^m x - T^m p\|^2 \le a_m^2 \|x - p\|^2 + k\|x - T^m x\|^2 + \nu_m. \tag{3.18}$$

Hence T is an asymptotically demicontractive mapping in the intermediate sense. Moreover, let $y \in \cap_{n=1}^{\infty} F(T_n)$. Since T_n is pointwise convergent for each $n \in \mathbb{N}$, we have $T^m y = \lim_{n\to\infty} T_n^m y = y$. Using the continuity of T, we have $Ty = \lim_{n\to\infty} T_n y = y$. Hence, $y \in F(T)$. This implies that $\cap_{n=1}^{\infty} F(T_n) \subset F(T)$.

(b) Suppose that $\{x_n\}$ is bounded. Then $\{z_n\}$ is also bounded and hence there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z \in C$. We now show that z is a fixed point of T. Since T_n is an asymptotically demicontractive mapping in the intermediate sense and $x_{n+1} = T_n^m x_n$, we obtain

$$\begin{aligned} \|x_{n+1} - T^m z\|^2 &= \|x_{n+1} - T_n^m z + T_n^m z - T^m z\|^2 \\ &= \|x_{n+1} - T_n^m z\|^2 + \|T_n^m z - T^m z\|^2 \\ &\quad + 2\langle x_{n+1} - T_n^m z, T_n^m z - T^m z\rangle \\ &\le a_m^2 \|x_n - z\|^2 + k\|x_n - T_n^m x_n\|^2 + \nu_m \\ &\quad + \|T_n^m z - T^m z\|(\|T_n^m z - T^m z\| \\ &\quad + 2\|x_{n+1} - T_n^m z\|). \end{aligned} \tag{3.19}$$

Hence,

$$\mu_n + \varepsilon_n \le a_m^2 \|x_n - z\|^2 - \|x_{n+1} - T^m z\|^2, \tag{3.20}$$

where $\mu_n = k\|x_n - T_n^m x_n\|^2 + \nu_m$ and

$$\varepsilon_n = -\|T_n^m z - T^m z\|(\|T_n^m z - T^m z\| + 2\|x_{n+1} - T_n^m z\|).$$

Taking limit as $m \to \infty$, using the continuity of T and the fact that $\{x_n\}$ is bounded and $\{T_n z\}$ is convergent. Then by Lemma 2.6, we have $\frac{1}{n}\sum_{k=1}^{n}\mu_k \to 0$, and hence

$$\frac{1}{n}\sum_{k=1}^{n}(\mu_k + \varepsilon_k) \to 0.$$

Hence by Lemma 2.7, z is a fixed point of T. This completes the proof. □

4. Mean Convergence Theorems

In this section, we prove some mean convergence theorems for the class of asymptotically demicontractive mappings in the intermediate sense in Hilbert spaces.

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T_n : C \to C$ be a continuous asymptotically demicontractive mappings in the intermediate sense for each $n \in \mathbb{N}$ such that*

$$F := \bigcap_{i=1}^{\infty} F(T_n) \neq \emptyset.$$

Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,

$$x_{n+1} = T_n^m x_n \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k \tag{4.1}$$

for $m, n \in \mathbb{N}$. Then we have the followings:

(a) *The sequence $\{x_n\}$ is bounded and $\{Px_n\}$ converges strongly, where P is the metric projection of H onto F.*

(b) *If each weak cluster point of $\{z_n\}$ belongs to F, then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$.*

Proof. (a) Since T_n is an asymptotically demicontractive mapping in the intermediate sense, we have

$$\begin{aligned} \|x_{n+1} - u\|^2 &= \|T_n^m x_n - u\|^2 \\ &\leq a_m^2\|x_n - u\|^2 + k\|x_n - T_n^m x_n\|^2 + \nu_m, \end{aligned} \tag{4.2}$$

for all $u \in F$ and $m, n \in \mathbb{N}$. Taking limit as $m \to \infty$ and using the continuity of T, we see that $\{x_n\}$ is bounded and by Lemma 2.8, we have that $\{Px_n\}$ converges strongly.

