Nonlinear Functional Analysis and Applications
Vol. 23, No. 4 (2018), pp. 661-672
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

L-FUZZY MAPPINGS AND COMMON FIXED POINT THEOREMS

Alrazi Abdeljabbar[1] and Jamshaid Ahmad[2]

[1]Department of Mathematics, Khalifa University of Science and Technology,
The Petroleum Institute, P.O Box 2533, Abu Dhabi, UAE
e-mail: aabdeljabbar@pi.ac.ae

[2]Department of Mathematics, University of Jeddah,
P.O.Box 80327, Jeddah 21589, Saudi Arabia
e-mail: jkhan@uj.edu.sa, jamshaid_jasim@yahoo.com

Abstract. The aim of this paper is to develop a new common fixed point theorem of L-fuzzy mappings under generalized Θ-contraction in the context of complete metric space. We also provide an example to show the significance of the investigation of this paper.

1. Introduction and Preliminaries

Solving real-world problems becomes apparently easier with the introduction of fuzzy set theory in 1965 by Zadeh [32], as it helps in making the description of vagueness and imprecision clear and more precise. Later in 1967, Goguen [16] extended this idea to L-fuzzy set theory by replacing the interval $[0, 1]$. There are basically two understandings of the meaning of L, one is when L is a complete lattice equipped with a multiplication $*$ operator satisfying certain conditions as shown in the initial paper [16] and the second understanding of the meaning of L is that L is a completely distributive complete lattice with an order-reversing involution.

In 2014, Rashid et al. [25] introduced the notion of β_{F_L}-admissible for a pair of L-fuzzy mappings and utilized it to proved a common L-fuzzy fixed point theorem. For more details on this direction, we refer the reader to [1, 9, 26].

[0]Received February 21, 2018. Revised May 7, 2018.
[0]2010 Mathematics Subject Classification: 46S40, 47H10.
[0]Keywords: L-fuzzy mappings, common fixed point, Θ-contraction.
[0]Corresponding author: A. Abdeljabbar(aabdeljabbar@pi.ac.ae).

Let (X, d) be a metric space and $CB(X)$ be the family of nonempty, closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where

$$d(x,A) = \inf_{y\in A} d(x,y).$$

Definition 1.1. ([16]) A partially ordered set $(L, \precsim_L)$ is called

(i) a lattice, if $a \vee b \in L$, $a \wedge b \in L$ for any $a, b \in L$.
(ii) a complete lattice, if $\vee A \in L$, $\wedge A \in L$ for any $A \subseteq L$.
(iii) distributive if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.

Definition 1.2. ([16]) Let L be a lattice with top element 1_L and bottom element 0_L and let $a, b \in L$. Then b is called a complement of a, if $a \vee b = 1_L$, and $a \wedge b = 0_L$.

If $a \in L$ has a complement element, then it is unique. It is denoted by $\acute{a}$.

Definition 1.3. ([16]) A $L-$fuzzy set A on a nonempty set X is a function $A : X \to L$, where L is complete distributive lattice with 1_L and 0_L.

Remark 1.4. The class of $L-$fuzzy sets is larger than the class of fuzzy sets as an $L-$fuzzy set is a fuzzy set if $L = [0, 1]$.

The α_L-level set of $L-$fuzzy set A, is denoted by A_{α_L}, and is defined as follows

$$A_{\alpha_L} = \{x : \alpha_L \precsim_L A(x)\} \text{ if } \alpha_L \in L\backslash\{0_L\},$$
$$A_{0_L} = \overline{\{x : 0_L \precsim_L A(x)\}}.$$

Here $cl(B)$ and $\Im_L(Y)$ denote the closure of the set B and L-fuzzy set on Y, respectively.

We denote and define the characteristic function χ_{L_A} of a L-fuzzy set A as follows:

$$\chi_{L_A} := \begin{cases} 0_L & \text{if } x \notin A \\ 1_L & \text{if } x \in A \end{cases}.$$

Definition 1.5. Let X be an arbitrary set and Y be a metric space. A mapping T is called $L-$fuzzy mapping if T is a mapping from X into $\Im_L(Y)$. A $L-$fuzzy mapping T is a $L-$fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 1.6. Let (X, d) be a metric space and S, T be $L-$fuzzy mappings from X into $\Im_L(X)$. A point $z \in X$ is called a $L-$fuzzy fixed point of T if $z \in [Tz]_{\alpha_L}$, where $\alpha_L \in L\backslash\{0_L\}$. The point $z \in X$ is called a common $L-$fuzzy fixed point of S and T if $z \in [Sz]_{\alpha_L} \cap [Tz]_{\alpha_L}$. When $\alpha_L = 1_L$, it is called a common fixed point of $L-$fuzzy mappings.

