Nonlinear Functional Analysis and Applications
Vol. 23, No. 4 (2018), pp. 629-641
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

COMMON FIXED POINT THEOREM FOR THE R-WEAKLY COMMUTING MAPPINGS IN $\mathcal{M}$-FUZZY METRIC SPACES

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Abstract. In this paper we prove common fixed point theorems for two mappings under the condition of R-weakly commuting complete $\mathcal{M}-$fuzzy metric spaces. A lot of fixed point theorems on ordinary metric spaces are special cases of our main result.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh ([23]) in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani ([7]) and Kramosil and Michalek ([9]) have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and E-infinity theory which were given and studied by El Naschie [3,

[0]Received January 2, 2018. Revised September 5, 2018.
[0]2010 Mathematics Subject Classification: 54E70, 47H25.
[0]Keywords: $\mathcal{M}$-fuzzy metric space, fuzzy contractive mapping, complete fuzzy metric space, common fixed point theorem, contractive mapping, R-weakly commuting self-mappings, sequentially convergent.
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6, 22]. Many authors [8, 11, 14, 15] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. One should there exists a space between spaces.

2. Preliminaries

Definition 2.1. ([7]) A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 2.2. ([19]) Let X be a nonempty set. A function $S : X^3 \to [0,\infty)$ is said to be an S-metric on X, if for each $x, y, z, a \in X$,

(1) $S(x,y,z) \geq 0$,
(2) $S(x,y,z) = 0$ if and only if $x = y = z$,
(3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Example 2.3. ([19]) We can easily check that the following examples are S-metric spaces.

(1) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X . Then $S(x,y,z) = \|y + z - 2x\| + \|y - z\|$ is an $S-$metric on X.
In general, if X is a vector space over $\mathbb{R}$ and $\|.\|$ a norm on X. Then it is easy to see that
$$S(x,y,z) = \|\alpha y + \beta z + \lambda x\| + \|y - z\|,$$
where $\alpha + \beta = \lambda$ for every $\alpha, \beta \geq 1$, is an S-metric on X.

(2) Let X be a nonempty set and d_1, d_2 be two ordinary metrics on X. Then
$$S(x,y,z) = d_1(x,z) + d_2(y,z),$$
is an S-metric on X.

Here we introduce a $\mathcal{M}$-fuzzy metric. We describe the space along with some associated concepts in the following.

Definition 2.4. A 3-tuple $(X, \mathcal{M}, *)$ is called a $\mathcal{M}$-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s, r > 0$,

(1) $\mathcal{M}(x,y,z,t) > 0$,
(2) $\mathcal{M}(x,y,z,t) = 1$ if and only if $x = y = z$,
(3) $\mathcal{M}(x,y,z,\vee\{t,r,s\}) \geq \mathcal{M}(x,x,a,t) * \mathcal{M}(y,y,a,s) * \mathcal{M}(z,z,a,r)$ where $\vee\{t,s,r\} = \max\{t,s,r\}$,
(4) $\mathcal{M}(x,y,z,.) : (0,\infty) \longrightarrow [0,1]$ is continuous.

Example 2.5. Let $a * b = ab$ for all $a,b \in [0,1]$ we define

$$\mathcal{M}(x,y,z,t) = exp^{\frac{-S(x,y,z)}{t}}$$

where S is an $S-$metric on set X. Then $(X,\mathcal{M},*)$ is a $\mathcal{M}$-fuzzy metric space.

Proof. (i) $\mathcal{M}(x,y,z,t) > 0$ for all $x,y,z \in X$ and $t > 0$ is trivial.
(ii)

$$\begin{aligned} \mathcal{M}(x,y,z,t) = 1 &\iff S(x,y,z) = 0 \\ &\iff x = y = z. \end{aligned}$$

(iii) Since $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$,

$$\begin{aligned} \frac{S(x,y,z)}{t \vee s \vee r} &\leq \frac{S(x,x,a) + S(y,y,a) + S(z,z,a)}{t \vee s \vee r} \\ &\leq \frac{S(x,x,a)}{t} + \frac{S(y,y,a)}{s} + \frac{S(z,z,a)}{r}. \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{M}(x,y,z,\vee\{t,r,s\}) &= exp^{\frac{-S(x,y,z)}{t\vee s\vee r}} \\ &\geq exp^{-\frac{S(x,x,a)}{t}-\frac{S(y,y,a)}{s}-\frac{S(z,z,a)}{r}} \\ &= exp^{\frac{-S(x,x,a)}{t}}.exp^{\frac{-S(y,y,a)}{s}}.exp^{\frac{-S(z,z,a)}{r}} \\ &= \mathcal{M}(x,x,a,t).\mathcal{M}(y,y,a,s).\mathcal{M}(z,z,a,r) \\ &= \mathcal{M}(x,x,a,t) * \mathcal{M}(y,y,a,s) * \mathcal{M}(z,z,a,r). \end{aligned}$$

