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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE KAEHLER MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We study lightlike hypersurfaces M of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection subject such that the characteristic vector field ζ of $\bar{M}$ belongs to our screen distribution $S(TM)$. First, we provide several new results on such a lightlike hypersurface. Next, we investigate lightlike hypersurfaces of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$.

1. Introduction

In 1924, Friedmann-Schouten [3] introduced the idea of a semi-symmetric connection: A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *semi-symmetric connection* if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}, \tag{1.1}$$

where θ is a 1-form associated with a smooth unit spacelike vector field ζ, which is called the *characteristic vector field* of $\bar{M}$, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Moreover, if this connection is a metric one, *i.e.*, it satisfies $\bar{\nabla}\bar{g} = 0$, then $\bar{\nabla}$ is called a *semi-symmetric metric connection* on $\bar{M}$. The notion of a semi-symmetric metric connection was introduced by Yano [8] and studied by this author [4, 6]. In the followings, denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

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Remark 1.1. Denote $\widetilde{\nabla}$ by the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. It is well known that *a linear connection* $\bar{\nabla}$ *on* $\bar{M}$ *is a semi-symmetric metric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta. \tag{1.2}$$

The object of this paper is to study lightlike hypersurfaces M of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection $\bar{\nabla}$ subject such that the characteristic vector field ζ of $\bar{M}$ belongs to our screen distribution $S(TM)$ of M. In Section 3, we provide several new results on such a lightlike hypersurface. In Section 4, we characterize lightlike hypersurfaces of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric connection subject to the condition that ζ belongs to $S(TM)$.

2. Structure equations

Let $(\bar{M}, \bar{g}, J)$ be an indefinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure ([7]) such that

$$J^2 = -I, \qquad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \qquad (\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = 0. \tag{2.1}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the semi-symmetric metric connection $\bar{\nabla}$, the third equation of three equations in (2.1) is reduced to

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X} - \bar{g}(\bar{X}, J\bar{Y})\zeta + \bar{g}(\bar{X}, \bar{Y})J\zeta. \tag{2.2}$$

Let (M, g) be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M} = (\bar{M}, \bar{g}, J)$. Then the normal bundle $TM^{\perp}$ of M is a subbundle of the tangent bundle TM. A complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM is non-degenerate and called a *screen distribution* of M such that

$$TM = TM^{\perp} \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use same notations for any others. For any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$, respectively [1, Section 4.1]. Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM).$$

Denote by X, Y and Z the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.3}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.4}$$
$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.5}$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.6}$$

where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators and τ is a 1-form on TM.

The connection ∇ is a semi-symmetric non-metric connection and satisfy

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y). \tag{2.7}$$
$$T(X,Y) = \theta(Y)X - \theta(X)Y, \tag{2.8}$$

and we see that B is symmetric on TM, where T is the torsion tensor with respect to the induced connection ∇ and η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N).$$

From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and satisfies

$$B(X,\xi) = 0. \tag{2.9}$$

The above two local second fundamental forms B and C for TM and $S(TM)$ respectively are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.10}$$
$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0. \tag{2.11}$$

From (2.10), A^*_ξ is $S(TM)$-valued real self-adjoint operator and satisfies

$$A^*_\xi \xi = 0, \tag{2.12}$$

3. Some results

For a lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$, it is known ([1, Section 6.2], [4]) that $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$ such that $TM^\perp \cap J(TM^\perp) = \{0\}$ and $TM^\perp \cap J(tr(TM)) = \{0\}$. Therefore $J(TM^\perp) \oplus J(tr(TM))$ is a vector subbundle of $S(TM)$, of rank 2. Thus there exist two non-degenerate almost complex distributions D_o and D

on M with respect to J, *i.e.*, $J(D_o) \subset D_o$ and $J(D) \subset D$, such that

$$S(TM) = J(TM^{\perp}) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = \{TM^{\perp} \oplus_{orth} J(TM^{\perp})\} \oplus_{orth} D_o.$$