Very recently, Jleli and Samet [20] introduced a new type of contraction called Θ-contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.7. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

(Θ_1) Θ is nondecreasing;
(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty} \alpha_n = 0$;
(Θ_3) there exists $0 < h < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^h} = l$.

A mapping $S : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(Sx, Sy) > 0 \Longrightarrow \Theta(d(Sx, Sy)) \leq [\Theta(d(x, y))]^k. \tag{1.1}$$

Theorem 1.8. ([20]) *Let (X, d) be a complete metric space and $S : X \to X$ be a Θ-contraction, Then S has a unique fixed point.*

They showed that any Banach contraction is a particular case of Θ-contraction while there are Θ-contractions which are not Banach contractions. To be consistent with Samet *et al.* [20], we denote by the Ψ set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions (Θ_1)-(Θ_3).

Later on Altune *et al.* [17] modified the above definitions by adding a general condition (Θ_4) which is given in this way:

(Θ_4) $\Theta(\inf A) = \inf \Theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

Following Altune *et al.* [17], we represent the set of all continuous functions $\Theta : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(\Theta_1) - (\Theta_4)$ conditions by Ω.
For more details on Θ-contraction, we refer the reader to [3, 4, 19, 21, 23, 30].

In this paper, we use a generalized Θ-contraction to obtain common fixed points for L- fuzzy mappings in the setting of metric spaces.

For the sake of convenience, we first state some known results for subsequent use in the next section.

Lemma 1.9. *Let (X, d) be a metric space and $A, B \in CB(X)$. Then for each $a \in A$, we have*

$$d(a, B) \leq H(A, B).$$

2. Main Results

In this way, we state and prove a common fixed point theorem for L-fuzzy mappings.

Theorem 2.1. *Let (X,d) be a complete metric space, S,T be L-fuzzy mappings from X into $\Im_L(X)$, and for each $\alpha_L \in L\backslash\{0_L\}$, $[Sx]_{\alpha_L(x)}$, $[Ty]_{\alpha_L(y)}$ be nonempty closed bounded subsets of X. If there exist some $\Theta \in \Omega$ and $k \in (0,1)$ such that*

$$\Theta\left(H\left([Sx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}\right)\right) \leq \Theta(M(x,y))^k \tag{2.1}$$

for all $x,y \in X$ with $H\left([Sx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}\right) > 0$, where

$$\begin{aligned} M(x,y) = \max\Big\{&d(x,y), d\big(x,[Sx]_{\alpha_L(x)}\big), d\big(y,[Ty]_{\alpha_L(y)}\big),\\ &\frac{1}{2}\Big[d(x,[Ty]_{\alpha_L(y)} + d(y,[Sx]_{\alpha_L(x)})\Big]\Big\}. \end{aligned} \tag{2.2}$$

Then S and T have a common L-fuzzy fixed point.

Proof. Let x_0 be an arbitrary point in X, then by hypotheses there exists $\alpha_L(x_0) \in L\backslash\{0_L\}$ such that $[Sx_0]_{\alpha_L(x_0)}$ is a nonempty closed bounded subset of X and let $x_1 \in [Sx_0]_{\alpha_L(x_0)}$. For this x_1, there exists $\alpha_L(x_1) \in L\backslash\{0_L\}$ such that $[Tx_1]_{\alpha_L(x_1)}$ is a nonempty, closed and bounded subset of X. By Lemma 1.9, (Θ_1) and (2.1), we have

$$\begin{aligned} &\Theta(d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\\ &\leq \Theta\left(H\left([Sx_0]_{\alpha_L(x_0)}, [Tx_1]_{\alpha_L(x_1)}\right)\right)\\ &\leq \Theta(M(x_0,x_1))^k\\ &= \left[\Theta\left(\max\left\{\begin{array}{c} d(x_0,x_1), d\left(x_0,[Sx_0]_{\alpha_L(x_0)}\right), d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right),\\ \frac{1}{2}[d\left(x_0,[Tx_1]_{\alpha_L(x_1)}\right) + d\left(x_1,[Sx_0]_{\alpha_L(x_0)}\right)] \end{array}\right\}\right)\right]^k\\ &= \left[\Theta\left(\max\left\{\begin{array}{c} d(x_0,x_1), d\left(x_0,[Sx_0]_{\alpha_L(x_0)}\right), d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right),\\ \frac{1}{2}d\left(x_0,[Tx_1]_{\alpha_L(x_1)}\right) \end{array}\right\}\right)\right]^k. \end{aligned} \tag{2.3}$$