Hence $(X,\mathcal{M},*)$ is a $\mathcal{M}$-fuzzy metric space. □

A sequence $\{x_n\}$ in X is said to be convergent to x if $\mathcal{M}(x_n,x_n,x,t) \longrightarrow 1$ as $n \longrightarrow \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n,x_n,x_m,t) > 1 - \varepsilon$ for each $n,m \geq n_0$. The $\mathcal{M}$-fuzzy metric space $(X,\mathcal{M},*)$ is said to be complete if every Cauchy sequence is convergent.

The following properties of $\mathcal{M}$ noted in the theorem below are easy consequences of the definition.

Lemma 2.6. *Let $(X,\mathcal{M},*)$ be an $\mathcal{M}$-fuzzy metric space. Then*
(i) $\mathcal{M}(x,x,y,t) = \mathcal{M}(y,y,x,t)$.

(ii) $\mathcal{M}(x,x,y,t)$ *is nondecreasing with respect to* t *for each* $x,y \in X$.

Proof. (i) For every $t \in (0,\infty)$, we have

$$\begin{aligned}\mathcal{M}(x,x,y,t) &= \mathcal{M}(x,x,y,t \vee t \vee t)\\ &\geq \mathcal{M}(x,x,x,t) * \mathcal{M}(x,x,x,t) * \mathcal{M}(y,y,x,t)\\ &= \mathcal{M}(y,y,x,t).\end{aligned}$$

Similarly, we can show that $\mathcal{M}(y,y,x,t) \geq \mathcal{M}(x,x,y,t)$. Hence, we have

$$\mathcal{M}(x,x,y,t) = \mathcal{M}(y,y,x,t).$$

(ii) For every $t,s \in (0,\infty)$, let $t \geq s$, we have

$$\begin{aligned}\mathcal{M}(x,x,y,t) &= \mathcal{M}(x,x,y,t \vee s \vee s)\\ &\geq \mathcal{M}(x,x,x,t) * \mathcal{M}(x,x,x,s) * \mathcal{M}(y,y,x,s)\\ &= \mathcal{M}(y,y,x,s)\\ &= \mathcal{M}(x,x,y,s).\end{aligned}$$

This completes the proof. □

Example 2.7. Let $a * b = ab$ for all $a,b \in [0,1]$ and M_1 and M_2 be two fuzzy sets on $X \times X \times (0,\infty)$ define by

$$\mathcal{M}(x,y,z,t) = M_1(x,z,t) * M_2(y,z,t),$$

for all $x,y,z \in X$. Then $(X,\mathcal{M},*)$ is a $\mathcal{M}$-fuzzy metric space.

Proof. (i) $\mathcal{M}(x,y,z,t) > 0$ for all $x,y,z \in X$ and $t > 0$ is trivial.
(ii)

$$\begin{aligned}\mathcal{M}(x,y,z,t) = 1 &\iff M_1(x,z,t) = M_2(y,z,t) = 1\\ &\iff x = y = z.\end{aligned}$$

(iii) Let $t \geq s \geq r$, it follows that,

$$\begin{aligned}\mathcal{M}(x,y,z,t \vee s \vee r) &= \mathcal{M}(x,y,z,t)\\ &= M_1(x,z,t) * M_2(y,z,t)\\ &\geq M_1(x,a,t) * M_1(a,z,t) * M_2(y,a,t) * M_2(a,z,t)\\ &\geq M_1(x,a,t) * M_2(x,a,t) * M_1(y,a,t)\\ &\quad * M_2(y,a,t) * M_1(z,a,t) * M_2(z,a,t)\\ &= \mathcal{M}(x,x,a,t) * \mathcal{M}(y,y,a,t)\mathcal{M}(z,z,a,t)\\ &\geq \mathcal{M}(x,x,a,t) * \mathcal{M}(y,y,a,s)\mathcal{M}(z,z,a,r).\end{aligned}$$