In this case, the decomposition form of TM is reduced to

$$TM = D \oplus J(tr(TM)). \tag{3.1}$$

Consider two local lightlike vector fields U and V on $S(TM)$ such that

$$U = -JN, \qquad V = -J\xi. \tag{3.2}$$

Denote by S the projection morphism of TM on D with respect to the decomposition (3.1). Then any vector field X on M is expressed as follows:

$$X = SX + u(X)U,$$

where u and v are 1-forms locally defined on TM by

$$u(X) = g(X, V), \qquad v(X) = g(X, U). \tag{3.3}$$

Using (3.2), the action JX of any $X \in \Gamma(TM)$ by J is xpressed as follows:

$$JX = FX + u(X)N, \tag{3.4}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Applying J to (3.4) and using (2.1) and (3.2), we have

$$F^2X = -X + u(X)U. \tag{3.5}$$

As $u(U) = 1$ and $FU = 0$, the set (F, u, U) defines an indefinite almost contact structure on M. Then F is called the *structure tensor field* of M.

In the sequel, we shall assume that ζ belongs to $S(TM)$. Applying $\bar{\nabla}_X$ to (3.2) and (3.4) and using (2.2)∼(2.6), (2.9)∼(2.11), and (3.4), we have

$$B(X, U) = C(X, V) - \theta(V)\eta(X), \tag{3.6}$$
$$\nabla_X U = F(A_N X) + \tau(X)U + \theta(U)X - v(X)\zeta - \eta(X)F\zeta. \tag{3.7}$$
$$\nabla_X V = F(A^*_{\xi} X) - \tau(X)V + \theta(V)X - u(X)\zeta, \tag{3.8}$$
$$(\nabla_X F)Y = u(Y)A_N X - B(X, Y)U + \theta(FY)X - \theta(Y)FX - \bar{g}(X, JY)\zeta + g(X, Y)F\zeta. \tag{3.9}$$

Theorem 3.1. *There exist no lightlike hypersurfaces of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$ and V is parallel with respect to the connection ∇ on M.*

Proof. Assume that V is parallel with respect to the connection ∇. Taking the scalar product with N to (3.8) and using (2.10) and (3.4), we obtain

$$B(X, U) + \theta(V)\eta(X) = 0.$$

Replacing X by ξ to this equation and using (2.9), we have $\theta(V) = 0$. Thus

$$B(X, U) = 0.$$

Taking the scalar product with ζ to (3.8) and using $\theta(V) = 0$, we obtain

$$B(X, F\zeta) = -u(X).$$

From the last two equations, we have the following impossible result:

$$-1 = -u(U) = B(U, F\zeta) = B(F\zeta, U) = 0.$$

Thus we have our theorem. □

Theorem 3.2. *There exist no lightlike hypersurfaces of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$ and F is parallel with respect to the connection ∇ on M.*

Proof. Assume that F is parallel with respect to ∇. Replacing Y by ξ to (3.9) and using (2.9) and the fact that $F\xi = -V$, we get

$$\theta(V)X = u(X)\zeta.$$

Taking the scalar product with N to this, we have $\theta(V)\eta(X) = 0$. It follows that $\theta(V) = 0$. Taking $X = U$ to the last equation: $u(X)\zeta = 0$, we get $\zeta = 0$. It is a contradiction to $\zeta \neq 0$. □

Definition 3.3. ([5]) The structure tensor field F of M is said to be *recurrent* if there exists a 1-form ϖ on TM such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

Theorem 3.4. *There exist no lightlike hypersurfaces of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that ζ belongs to $S(TM)$ and the structure tensor field F of M is recurrent.*

Proof. From the above definition and (3.9), we obtain

$$\varpi(X)FY = u(Y)A_N X - B(X,Y)U + \theta(FY)X - \theta(Y)FX \\ - \bar{g}(X, JY)\zeta + g(X,Y)F\zeta.$$