By triangle inequality and (Θ_1), we get

$$\begin{aligned}
&\Theta(d\left(x_1, [Tx_1]_{\alpha_L(x_1)}\right) \\
&\leq \left[\Theta\left(\max\left\{\begin{array}{c} d(x_0,x_1), d\left(x_0,[Sx_0]_{\alpha_L(x_0)}\right), d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right), \\ \frac{1}{2}\left(d(x_0,x_1)+d(x_1,[Tx_1]_{\alpha_L(x_1)})\right) \end{array}\right\}\right)\right]^k \\
&\leq \left[\Theta\left(\max\left\{d(x_0,x_1), d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\right\}\right)\right]^k.
\end{aligned}$$

If $\max\left\{d(x_0,x_1), d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\right\} = d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)$. Then from (2.3), we get

$$\begin{aligned}
\Theta\left(d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\right) &\leq \left[\Theta\left(d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\right)\right]^k \\
&\leq \left[\Theta\left(d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\right)\right],
\end{aligned}$$

which is a contradiction. So, $\max\left\{d(x_0,x_1), d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\right\} = d(x_0,x_1)$. Then

$$\Theta\left(d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right)\right) \leq [\Theta(d(x_0,x_1)]^k. \tag{2.4}$$

From (Θ_4), we know that

$$\Theta\left(d\left(x_1,[Tx_1]_{\alpha_T(x_1)}\right)\right) = \inf_{y\in[Tx_1]_{\alpha_L(x_1)}} \Theta(d(x_1,y)).$$

Thus, from (2.4), we get

$$\inf_{y\in[Tx_1]_{\alpha_L(x_1)}} \Theta(d(x_1,y)) \leq [\Theta(d(x_0,x_1)]^k. \tag{2.5}$$

Then, from (2.5), there exists $x_2 \in [Tx_1]_{\alpha_L(x_1)}$ such that

$$\Theta(d(x_1,x_2)) \leq [\Theta(d(x_0,x_1)]^k. \tag{2.6}$$

For this x_2, there exists $\alpha_L(x_2) \in L\backslash\{0_L\}$ such that $[Sx_2]_{\alpha_L(x_2)}$ is a nonempty closed bounded subset of X. By Lemma 1.9, (Θ_1) and (2.1), we have

$$\begin{aligned}
&\Theta\left(d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right) \\
&\leq \Theta(H\left([Tx_1]_{\alpha_L(x_1)}, [Sx_2]_{\alpha_L(x_2)}\right) = \Theta(H\left([Sx_2]_{\alpha_L(x_2)}, [Tx_1]_{\alpha_L(x_1)}\right) \\
&\leq \Theta(M(x_2, x_1))^k \\
&= \left[\Theta\left(\max\left\{\begin{array}{c} d(x_2, x_1), d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right), d\left(x_1, [Tx_1]_{\alpha_L(x_1)}\right), \\ \frac{1}{2}[d\left(x_2, [Tx_1]_{\alpha_L(x_1)}\right) + d\left(x_1, [Sx_2]_{\alpha_L(x_2)}\right)] \end{array}\right\}\right)\right]^k \\
&= \left[\Theta\left(\max\left\{\begin{array}{c} d(x_2, x_1), d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right), d(x_1, x_2), \\ \frac{1}{2}d\left(x_1, [Sx_2]_{\alpha_L(x_2)}\right) \end{array}\right\}\right)\right]^k.
\end{aligned}$$

By triangle inequality and (Θ_1), we get

$$\begin{aligned}
&\Theta\left(d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right) \\
&\leq \left[\Theta\left(\max\left\{\begin{array}{c} d(x_1, x_2), d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right), \\ \frac{1}{2}\left(d(x_1, x_2) + d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right), \end{array}\right\}\right)\right]^k,
\end{aligned}$$

which further implies that

$$\Theta\left(d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right) \leq \left[\Theta\left(\max\left\{d(x_1, x_2), d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right\}\right)\right]^k. \quad (2.7)$$