Hence $(X,\mathcal{M},*)$ is a $\mathcal{M}$-fuzzy metric space. □

Lemma 2.8. *Let* $(X,\mathcal{M},*)$ *be a* $\mathcal{M}$*-fuzzy metric space. If sequence* $\{x_n\}$ *in* X *converges to* x *then* x *is unique.*

Proof. Let $\{x_n\}$ converges to x and y. Then for each $0 < \varepsilon < 1$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\forall n \geq n_1 \Longrightarrow \mathcal{M}(x_n, x_n, x, t) > 1 - \varepsilon,$$

and

$$n \geq n_2 \Longrightarrow \mathcal{M}(x_n, x_n, y, t) > 1 - \varepsilon.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ we have

$$\begin{aligned} \mathcal{M}(x, x, y, t) &\geq \mathcal{M}(x, x, x_n, t) * \mathcal{M}(x, x, x_n, t) * \mathcal{M}(y, y, x_n, t) \\ &> (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon). \end{aligned}$$

By taking the limit $\varepsilon \longrightarrow 0$ in above inequality we get $\mathcal{M}(x, x, y, t) \geq 1$. Hence $\mathcal{M}(x, x, y, t) = 1$ so $x = y$. □

Lemma 2.9. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then the convergent sequence $\{x_n\}$ in X is Cauchy.*

Proof. Since $\lim_{n\to\infty} x_n = x$, for each $0 < \varepsilon < 1$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Longrightarrow \mathcal{M}(x_n, x_n, x, t) > 1 - \varepsilon,$$

and

$$m \geq n_2 \Longrightarrow \mathcal{M}(x_m, x_m, x, t) > 1 - \varepsilon.$$

If set $n_0 = \; max\{n_1, n_2\}$, then for every $n, m \geq n_0$ we have

$$\begin{aligned} \mathcal{M}(x_n, x_n, x_m, t) &\geq \mathcal{M}(x_n, x_n, x, t) * \mathcal{M}(x_n, x_n, x, t) * \mathcal{M}(x_m, x_m, x, t) \\ &> (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon). \end{aligned}$$

By taking the limit $\varepsilon \longrightarrow 0$ in above inequality we get $\mathcal{M}(x_n, x_n, x_m, t) \geq 1$. Hence $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.10. *Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} \mathcal{M}(x_n, x_n, y_n, t) = \mathcal{M}(x, x, y, t).$$

Proof. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, for each $0 < \varepsilon < 1$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Rightarrow \mathcal{M}(x_n, x_n, x, t) > 1 - \varepsilon$$

and

$$n \geq n_2 \Rightarrow \mathcal{M}(y_n, y_n, y, t) > 1 - \varepsilon.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ we have

$$\begin{aligned}\mathcal{M}(x_n, x_n, y_n, t) &\geq \mathcal{M}(x_n, x_n, x, t) * \mathcal{M}(x_n, x_n, x, t) * \mathcal{M}(y_n, y_n, x, t)\\ &\geq \mathcal{M}(x_n, x_n, x, t) * \mathcal{M}(x_n, x_n, x, t)\\ &\quad * \mathcal{M}(y_n, y_n, y, t) * \mathcal{M}(y_n, y_n, y, t) * \mathcal{M}(x, x, y, t)\\ &> \ (1-\varepsilon\ ) * \ (1-\varepsilon\ ) * \ (1-\varepsilon\ ) * \mathcal{M}(x, x, y, t).\end{aligned}$$

By taking the limit when $\varepsilon \to 0$ in above inequality we get

$$\mathcal{M}(x_n, x_n, y_n, t) \geq \mathcal{M}(x, x, y, t) \tag{2.1}$$

On the other hand , we have

$$\begin{aligned}\mathcal{M}(x, x, y, t) &\geq \mathcal{M}(x, x, x_n, t) * \mathcal{M}(y, y, x_n, t) * \mathcal{M}(y, y, x_n, t)\\ &\geq \mathcal{M}(x, x, x_n, t) * \mathcal{M}(x, x, x_n, t)\\ &\quad * \mathcal{M}(y, y, y_n, t) * \mathcal{M}(y, y, y_n, t) * \mathcal{M}(x, x, y, t)\\ &> \ (1-\varepsilon\ ) * \ (1-\varepsilon\ ) * \ (1-\varepsilon\ ) * \mathcal{M}(x, x, y, t),\end{aligned}$$