Replacing Y by ξ and using (2.9) and the fact that $F\xi = -V$, we get

$$\varpi(X)V = \theta(V)X - u(X)\zeta.$$

Taking the scalar product with N, ζ and U to this by turns, we have

$$\theta(V) = 0, \qquad u(X) = 0, \qquad \varpi(X) = 0.$$

As $\varpi(X) = 0$, the structure tensor F is parallel with respect to the induced connection ∇ on M. Thus we have our theorem by Theorem 3.2. □

Definition 3.5. ([5]) The structure tensor field F of M is said to be *Lie recurrent* if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$. A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 3.6. *Let M be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric metric connection such that the characteristic vector field ζ of $\bar{M}$ belongs to $S(TM)$. Then*

(1) *F is Lie parallel,*
(2) *the 1-form τ satisfies $\tau = 0$, and*
(3) *the shape operator A^*_ξ satisfies $A^*_\xi U = A^*_\xi V = 0$.*

Proof. (1) Using the above definition, (2.8), (3.4) and (3.9), we obtain

$$\begin{aligned}\vartheta(X)FY = & -\nabla_{FY}X + F\nabla_Y X + u(Y)A_N X - B(X,Y)U \\ & - \bar{g}(X, JY)\zeta + g(X,Y)F\zeta.\end{aligned} \tag{3.10}$$

Taking $Y = \xi$ to (3.10) and using (2.9) and the fact that $F\xi = -V$, we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X + u(X)\zeta. \tag{3.11}$$

Taking the scalar product with V to (3.11), we have

$$u(\nabla_V X) = -\theta(V)u(X). \tag{3.12}$$

Replacing Y by V to (3.10) and using the fact that $FV = \xi$, we have

$$\vartheta(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U + u(X)F\zeta.$$

Applying F to this equation and using (3.5) and (3.12), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X + u(X)\zeta.$$

Comparing this equation with (3.11), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N to (3.10) and using $(2.11)_2$, we get

$$-\bar{g}(\nabla_{FY}X, N) + g(\nabla_Y X, U) + \theta(U)g(X,Y) = 0. \tag{3.13}$$

Replacing X by V to (3.13) and using (2.10) and (3.8), we have

$$B(FY, U) + \tau(Y) = 0.$$

Taking $Y = U$ to this equation and using the fact that $FU = 0$, we obtain

$$\tau(U) = 0. \tag{3.14}$$

Replacing X by ξ to (3.13) and using (2.6) and (2.10), we have

$$B(X,U)=\tau(FX). \tag{3.15}$$

From this equation and (3.6), we see that

$$u(A_N X)=\tau(FX)+\theta(V)\eta(X). \tag{3.16}$$

Replacing X by U to (3.10) and using (2.11), (3.6) and (3.7), we obtain

$$u(Y)A_N U-F(A_N FY)-A_N Y-\tau(FY)U+\eta(Y)\zeta+v(Y)F\zeta=0.$$

Taking the scalar product with V to this and using (3.16), we get $\tau(FY)=0$. Taking $Y=FX$ to $\tau(FY)=0$ and using (3.5) and (3.14), we have

$$\tau(X)=0, \qquad \forall\, X\in\Gamma(TM).$$

(3) As $\tau=0$, using (2.10) and (3.15), we have $g(A^*_\xi U,X)=0$, As $S(TM)$ is non-degenerate, we get $A^*_\xi U=0$. Replacing X by ξ to (3.11) and using (2.12) and the fact that $\tau=0$, we obtain $A^*_\xi V=0$. □