If $\max\left\{d(x_1, x_2), d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right\} = d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)$. Then from (2.7), we get

$$\Theta\left[d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right] \leq \Theta\left[d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right]^k \leq \Theta\left[d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right]$$

which is a contradiction. So, $\max\left\{d(x_1, x_2), d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right\} = d(x_1, x_2)$. Then

$$\Theta\left[d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right] \leq \Theta\left[d(x_1, x_2)\right]^k. \quad (2.8)$$

From (Θ_4), we know that

$$\Theta\left[d\left(x_2, [Sx_2]_{\alpha_L(x_2)}\right)\right] = \inf_{y_1 \in [Sx_2]_{\alpha_L(x_2)}} \Theta(d(x_2, y_1)).$$

Thus

$$\inf_{y_1 \in [Sx_2]_{\alpha_L(x_2)}} \Theta(d(x_2, y_1)) \leq \Theta\left[d(x_1, x_2)\right]^k. \quad (2.9)$$

Then, from (2.9), there exists $x_3 \in [Sx_2]_{\alpha_L(x_2)}$ such that

$$\Theta(d(x_2, x_3)) \leq [\Theta(d(x_1, x_2)]^k. \tag{2.10}$$

So, continuing recursively, we obtain a sequence $\{x_n\}$ in X such that $x_{2n+1} \in [Sx_{2n}]_{\alpha_L(x_{2n})}$ and $x_{2n+2} \in [Tx_{2n+1}]_{\alpha_L(x_{2n+1})}$, and

$$\Theta(d(x_{2n+1}, x_{2n+2})) \leq [\Theta(d(x_{2n}, x_{2n+1})]^k \tag{2.11}$$

and

$$\Theta(d(x_{2n+2}, x_{2n+3})) \leq [\Theta(d(x_{2n+1}, x_{2n+2})]^k \tag{2.12}$$

for all $n \in \mathbb{N}$. From (2.11) and (2.12), we have

$$\Theta(d(x_n, x_{n+1})) \leq [\Theta(d(x_{n-1}, x_n)]^k, \tag{2.13}$$

which further implies that

$$\begin{aligned} \Theta(d(x_n, x_{n+1})) &\leq [\Theta(d(x_{n-1}, x_n)]^k \\ &\leq [\Theta(d(x_{n-2}, x_{n-1})]^{k^2} \\ &\;\;\vdots \\ &\leq [\Theta(d(x_0, x_1)]^{k^n}. \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}$. Since $\Theta \in \Omega$, so by taking limit as $n \to \infty$ in (2.14) we have,

$$\lim_{n\to\infty} \Theta(d(x_n, x_{n+1})) = 1 \tag{2.15}$$

which implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.16}$$

by (Θ_2). From the condition (Θ_3), there exist $0 < r < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} = l. \tag{2.17}$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} - l\right| \leq B$$

for all $n > n_0$. This implies that

$$\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} \geq l - B = \frac{l}{2} = B$$

for all $n > n_0$. Then

$$nd(x_n, x_{n+1})^r \leq An[\Theta(d(x_n, x_{n+1})) - 1] \tag{2.18}$$

for all $n > n_0$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$B \le \frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r}$$

for all $n > n_0$. This implies that

$$nd(x_n, x_{n+1})^r \le An[\Theta(d(x_n, x_{n+1})) - 1]$$

for all $n > n_0$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$nd(x_n, x_{n+1})^r \le An[\Theta(d(x_n, x_{n+1})) - 1] \tag{2.19}$$

for all $n > n_0$. Thus by (2.14) and (2.19), we get

$$nd(x_n, x_{n+1})^r \le An([(\Theta d(x_0, x_1))]^{r^n} - 1). \tag{2.20}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} nd(x_n, x_{n+1})^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \le \frac{1}{n^{1/r}} \tag{2.21}$$

for all $n > n_1$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have,

$$d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \le \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}. \tag{2.22}$$