as $\varepsilon \to 0$ we have

$$\mathcal{M}(x, x, y, t) > \mathcal{M}(x_n, x_n, y_n, t) \tag{2.2}$$

Therefore, by (2.1) and (2.2) we have

$$\lim_{n\to\infty} \mathcal{M}(x_n, x_n, y_n, t) = \mathcal{M}(x, x, y, t).$$

This completes the proof. □

Definition 2.11. Let $(X, \mathcal{M}_1, *)$ and $(Y, \mathcal{M}_2, *)$ be two $\mathcal{M}$-fuzzy metric spaces and $T : X \longrightarrow Y$ be a map. T is called sequentially convergent if $\{x_n\}$ is $\mathcal{M}_1$-convergent in X provided $\{Tx_n\}$ is $\mathcal{M}_2$-convergent in Y.

Definition 2.12. Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space and let f and g be maps from X into itself. The maps f and g are said to be weakly commuting if

$$\mathcal{M}(fgx, fgx, gfx, t) \geq \mathcal{M}(fx, fx, gx, t)$$

for each $x \in X$.

Definition 2.13. Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space and f and g be maps from X into itself. The maps f and g are said to be $R-$weakly commuting if there exist a positive real number R such that

$$\mathcal{M}(fgx, fgx, gfx, t) \geq \mathcal{M}(fx, fx, gx, \frac{t}{R})$$

for each $x \in X$ and $t > 0$.

Weak commutativity implies $R-$weak commutivity in an $\mathcal{M}$-fuzzy metric space. However, $R-$weak commutativity implies weak commutativity only when $R \leq 1$.

Example 2.14. Let $X = R^2$ and $\mathcal{M}$ be the $\mathcal{M}$-fuzzy metric on X^3 defined by

$$\mathcal{M}(x, y, z, t) = e^{\frac{-\{\|x-y\|+\|y-z\|\}}{t}}.$$

Then $(X, \mathcal{M}, *)$ is an $\mathcal{M}$-fuzzy metric space. Define $f(x, y) = (x^2, \sin y)$ and $g(x, y) = (2x - 1, \sin y)$. Then, we have

$$\begin{aligned}
\mathcal{M}(fg(x, y), fg(x, y), gf(x, y), t) &= \mathcal{M}(((2x-1)^2, \sin(\sin y)), ((2x-1)^2, \\
&\quad \sin(\sin y)), (2x^2 - 1, \sin(\sin y)), t) \\
&= e^{\frac{-\sqrt{(2(x-1)^2)^2}}{t}} = e^{\frac{-2(x-1)^2}{t}} \\
&= \mathcal{M}(f(x, y), f(x, y), g(x, y), \frac{t}{2}) \\
&< \mathcal{M}(f(x, y), f(x, y), g(x, y), t)
\end{aligned}$$

Therefore, for $R = 2$, f and g are R-weakly commuting. But f and g are not weakly commuting.

Now we introduce and prove the main theorem in this paper.

3. Main Results

Theorem 3.1. *Let $(X, \mathcal{M}_1, *)$ be an $\mathcal{M}-$ fuzzy metric space, $(Y, \mathcal{M}_2, *)$ be a complete $\mathcal{M}-$fuzzy metric space such that $a * b = a.b$ for every $a, b \in [0, 1]$. Let f and g be R-weakly commuting self mappings of X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$;
(b) f *or* g *is continuous;*
(c) $\mathcal{M}_1(Ffx, Ffx, Ffy, t) \geq \mathcal{M}_2(Fgx, Fgx, Fgy, t)^k$,
where $0 < k < 1$ *and* $F : X \longrightarrow Y$ *is one-to-one, continuous and sequentially convergent.*

Then there exist a unique common fixed point $z \in X$ of f and g.