(b) Since $\{z_n\}$ is bounded from (a), there exists a weak cluster point z of $\{z_n\}$. Suppose that $\{z_{n_i}\}$ is a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and w is the strong limit of $\{Px_n\}$. Then we have to show that $z = w$. Since P is the metric projection of H onto F and $z \in F$, we see from (2.1) that

$$\langle z - Px_k, x_k - Px_k\rangle \leq 0, \tag{4.3}$$

for each $k \in \mathbb{N}$. Since T_k is an asymptotically demicontractive mapping in the intermediate sense, for each $k \in \mathbb{N}$ and $Px_k \in F$, we have

$$\begin{aligned} \|x_{k+1} - Px_{k+1}\|^2 &\leq \|x_{k+1} - Px_k\|^2 \\ &= \|T_k^m x_k - Px_k\|^2 \\ &\leq a_m^2 \|x_k - Px_k\|^2 \\ &\quad + k\|x_k - T_k^m x_k\|^2 + \nu_m, \end{aligned} \tag{4.4}$$

for each $k, m \in \mathbb{N}$. Taking limit as $m \to \infty$ and by continuity of T, then we have

$$\begin{aligned} \langle z - w, x_k - Px_k \rangle &= \langle z - Px_k, x_k - Px_k \rangle \\ &\quad + \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \|Px_k - w\| \|x_k - Px_k\| \\ &\leq \|Px_k - w\| \|x_1 - Px_1\|, \end{aligned} \tag{4.5}$$

for each $k \in \mathbb{N}$. Next, we sum these inequalities from $k = 1$ to n_i, and dividing by n_i, we obtain

$$\left\langle z - w, z_{n_i} - \frac{1}{n_i} \sum_{k=1}^{n_i} Px_k \right\rangle \leq \frac{1}{n_i} \sum_{k=1}^{n_i} \|Px_k - w\| \|x_1 - Px_1\|. \tag{4.6}$$

Since $z_{n_i} \rightharpoonup z$ as $i \to \infty$ and $Px_n \to w$ as $n \to \infty$, we have $\langle z - w, z - w \rangle \leq 0$. Hence, $z = w$. This completes the proof. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T_n : C \to C$ be a continuous asymptotically demicontractive mapping in the intermediate sense for each $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n^m x_n \text{ and } z_n = \frac{1}{n} \sum_{k=1}^{n} x_k,$$

for $m, n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent to T and

$$F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset.$$

Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$, where P is the metric projection of H onto F.

Proof. Since T_n is a continuous asymptotically demicontractive mapping in the intermediate sense and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ for each $n \in \mathbb{N}$, by Lemma 4.1, we know that $\{x_n\}$ is bounded. Hence, by Theorem 3.6 (b) we know that every cluster point of $\{z_n\}$ belongs to F. It follows from Lemma 4.1 (b) that $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$. This completes the proof. □

Acknowledgments: This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

References

[1] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings,* J. Nonlinear Convex Anal., **8**(1) (2007), 61-79.

[2] K. Aoyama and F. Kohsaka, *Fixed point and mean convergence theorems for a family of λ-hybrid mappings,* J. of Nonlinear Anal. and Optim., **2** (2011), 85-92.

[3] R.E. Bruck, T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property,* Colloq. Math., **65** (1993), 169-179.

[4] C.E. Chidume, N. Shahzad and H. Zegeye, *Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense,* Numer. Funct. Anal. Optim., **25** (2004), 239-257.

[5] C-S. Ge, *A hybrid algorithm with variable coefficients for asymptotically pseudocontractive mappings in the intermediate sense on unbounded domains,* Nonlinear Anal. TMA. **75** (2012), 2859-2866.

[6] T.L. Hicks and J.D. Kubicek, *On the Mann iteration process in a Hilbert spaces,* J. Math. Anal. Appl., **59** (1977), 498-504.

[7] C.S. Hu and G. Cai, *Convergence theorems for equilibrium problems and fixed point problems of a finite family of asymptotically k-strictly pseudocontractive mappings in the intermediate sense,* Comput. Math. with Appl., **61** (2011), 79-93.

[8] G.E. Kim and T.H. Kim, *Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces,* Comput. Math. with Appl., **42** (2001), 1565-1570.

[9] J.K. Kim and D.R. Sahu and Y.M. Nam, *Convergence theorem for fixed points of nearly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings,* Nonlinear Anal. TMA, **71** (2009), 2833-2838.

[10] J.K. Kim and Ng. Buong, *A new iterative method for equilibrium problems and fixed point problems for infinite family of nonself strictly pseudocontractive mappings,* Fixed Point Theory and Appl., 2013, 2013:286, doi:10.1186/1687-1812-2013-286.