4. Hypersurfaces of an indefinite complex space form

Denote by $\bar{R}$, R and R^* the curvature tensors of the semi-symmetric metric connection $\bar\nabla$ on $\bar{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively, such that

$$\begin{aligned}\bar{R}(X,Y)Z&=R(X,Y)Z+B(X,Z)A_N Y-B(Y,Z)A_N X\\&\quad+\{(\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)\\&\quad+[\tau(X)-\theta(X)]B(Y,Z)-[\tau(Y)-\theta(Y)]B(X,Z)\}N,\end{aligned} \tag{4.1}$$

$$\begin{aligned}R(X,Y)PZ&=R^*(X,Y)PZ+C(X,PZ)A^*_\xi Y-C(Y,PZ)A^*_\xi X\\&\quad+\{(\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)\\&\quad-[\tau(X)+\theta(X)]C(Y,PZ)+[\tau(Y)+\theta(Y)]C(X,PZ)\}\xi.\end{aligned} \tag{4.2}$$

Definition 4.1. An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c;

$$\begin{aligned}\widetilde{R}(\bar X,\bar Y)\bar Z&=\frac{c}{4}\{\bar g(\bar Y,\bar Z)\bar X-\bar g(\bar X,\bar Z)\bar Y\\&\quad+\bar g(J\bar Y,\bar Z)J\bar X-\bar g(J\bar X,\bar Z)J\bar Y+2\bar g(\bar X,J\bar Y)J\bar Z\},\end{aligned} \tag{4.3}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

By directed calculations from (1.1) and (1.2), we see that

$$\begin{aligned}\bar{R}(\bar X,\bar Y)\bar Z&=\widetilde{R}(\bar X,\bar Y)\bar Z+\bar g(\bar X,\bar Z)\bar\nabla_{\bar Y}\zeta-\bar g(\bar Y,\bar Z)\bar\nabla_{\bar X}\zeta\\&\quad+\{(\bar\nabla_{\bar X}\theta)(\bar Z)-\bar g(\bar X,\bar Z)\}\bar Y-\{(\bar\nabla_{\bar Y}\theta)(\bar Z)-\bar g(\bar Y,\bar Z)\}\bar X.\end{aligned} \tag{4.4}$$

Taking the scalar product with ξ and N to (4.4) by turns and then, substituting (4.1) and (4.3) into the resulting equation and using $(2.11)_2$, (3.4) and the fact that $\bar{\nabla}$ is a metric connection, we obtain

$$\begin{aligned}(\nabla_X B)&(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \{\tau(X)-\theta(X)\}B(Y,Z) - \{\tau(Y)-\theta(Y)\}B(X,Z) \\ &- g(X,Z)B(Y,\zeta) + g(Y,Z)B(X,\zeta) \\ = &\ \frac{c}{4}\{u(X)g(FY,Z) - u(Y)g(FX,Z) + 2u(Z)\bar{g}(X,JY)\},\end{aligned} \tag{4.5}$$

$$\begin{aligned}(\nabla_X C)&(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &- \{\tau(X)+\theta(X)\}C(Y,PZ) + \{\tau(Y)+\theta(Y)\}C(X,PZ) \\ &- g(X,PZ)C(Y,\zeta) + g(Y,PZ)C(X,\zeta) \\ &- (\bar{\nabla}_X\theta)(PZ)\eta(Y) + (\bar{\nabla}_Y\theta)(PZ)\eta(X) \\ = &\ (\frac{c}{4}+1)\{\eta(X)g(Y,PZ) - \eta(Y)g(X,PZ)\} \\ &+ \frac{c}{4}\{v(X)g(FY,PZ) - v(Y)g(FX,PZ) + 2v(PZ)\bar{g}(X,JY)\}.\end{aligned} \tag{4.6}$$

Definition 4.2. ([5]) The structure vector field U is called *principal* (with respect to A^*_ξ) if there exists a smooth function α on $\mathcal{U}$ such that

$$A^*_\xi U = \alpha U. \tag{4.7}$$

A lightlike hypersurface M of an indefinite Kaehler manifold is called a *Hopf lightlike hypersurface* [5] if it admits a principal structure vector field U.