Proof. Let x_0 be an arbitrary point in X. By (a), there exist a point x_1 in X such that $fx_0 = gx_1$. Continuing in this process, we can choose x_{n+1} such that $fx_n = gx_{n+1}$. Set $y_n = Ffx_n$. Then

$$\begin{aligned}
\mathcal{M}_2(y_n, y_n, y_{n+1}, t) &= \mathcal{M}_2(Ffx_n, Ffx_n, Ffx_{n+1}, t) \\
&\geq \mathcal{M}_2(Fgx_n, Fgx_n, Fgx_{n+1}, t)^k \\
&= \mathcal{M}_2(Ffx_{n-1}, Ffx_{n-1}, Ffx_n, t)^k \\
&= \mathcal{M}_2(y_{n-1}, y_{n-1}, y_n, t)^k \\
&\geq \mathcal{M}_2(y_{n-2}, y_{n-2}, y_{n-1}, t)^{k^2} \\
&\vdots \\
&\geq \mathcal{M}_2(y_0, y_0, y_1, t)^{k^n}.
\end{aligned}$$

Thus for all $m > n$, we have

$$\begin{aligned}
\mathcal{M}_2(y_n, y_n, y_m, t) &= \mathcal{M}_2(y_n, y_n, y_{n+1}, t \vee t \vee t) \\
&\geq \mathcal{M}_2(y_n, y_n, y_{n+1}, t) * \mathcal{M}_2(y_n, y_n, y_{n+1}, t) \\
&\quad * \mathcal{M}_2(y_m, y_m, y_{n+1}, t) \\
&= \mathcal{M}_2(y_n, y_n, y_{n+1}, t).\mathcal{M}_2(y_n, y_n, y_{n+1}, t).\mathcal{M}_2(y_{n+1}, y_{n+1}, y_m, t) \\
&\geq \mathcal{M}_2(y_0, y_0, y_1, t)^{k^n}.\mathcal{M}_2(y_0, y_0, y_1, t)^{k^n}.\mathcal{M}_2(y_m, y_m, y_{n+1}, t) \\
&= \mathcal{M}_2(y_0, y_0, y_1, t)^{k^n}.\mathcal{M}_2(y_0, y_0, y_1, t)^{k^n}.\mathcal{M}_2(y_{n+1}, y_{n+1}, y_m, t) \\
&\ \vdots \\
&\geq \mathcal{M}_2(y_0, y_0, y_1, t)^{2[k^n + k^{n+1} + \ldots + k^{m-1}]} \\
&\geq \mathcal{M}_2(y_0, y_0, y_1, t)^{\frac{2k^n}{1-k}}.
\end{aligned}$$

Taking the limit as $n, m \longrightarrow \infty$, we get $\mathcal{M}_2(y_n, y_n, y_m, t) \longrightarrow 1$. This means that $\{y_n\}$ is a Cauchy sequence. Since $(Y, \mathcal{M}_2, *)$ is complete, the sequence $\{y_n\}$ converges to some $y \in Y$. Since F is sequentially convergent, $\{fx_n\}$ converges to some $z \in X$ and also from the continuity of F, $\{Ffx_n\}$ converges to Fz. Note that $\{y_n\}$ converges to y, then $y_n = Ffx_n = Fgx_{n+1} \to Fz = y$. Also gx_n converges to z in X. Let us suppose that the mapping f is continuous. Then $\lim_{n \longrightarrow \infty} ffx_n = fz$ and $\lim_{n \longrightarrow \infty} fgx_n = fz$. Further, since f and g are R-weakly commuting, we have

$$\mathcal{M}_1(fgx_n, fgx_n, gfx_n, t) \geq \mathcal{M}_1(fx_n, fx_n, gx_n, \frac{t}{R}).$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, we have

$$\begin{aligned}\mathcal{M}_1(fz, fz, \lim_{n\longrightarrow\infty} gfx_n, t) &= \lim_{n\longrightarrow\infty} \mathcal{M}_1(fgx_n, fgx_n, gfx_n, t)\\ &\geq \lim_{n\longrightarrow\infty} \mathcal{M}_1(fx_n, fx_n, gx_n, \frac{t}{R})\\ &= \mathcal{M}_1(z, z, z, \frac{t}{R})\\ &= 1.\end{aligned}$$

Hence, we get $\lim_{n\longrightarrow\infty} gfx_n = fz$. We now prove that $z = fz$. By (c)

$$\begin{aligned}\mathcal{M}_2(Ffz, Ffz, Fz, t) &= \lim_{n\longrightarrow\infty} \mathcal{M}_2(Fffx_n, Fffx_n, Ffx_n, t)\\ &\geq \lim_{n\longrightarrow\infty} \mathcal{M}_2(Fgfx_n, Fgfx_n, Fgx_n, t)^k\\ &= \mathcal{M}_2(Ffz, Ffz, Fz, t)^k.\end{aligned}$$

