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COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF COMPLEX ORDER TYPE β

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Abstract. In the present work, we aim at determine the coefficient bounds for certain subclasses of convex functions of complex order, which are introduced here by means of a family of nonhomogeneous Cauchy-Euler differential equations.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of order $b(b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\})$ and type $\beta(0 \leq \beta < 1)$, that is $f(z) \in \mathcal{S}_b^*(\beta)$ if it satisfies the following

$$\Re e \left\{ 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > \beta, \quad (z \in \mathbb{U}, b \in \mathbb{C}^*) \tag{1.2}$$

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and is said to be convex of complex order $b(b \in \mathbb{C}^*)$ and type $\beta(0 \leq \beta < 1)$, denoted by $\mathcal{K}_b(\beta)$ if it satisfies the following

$$\Re e \left\{ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \right\} > \beta, \quad (z \in \mathbb{U}, b \in \mathbb{C}^*), \tag{1.3}$$

where $\Re e\{f(z)\}$ is the real part of $f(z)$. The classes $\mathcal{S}_b^*(\beta)$ and $\mathcal{S}_b(\beta)$ were defined by Frasin [1]. Note that $\mathcal{S}_b^*0) = \mathcal{S}_b^*$ and $\mathcal{K}_b(0) = \mathcal{K}_b$ defined by Nasr and Aouf [3] and Wiatrowski [5]. Also $\mathcal{S}_1^*(\beta) = \mathcal{S}^*(\beta)$ and $\mathcal{K}_1(\beta) = \mathcal{K}(\beta)$ which are, respectively, the class of starlike functions of order β and the class of convex functions of order β.

Let $\mathcal{M}(b, \lambda, \mu, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition

$$\Re e \left[1 + \frac{1}{b} \left(\frac{z[\lambda\mu z^3 f'''(z) + (\lambda - \mu)z^2 f''(z) + zf'(z)]'}{\lambda\mu z^3 f'''(z) + (\lambda - \mu)z^2 f''(z) + zf'(z)} - 1 \right) \right] > \beta, \tag{1.4}$$

where $0 \leq \mu \leq \lambda \leq 1$, $0 \leq \beta < 1, b \in \mathbb{C}^*$ and $z \in \mathbb{U}$. For $\mu = 0$, the class $\mathcal{M}(b, \lambda, \mu, \beta)$ reduces to the class introduced by Kamali [2].

Clearly, we have $\mathcal{M}(b, 0, 0, \beta) = \mathcal{K}_b(\beta)$. The main object of this paper is to derive some coefficient bounds for the class $\mathcal{M}(b, \lambda, \mu, \beta)$ also for functions in the subclass $\mathcal{J}(b, \lambda, \mu, \beta; m, \tau)$ of $\mathcal{A}$, which consists of functions $f(z) \in \mathcal{A}$ satisfying the following nonhomogeneous Cauchy-Euler differential equation

$$\begin{aligned} & z^m \frac{d^m w}{dz^m} + \binom{m}{1}(\tau + m - 1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots + \binom{m}{m} w \prod_{j=0}^{m-1}(\tau + j) \\ & = g(z) \prod_{j=0}^{m-1}(\tau + j + 1) \end{aligned} \tag{1.5}$$

$$(w = f(z); g(z) \in \mathcal{M}(b, \lambda, \mu, \beta); \tau \in \mathbb{R} \setminus (-\infty, -1]; m \in \mathbb{N}^* = \mathbb{N} \setminus \{1\} = \{2, 3, \dots\}).$$

2. Coefficient estimates for the functions class $\mathcal{M}(b, \lambda, \mu, \beta)$

Our first result given by Theorem 2.1 below:

Theorem 2.1. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{M}(b, \lambda, \mu, \beta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|b|(1 - \beta)]}{n![1 + (\lambda\mu(n - 2) + \lambda - \mu)(n - 1)]}, \quad n \in \mathbb{N}^*, \tag{2.1}$$

where $0 \leq \mu \leq \lambda \leq 1$; $0 \leq \beta < 1$ and $b \in \mathbb{C}^*$.

Proof. Let the function $f(z) \in \mathcal{A}$ be given by (1.1), and let the function $H(z)$ be defined by

$$H(z) = \lambda\mu z^3 f'''(z) + (\lambda - \mu)z^2 f''(z) + zf'(z). \tag{2.2}$$

Then the function $H(z)$ is analytic in $\mathbb{U}$ with $H(0) = H'(0) - 1 = 0$.
From (1.1) and (2.2) it is obvious that

$$H(z) = z + \sum_{k=2}^{\infty} S_k z^k, \quad z \in \mathbb{U},$$

where

$$S_k := [1 + (\lambda\mu(k-2) + \lambda - \mu)(k-1)]ka_k, \quad (k \in \mathbb{N}^*). \tag{2.3}$$

Now we define the function $q(z)$ by

$$q(z) = \frac{1 + \frac{1}{b}(\frac{zH'(z)}{H(z)} - 1) - \beta}{1 - \beta}. \tag{2.4}$$

Also, we assume that

$$q(z) = 1 + c_1 z + c_2 z^2 + \cdots . \tag{2.5}$$

So from (2.4) we obtain

$$1 + \frac{1}{b}\left(\frac{zH'(z)}{H(z)} - 1\right) - \beta = (1 - \beta)(1 + c_1 z + c_2 z^2 + \cdots), \tag{2.6}$$

or, equivalently,

$$zH'(z) - H(z) = H(z)b(1 - \beta)(c_1 z + c_2 z^2 + \cdots). \tag{2.7}$$

Using (2.7), we conclude that

$$\begin{aligned}
(2-1)S_2 &= b(1-\beta)c_1, \\
(3-1)S_3 &= b(1-\beta)[c_1 S_2 + c_2], \\
(4-1)S_4 &= b(1-\beta)[c_1 S_3 + c_2 S_2 + c_3], \\
&\vdots \\
(n-1)S_n &= b(1-\beta)[c_1 S_{n-1} + c_2 S_{n-2} + \cdots + c_{n-1}].
\end{aligned} \tag{2.8}$$