Definition 4.3. ([2]) A lightlike hypersurface M is said to be *screen conformal* if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that

$$C(X,PY) = \varphi B(X,Y). \tag{4.8}$$

Theorem 4.4. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection subject such that ζ belongs to $S(TM)$. If one of the following conditions is satisfied;*

(1) *M is Lie recurrent,*
(2) *M is Hopf lightlike hypersurface, and*
(3) *M is screen conformal,*

then $c = 0$, i.e., $\bar{M}(c)$ is flat.

Proof. (1) In case M is Lie recurrent. As $\tau = 0$, from (3.15) we obtain

$$B(Y,U) = 0. \tag{4.9}$$

Applying ∇_X to this equation and using (3.7) and (4.9), we have

$$\begin{aligned}(\nabla_X B)(Y,U) &= -B(Y,F(A_N X)) - \theta(U)B(X,Y) \\ &\quad + v(X)B(Y,\zeta) + \eta(X)B(Y,F\zeta).\end{aligned}$$

Substituting the last two equations into (4.5), we have

$$\begin{aligned} &B(X, F(A_N Y)) - B(Y, F(A_N X)) \\ &\quad + \eta(X)B(Y, F\zeta) - \eta(Y)B(X, F\zeta) \\ &= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}. \end{aligned}$$

Taking $X = \xi$ and $Y = U$ to this and using (2.9) and (4.9), we get $c = 0$.

(2) Taking the scalar product with X to (4.7) and using (2.10), we get

$$B(X, U) = \alpha v(X), \qquad C(X, V) = \alpha v(X) + \theta(V)\eta(X), \tag{4.10}$$

due to (3.6). Applying ∇_X to $v(Y) = g(X, U)$ and using (2.7), $(2.11)_2$, (3.4), (3.6), (3.7) and $(4.10)_1$, we obtain

$$\begin{aligned} (\nabla_X v)Y = v(Y)\tau(X) - \theta(Y)v(X) - g(A_N X, FY) \\ + \theta(U)g(X, Y) + \theta(FY)\eta(X). \end{aligned} \tag{4.11}$$

Applying ∇_X to $B(Y, U) = \alpha v(Y)$ and using (3.7) and (4.11), we have

$$\begin{aligned} (\nabla_X B)(Y, U) = (X\alpha)v(Y) - \; & B(Y, F(A_N X)) - \theta(U)B(X, Y) \\ & + v(X)B(Y, \zeta) + \eta(X)B(Y, F\zeta) \\ & - \alpha\{\theta(Y)v(X) + g(A_N X, FY) \\ & - \theta(U)g(X, Y) - \theta(FY)\eta(X)\}. \end{aligned}$$

Substituting this equation and $(4.10)_1$ into (4.5) such that $Z = U$, we have

$$\begin{aligned} &\{X\alpha + \alpha\tau(X)\}v(Y) - \{Y\alpha + \alpha\tau(Y)\}v(X) \\ &\quad + \{B(Y, F\zeta) + \alpha\theta(FY)\}\eta(X) - \{B(X, F\zeta) + \alpha\theta(FX)\}\eta(Y) \\ &\quad + g(A_N X,\ F(A^*_\xi Y) - \alpha FY) - g(A_N Y,\ F(A^*_\xi X) - \alpha FX) \\ &= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}. \end{aligned}$$

Taking $X = \xi$ and $Y = U$ and using (2.9), (2.12), (3.4), (4.7), $(4.10)_{1,2}$ and the facts that $B(U, F\zeta) = 0$, $C(U, V) = 0$, $FU = 0$, $F\xi = -V$, we get $c = 0$.