By the above inequality, we get $Ffz = Fz$. Since F is one-to-one, it follows that $fz = z$. Since $f(X) \subseteq g(X)$, we can fined $z_1 \in X$ such that $z = fz = gz_1$. Now,

$$\mathcal{M}_2(Fffx_n, Fffx_n, Ffz_1, t) \geq \mathcal{M}_2(Fgfx_n, Fgfx_n, Fgz_1, t)^k.$$

Taking the limit as $n \longrightarrow \infty$, we get

$$\begin{aligned}\mathcal{M}_2(Ffz, Ffz, Ffz_1, t) &= \lim_{n\longrightarrow\infty} \mathcal{M}_2(Fffx_n, Fffx_n, Ffz_1, t)\\ &\geq \lim_{n\longrightarrow\infty} \mathcal{M}_2(Fgfx_n, Fgfx_n, Fgz_1, t)^k\\ &= \mathcal{M}_2(Ffz, Ffz, Fgz_1, t)^k\\ &= 1,\end{aligned}$$

which implies that $Ffz = Ffz_1$, i.e., $z = fz = fz_1 = gz_1$. Also, we have

$$\mathcal{M}_1(fz, fz, gz, t) = \mathcal{M}_1(fgz_1, fgz_1, gfz_1, t) \geq \mathcal{M}_1(fz_1, fz_1, gz_1, \frac{t}{R}) = 1,$$

which implies that $fz = gz$. Thus z is a common fixed point of f and g.

Now in order to prove the uniquenees, let $z' \neq z$ be another common fixed point of f and g. Then

$$\begin{aligned}\mathcal{M}_2(Fz, Fz, Fz', t) &= \mathcal{M}_2(Ffz, Ffz, Ffz', t)\\ &\geq \mathcal{M}_2(Fgz, Fgz, Fgz', t)^k\\ &= \mathcal{M}_2(Fz, Fz, Fz', t)^k\\ &> \mathcal{M}_2(Fz, Fz, Fz', t),\end{aligned}$$

which is a contradiction. Therefore, $Fz = Fz'$, i.e., $z = z'$ is unique common fixed point of f and g. This completes the proof. □

Example 3.2. Let $X = [1, \infty)$, $Y = \mathbb{R}^2$ and $\mathcal{M}_1(x, y, z, t) = e^{\frac{-max\{|x-y|,|y-z|\}}{t}}$, $\mathcal{M}_2(x, y, z, t) = e^{\frac{-\{\|x-y\|+\|y-z\|\}}{t}}$. Then $(X, \mathcal{M}_1, *)$ is an $\mathcal{M}-$fuzzy metric space. Define $f(x) = 2x - 1$ and $g(x) = x^2$ on X. It is evident that $f(X) \subseteq g(X)$, f is continuous,

$$\mathcal{M}_1(fgx, fgx, gfx, t) = e^{\frac{-2(x-1)^2}{t}} = \mathcal{M}_1(fx, fx, gx, \frac{t}{2})$$

for all $x \in X$. It is easy to see that f and g are R-weakly commuting for $R = 2$. If define $F : X \longrightarrow Y$ by $F(x) = (\frac{x}{2}, \frac{x-1}{2})$, then F is one-to-one, continuous and sequentially convergent. Also, we have

$$\begin{aligned}
& \mathcal{M}_2(Ffx, Ffy, Ffz, t) \\
= \; & \mathcal{M}_2\Big((\frac{2x-1}{2}, \frac{2x-2}{2}), (\frac{2y-1}{2}, \frac{2y-2}{2}), (\frac{2z-1}{2}, \frac{2z-2}{2}), t\Big) \\
= \; & e^{\frac{-\{\sqrt{(x-y)^2+(x-y)^2}+\sqrt{(y-z)^2+(y-z)^2}\}}{t}} \\
= \; & e^{\frac{-\{\sqrt{2}(|x-y|)+|y-z|\}}{t}}
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{M}_2(Fgx, Fgy, Fgz, t) \\
= \; & \mathcal{M}_2\Big((\frac{x^2}{2}, \frac{x^2-1}{2}), (\frac{y^2}{2}, \frac{y^2-1}{2}), (\frac{z^2}{2}, \frac{z^2-1}{2}), t\Big) \\
= \; & e^{\frac{-\{\sqrt{(x^2-y^2)^2+(x^2-y^2)^2}+\sqrt{(y^2-z^2)^2+(y^2-z^2)^2}\}}{t}} \\
= \; & e^{\frac{-\{\sqrt{2}(|x-y||x+y|)+|y-z||y+z|\}}{t}} \\
\leq \; & e^{\frac{-\{2\sqrt{2}(|x-y|)+|y-z|\}}{t}} \\
= \; & \mathcal{M}_2(Ffx, Ffy, Ffz, t)^2.
\end{aligned}$$

Therefore

$$\mathcal{M}_2(Ffx, Ffy, Ffz, t) \geq \mathcal{M}_2(Fgfx, Fgy, Fgz, t)^k$$

for $k = \frac{1}{2}$. Thus all the conditions of Theorem 3.1 are satisfied and 1 is a common fixed point of f and g.