[11] J.K. Kim, *Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quisi--nonexpansive mappings,* Fixed Point Theory and Appl., 2011, 2011-10, doi: 10,1186/1687-1812-2011-10,

[12] J.K .Kim, S.Y. Cho and X.L. Qin, *Some results on generalized equilibrium problems involving strictly pseudocontractive mappings,* Acta Math. Sci., Series B, **31**(5) (2011), 2041-2057.

[13] C. Martinez-Yanes and H.K. Xu, *Strong convergence of the CQ method for fixed point iteration processes,* Nonlinear Anal. TMA, **64**(11) (2006), 2400-2411.

[14] G.A. Okeke and J.O. Olaleru, *Common fixed points of a three-step iteration with errors of asymptotically quasi-nonexpansive nonself-mappings in the intermediate sense in Banach spaces,* Fasciculi Mathematici Nr. **52** (2014), 93-115.

[15] G.A. Okeke and J.O. Olaleru, *Existence of fixed points of certain classes of nonlinear mappings,* Inter. Jour. of Math. Model. Comput., **4**(4) (2014), 357-364.

[16] J.O. Olaleru and A.A. Mogbademu, *On the modified Noor iteration scheme for nonlinear maps,* Acta Math. Univ. Comenianae, **LXXX**2(2) (2011), 221-228.

[17] J.O. Olaleru and A.A. Mogbademu, *Approximation of fixed points of strongly successively pseudocontractive maps in Banach space,* Inter. Jour.l of Comput. and Appl. Math., **7**(2) (2012), 121-132.

[18] J.O. Olaleru and G.A. Okeke, *Strong convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense,* British Jour. of Math. Comput. Sci., **2**(3) (2012), 151-162.

[19] J.O. Olaleru and G.A. Okeke, *Convergence theorems on asymptotically demicontractive and hemicontractive mappings in the intermediate sense,* Fixed Point Theory and Appl., 2013, 2013:352,

[20] M.O. Osilike, *Iterative approximation of fixed points of asymptotically demicontractive mappings,* Indian J. Pure Appl. Math., **29**(12) (1998), 1291-1300.

[21] Q.H. Liu, *Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings,* Nonlinear Anal. TMA, **26**(11) (1996), 1835-1842.

[22] X. Qin, S.Y. Cho and J.K. Kim, *Convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense,* Fixed Point Theory and Appl., (2010), Article ID 186874, 14 pages, doi:10.1155/2010/186874.

[23] Y. Qing and J.K. Kim, *Weak convergence of algorithms for asymptotically strict pseudo-contractions in the intermediate sense and equilibrium problems,* Fixed Point Theory and Appl., 2012, 2012-132, doi: 10,1186/1687-1812-2012-132.

[24] D.R. Sahu, H.K. Xu and J.-C. Yao, *Asymptotically strict pseudocontractive mappings in the intermediate sense,* Nonlinear Anal. TMA, **70**(10) (2009), 3502-3511.

[25] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings,* J. Optim. Theory Appl., **118** (2003), 417-428.

[26] W. Takahashi, *Introduction to nonlinear and convex analysis,* Yokohama Publishers, 2009.

[27] K.K. Tan and H.K. Xu, *Nonlinear ergodic theorems for asymptotically nonexpansive mappings in Banach spaces,* Proc. Amer. Math. Soc., **114** (1992), 399-404.

[28] L. Wang, Z. Xiao and H. Zhou, *Iterative algorithms to common fixed point for a finite family of asymptotically demicontractive mappings in Banach spaces,* Nonlinear Funct. Anal. and Appl., **12** (2007), 317-323.

[29] L-P. Yang, X. Xie, S. Peng and G. Hu, *Demiclosed principle and convergence for modified three step iterative process with errors of non-Lipschitzian mappings,* J. of Comput. and Appl. Math., **234** (2010), 972-984.

[30] H. Zegeye, M. Robdera and B. Choudhary, *Convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense,* Comput. and Math. with Appl., **62** (2011), 326-332.

[31] H. Zhou, *Demiclosedness principle with applications for asymptotically pseudocontractions in Hilbert spaces,* Nonlinear Anal. TMA, **70** (2009), 3140-3145.