(3) If M is screen conformal, then, using (3.6) and (4.8), we have

$$B(X, U - \varphi V) = -\,\theta(V)\eta(X).$$

Replacing X by ξ to this equation and using (2.9), we obtain

$$\theta(V) = 0, \qquad\qquad B(X, U - \varphi V) = 0. \tag{4.12}$$

Applying $\bar{\nabla}_X$ to $\theta(V) = 0$ and using (2.10), (3.4), (3.8) and $(4.12)_1$, we get

$$(\bar{\nabla}_X\theta)(V) = B(X, F\zeta) + u(X). \tag{4.13}$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation into (4.6) and using (4.5), we have

$$\begin{aligned}&\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ)\\&\quad - (\bar{\nabla}_X\theta)(PZ)\eta(Y) + (\bar{\nabla}_Y\theta)(PZ)\eta(X)\\&= (\frac{c}{4} + 1)\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ)\}\\&\quad + \frac{c}{4}\{[v(X) - \varphi u(X)]g(FY, PZ) - [v(Y) - \varphi u(Y)]g(FX, PZ)\\&\quad + 2[v(PZ) - \varphi u(PZ)]\bar{g}(X, JY)\}.\end{aligned}$$

Taking $Y = \xi$ and $PZ = V$ to this and using (2.9) and (4.13), we have

$$\{\xi\varphi - 2\varphi\tau(\xi)\}B(X, V) + B(X, F\zeta) = \frac{3}{4}cu(X).$$

Taking $X = U - \varphi V$ to this equation and using $(4.12)_2$, we obtain $c = 0$. □

Definition 4.5. ([1]) A screen distribution $S(TM)$ is called *totally umbilical* if there exists a smooth function γ on a neighborhood $\mathcal{U}$ such that

$$C(X, PY) = \gamma g(X, PY). \tag{4.14}$$

Theorem 4.6. *Let M be a lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection subject such that ζ belongs to $S(TM)$. If $S(TM)$ is totally umbilical, then $c = 0$, i.e., $\bar{M}(c)$ is flat, and $\gamma = 0$, i.e., $S(TM)$ is totally geodesic*

Proof. From (2.11), (3.6) and (4.14), we see that $A_N X = \gamma PX$ and

$$B(X, U) = \gamma u(X) - \theta(V)\eta(X).$$

Replacing X by ξ, V, U and ζ to this by turns and using (2.9), we obtain

$$\theta(V) = 0, \quad B(V, U) = 0, \quad B(U, U) = \gamma, \quad B(U, \zeta) = 0, \tag{4.15}$$

$$B(X, U) = \gamma u(X). \tag{4.16}$$

Applying $\bar{\nabla}_X$ to $\theta(V) = 0$ and using (2.10), (3.4), (3.8) and $(4.15)_1$, we get

$$(\bar{\nabla}_X\theta)(V) = B(X, F\zeta) + u(X).$$

Taking $Y = F\zeta$ to (4.16), we get $B(U, F\zeta) = 0$. Replacing X by U to the last equation and using the fact that $B(U, F\zeta) = 0$, we obtain

$$(\bar{\nabla}_U\theta)(V) = 1. \tag{4.17}$$

Applying ∇_X to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (2.7), we obtain

$$(\nabla_X C)(Y, PZ) = (X\gamma)g(Y, PZ) + \gamma B(X, PZ)\eta(Y).$$

Substituting this equation and (4.14) into (4.6), we have

$$\begin{aligned}
&\{X\gamma - \gamma\tau(X) - [\frac{c}{4} + 1]\eta(X)\}g(Y, PZ) \\
&\quad - \{Y\gamma - \gamma\tau(Y) - [\frac{c}{4} + 1]\eta(Y)\}g(X, PZ) \\
&\quad + \gamma\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\} \\
&\quad - (\bar{\nabla}_X\theta)(PZ)\eta(Y) + (\bar{\nabla}_Y\theta)(PZ)\eta(X) \\
&= \frac{c}{4}\{v(X)g(FY, PZ) - v(Y)g(FX, PZ) + 2v(PZ)\bar{g}(X, JY)\}.
\end{aligned}$$