Corollary 3.3. *Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}-$fuzzy metric space and let f and g be R-weakly commuting self mappings of X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$;*
(b) *f or g is continuous;*
(c) *$\mathcal{M}(Ffx, Ffx, Ffy, t) \geq \mathcal{M}(Fgx, Fgx, Fgy, t)^k$ where $0 < k < 1$ and $F : X \longrightarrow Y$ is one-to-one, continuous and sequentially convergent.*

Then f and g have a unique common fixed point $z \in X$. Moreover, if $Ff = fF$ and $Fg = gF$ then F and g have a unique common fixed point $z \in X$.

Proof. By Theorem 3.1, f and g have a unique common fixed point $z \in X$. Now we show that $Fz = z$.

$$\begin{aligned}\mathcal{M}(Fz, Fz, FFz, t) &= \mathcal{M}(Ffz, Ffz, FFfz, t)\\ &= \mathcal{M}(Ffz, Ffz, FfFz, t)\\ &\geq \mathcal{M}(Fgz, Fgz, FgFz, t)\\ &= \mathcal{M}(Fz, Fz, FFz, t)^k,\end{aligned}$$

it followes that $FFz = Fz$, hence $Fz = z$ from the injectivity of F. □

Corollary 3.4. *Let $(X, \mathcal{M}_1, *)$ be an $\mathcal{M}-$fuzzy metric space, $(Y, \mathcal{M}_2, *)$ be a complete $\mathcal{M}-$fuzzy metric space and let f be a self-mapping of X satisfying the following conditions:*

(a) *f is continuous;*
(b) *$\mathcal{M}(Ffx, Ffx, Ffy, t) \geq \mathcal{M}(Ffx, Ffx, Ffy, t)^k$ where $0 < k < 1$ and $F : X \longrightarrow Y$ is one-to-one, continuous and sequentially convergent.*

Then f has a unique common fixed point $z \in X$.

Corollary 3.5. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}-$fuzzy metric space and let f and g be R-weakly commuting self-mapping of X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$;*
(b) *f or g is continuous;*
(c) *$c\mathcal{M}(fx, fx, fy, t) \geq \mathcal{M}(gx, gx, gy, t)^k$ where $0 < k < 1$.*

Then f and g have a unique common fixed point $z \in X$.

Proof. If set $F = I$ identity map then by Corollary 3.3 follows that f and g have a unique common fixed point $z \in X$. □

Corollary 3.6. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}-$fuzzy metric space, F, f and g be self-mappings of X and let Ff and Fg be R-weakly commuting satisfying the following conditions:*

(a) *$Ff(X) \subseteq Fg(X)$;*
(b) *Ff or Fg is continuous;*
(c) *$\mathcal{M}(Ffx, Ffx, Ffy, t) \geq \mathcal{M}(Fgx, Fgx, Fgy, t)^k$ where $0 < k < 1$.*

If $Ff = fF$ and $Fg = gF$ then F, f and g have a unique common fixed point $z \in X$.

Proof. By Corollary 3.5, Ff and Fg have a unique common fixed point $z \in X$. That is $Ffz = Fgz = z$. Now we show that $fz = z$.

$$\begin{aligned} \mathcal{M}(Fz, Fz, z, t) &= \mathcal{M}(FFfz, FFfz, Ffz, t) \\ &= \mathcal{M}(FfFz, FfFz, Ffz, t) \\ &\geq \mathcal{M}(FgFz, FgFz, Fgz, t)^k \\ &= \mathcal{M}(FFgz, FFgz, Fgz, t)^k \\ &= \mathcal{M}(Fz, Fz, z, t)^k, \end{aligned}$$

it follows that $Fz = z$, hence $z = Ffz = fFz = fz$. Similarly we can show that $gz = z$. This completes the proof. □

Acknowledgments: This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

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