As $|c_n| \le 2$, $n = \{1, 2, 3, \cdots\}$, from (2.8) we have

$$\begin{aligned}
|S_2| &= |b(1-\beta)c_1| \\
&\le 2|b|(1-\beta),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
2|S_3| &= |b(1-\beta)[c_1 S_2 + c_2]| \\
&\le |b|(1-\beta)[2S_2 + 2] \\
&\le 2|b|(1-\beta)[1 + 2|b|(1-\beta)],
\end{aligned} \tag{2.10}$$

$$3|S_4| = |b(1-\beta)[c_1 S_3 + c_2 S_2 + c_3]| \tag{2.11}$$

or

$$\begin{aligned} 6|S_4| &\leq 2|b|(1-\beta)[S_3+S_2+1] \\ &\leq 2|b|(1-\beta)[1+2|b|(1-\beta)][2+2|b|(1-\beta)]. \end{aligned} \tag{2.12}$$

Using (2.9), (2.10) and (2.12), we get

$$|S_2| \leq \frac{\prod_j [j+2|b|(1-\beta)]}{(2-1)!}, j=0,$$

$$|S_3| \leq \frac{\prod_j [j+2|b|(1-\beta)]}{(3-1)!}, j=0,1,$$

$$|S_4| \leq \frac{\prod_j [j+2|b|(1-\beta)]}{(4-1)!}, j=0,1,2$$

and

$$|S_n| \leq \frac{\prod_{j=0}^{n-2} [j+2|b|(1-\beta)]}{(n-1)!}, j \in \mathbb{N}^*.$$

From (2.3), it is clear that

$$S_n := [1+(\lambda\mu(n-2)+\lambda-\mu)(n-1)]na_n, \quad (n \in \mathbb{N}^*) \tag{2.13}$$

implies

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j+2|b|(1-\beta)]}{n![1+(\lambda\mu(n-2)+\lambda-\mu)(n-1)]}.$$

□

Putting $\mu = \lambda = 1$, we get the following corollary.

Corollary 2.2. *Let the function $f(z) \in \mathcal{A}$ be given by* (1.1)*, and satisfies the condition*

$$\Re e\left[1+\frac{1}{b}\left(\frac{z[z^3 f'''(z) + +zf'(z)]'}{z^3 f'''(z) + +zf'(z)} - 1\right)\right] > \beta, \tag{2.14}$$

then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j+2|b|(1-\beta)]}{n!(n^2-3n+3)}, \quad n \in \mathbb{N}^*,$$

where $0 \leq \beta < 1$ and $b \in \mathbb{C}^*$.

Putting $\mu = \lambda = 0$, we get the following result obtained by Nasr and Aouf [4].

Corollary 2.3. *If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{K}_b$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j+2|b|]}{n!}.$$

3. Coefficient bounds for the class $\mathcal{J}(b, \lambda, \mu, \beta; m, \tau)$

Theorem 3.1. *Let the function* $f(z) \in \mathcal{A}$ *be given by* (1.1). *If* $f(z) \in \mathcal{J}(b, \lambda, \mu, \beta; m, \tau)$, *then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|b|(1-\beta)] \prod_{j=0}^{m-1}(\tau + j + 1)}{n![1 + (\lambda\mu(n-2) + \lambda - \mu)(n-1)] \prod_{j=0}^{m-1}(\tau + j + n)}, \quad m, n \in \mathbb{N}^*, \tag{3.1}$$

where $0 \leq \mu \leq \lambda \leq 1$; $0 \leq \beta < 1$, $b \in \mathbb{C}^*$ *and* $\tau \in \mathbb{R} \setminus (-\infty, -1]$.

Proof. Let the function $f(z) \in \mathcal{A}$ be given by (1.1), also let the function $g(z) = z + \sum_{k=2}^{\infty} d_k z^k \in \mathcal{M}(b, \lambda, \mu, \beta)$, implies

$$|d_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|b|(1-\beta)]}{n![1 + (\lambda\mu(n-2) + \lambda - \mu)(n-1)]}. \tag{3.2}$$

From (1.5), we have

$$a_n = \left(\frac{\prod_{j=0}^{m-1}(\tau + j + 1)}{\prod_{j=0}^{m-1}(\tau + j + n)} \right) d_n, \quad m, n \in \mathbb{N}^*, \ \ \tau \in \mathbb{R} \setminus (-\infty, -1].$$

Using (3.2), we get

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|b|(1-\beta)] \prod_{j=0}^{m-1}(\tau + j + 1)}{n![1 + (\lambda\mu(n-2) + \lambda - \mu)(n-1)] \prod_{j=0}^{m-1}(\tau + j + n)}.$$

□

Putting $\mu = \lambda = 1$, we get the following corollary.

Corollary 3.2. *Let the function* $f(z) \in \mathcal{A}$ *be given by* (1.1). *If* $f(z)$ *satisfies the equation* (1.5) *and* $g(z) = z + \sum_{k=2}^{\infty} d_k z^k$ *satisfies the condition* (2.14), *then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|b|(1-\beta)] \prod_{j=0}^{m-1}(\tau + j + 1)}{n!(n^2 - 3n + 3) \prod_{j=0}^{m-1}(\tau + j + n)}, \quad m, n \in \mathbb{N}^*,$$

where $0 \leq \beta < 1$, $b \in \mathbb{C}^*$ *and* $\tau \in \mathbb{R} \setminus (-\infty, -1]$.

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