Replacing Y by ξ to this equation and using (2.9), (3.2) and (3.3), we have

$$\begin{aligned}
\gamma B(X, PZ) \;=\; &\{\xi\gamma - \gamma\tau(\xi) - \frac{c}{4} - 1\}g(X, PZ) \\
&+ (\bar{\nabla}_X\theta)(PZ) - (\bar{\nabla}_\xi\theta)(PZ)\eta(X) \\
&- \frac{c}{4}\{v(X)u(PZ) + 2u(X)v(PZ)\}.
\end{aligned} \tag{4.18}$$

Taking $X = U$ and $PZ = V$ to (4.18) and using $(4.15)_2$ and (4.17), we have

$$\xi\gamma - \gamma\tau(\xi) = \frac{3}{4}c. \tag{4.19}$$

Applying $\bar{\nabla}_X$ to $g(\zeta, \zeta) = 1$ and using the fact that $\bar{\nabla}$ is metric, we obtain

$$(\bar{\nabla}_X\theta)(\zeta) = 0. \tag{4.20}$$

Taking $X = U$ and $PZ = \zeta$ to (4.18) and using $(4.15)_{1,4}$, (4.19) and (4.20), we get $\theta(U) = 0$. As $\bar{g}(J\zeta, \zeta) = 0$, we see that $g(F\zeta, \zeta) = 0$. Thus

$$\theta(U) = 0, \qquad g(F\zeta, \zeta) = 0. \tag{4.21}$$

Applying $\bar{\nabla}_X$ to $\theta(U) = 0$ and using (3.7), (4.14) and (4.21), we obtain

$$(\bar{\nabla}_X\theta)(U) = \gamma g(X, F\zeta) + v(X).$$

Taking $X = V$ and $X = U$ to this equation by turns, we obtain

$$(\bar{\nabla}_V\theta)(U) = 1, \qquad (\bar{\nabla}_U\theta)(U) = 0. \tag{4.22}$$

Taking $X = V$ and $PZ = U$ to (4.18) and using $(4.15)_2$, (4.19) and $(4.22)_1$, we have $c = 0$. As $c = 0$, the equation (4.18) reduces

$$\gamma B(X, PY) = -g(X, PZ) + (\bar{\nabla}_X\theta)(PZ) - (\bar{\nabla}_\xi\theta)(PZ)\eta(X).$$

Taking $X = Z = U$ to this equation and using $(4.15)_3$ and $(4.22)_2$, we have $\gamma = 0$. Thus $S(TM)$ is totally geodesic. □

Theorem 4.7. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection ∇ such that $\tau = 0$ and ζ belongs to $S(TM)$. If U is parallel with respect to ∇, then $c = 0$.*

Proof. Taking the scalar product with V to (3.7) and using $\tau = 0$, we get

$$\theta(U)u(X) - \theta(V)v(X) = 0.$$

Taking $X = U$ and $X = V$ to this equation by turns, we have

$$\theta(U) = 0, \qquad \theta(V) = 0. \tag{4.23}$$

Taking the scalar product with ζ, $F\zeta$ and N to (3.7) by turns and using (4.23) and the fact that $\tau = 0$, we obtain

$$g(F(A_{_N}X), \zeta) = v(X), \quad g(A_{_N}X, \zeta) = \eta(X), \quad C(X, U) = 0. \tag{4.24}$$

Applying $\bar{\nabla}_X$ to $\theta(U) = 0$ and using (3.7) and $(4.24)_1$, we have

$$(\bar{\nabla}_X\theta)(U) = 0. \tag{4.25}$$

Applying ∇_Y to $(4.24)_3$ and using the fact that $\nabla_Y U = 0$, we have

$$(\nabla_X C)(Y, U) = 0.$$

Substituting this equation and $(4.24)_3$ into (4.6) with $PZ = U$ and using $(4.24)_2$ and (4.25), we have

$$\frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$ to this equation, we obtain $c = 0$. □

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