Since, $0 < r < 1$, $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ is convergent. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence in (X, d). The completeness of (X, d) ensures that there exists $u \in X$ such that, $\lim_{n\to\infty} x_n = u$. Now, we prove that $u \in [Tu]_{\alpha_L(u)}$. We suppose on the contrary that $u \notin [Tu]_{\alpha_L(u)}$, then there exist a $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{2n_k+1}, [Tu]_{\alpha_L(u)}) > 0$ for all $n_k \ge n_0$. Since $d(x_{2n_k+1}, [Tu]_{\alpha_L(u)}) > 0$

for all $n_k \geq n_0$, by (Θ_1), we have

$$\begin{aligned}
&\Theta\left[d(x_{2n_k+1}, [Tu]_{\alpha_L(u)})\right] \\
&\leq \Theta\left[H([Sx_{2n_k}]_{\alpha_L(x_{2n_k})}, [Tu]_{\alpha_L(u)})\right] \\
&\leq [\Theta(M(x_{2n_k}, u))]^k \\
&= \left[\Theta\left(\max\left\{\begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, [Sx_{2n_k}]_{\alpha_L(x_{2n_k})}\right), d\left(u, [Tu]_{\alpha_L(u)}\right), \\ \frac{1}{2}[d\left(x_{2n_k}, [Tu]_{\alpha_L(u)}\right) + d\left(u, [Sx_{2n_k}]_{\alpha_L(x_{2n_k})}\right)] \end{array}\right\}\right)\right]^k \\
&\leq \left[\Theta\left(\max\left\{\begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, x_{2n_k+1}\right), d\left(u, [Tu]_{\alpha_L(u)}\right), \\ \frac{1}{2}[d\left(x_{2n_k}, [Tu]_{\alpha_L(u)}\right) + d\left(u, x_{2n_k+1}\right)] \end{array}\right\}\right)\right]^k.
\end{aligned}$$

Letting $n \to \infty$, in above inequality and using the continuity of Θ, we have

$$\Theta\left[d(u, [Tu]_{\alpha_L(u)})\right] \leq \left[\Theta(d(u, [Tu]_{\alpha_L(u)}))\right]^k$$

which is a conradiction because $k \in (0, 1)$. Hence $u \in [Tu]_{\alpha_L(u)}$. Similarly, we can easily prove that $u \in [Su]_{\alpha_L(u)}$. Thus $u \in [Su]_{\alpha_L(u)} \cap [Tu]_{\alpha_L(u)}$. □

The following result is a direct consequence of Theorem 2.1.

Theorem 2.2. *Let (X, d) be a complete metric space, S be an L-fuzzy mapping from X into $\Im_L(X)$, and for each $\alpha_L \in L\backslash\{0_L\}$, $[Sx]_{\alpha_L(x)}$, $[Sy]_{\alpha_L(y)}$ are nonempty closed bounded subsets of X. If there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta\left(H\left([Sx]_{\alpha_L(x)}, [Sy]_{\alpha_L(y)}\right)\right) \leq \Theta(M(x, y))^k$$

for all $x, y \in X$ with $H\Big([Sx]_{\alpha_L(x)}, [Sy]_{\alpha_L(y)}\Big) > 0$, where

$$\begin{aligned}
M(x, y) = \max\Big\{&d(x, y), d\big(x, [Sx]_{\alpha_L(x)}\big), d\big(y, [Sy]_{\alpha_L(y)}\big), \\
&\frac{1}{2}\Big[d(x, [Sy]_{\alpha_L(y)} + d(y, [Sx]_{\alpha_L(x)})\Big]\Big\}.
\end{aligned}$$

Then S has an L-fuzzy fixed point.

Corollary 2.3. *Let (X, d) be a complete metric space, S, T be fuzzy mappings from X into $\Im(X)$, and for each $\alpha(x) \in (0, 1]$, $[Sx]_{\alpha(x)}$, $[Ty]_{\alpha(y)}$ are nonempty closed bounded subsets of X. If there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta\left(H\left([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}\right)\right) \leq \Theta(M(x, y))^k$$

for all $x, y \in X$ with $H\left([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}\right) > 0$, where

$$M(x,y) = \max\Big\{d(x,y), d\big(x,[Sx]_{\alpha(x)}\big), d\big(y,[Ty]_{\alpha(y)}\big), \frac{1}{2}\Big[d(x,[Ty]_{\alpha(y)} + d(y,[Sx]_{\alpha(x)})\Big]\Big\}.$$

Then S and T have a common fuzzy fixed point.

Proof. Consider an L-fuzzy mapping $A : X \to \Im_L(X)$ defined by

$$Ax = \chi_{L_{S(x)}}.$$

Then for $\alpha_L \in L \backslash \{0_L\}$, we have

$$[Ax]_{\alpha_L(x)} = Sx.$$

Hence, by Theorem 2.1 we follow the result. □

Example 2.4. Let $X = [0,1]$, $d(x,y) = |x-y|$, for $x, y \in X$. Then (X,d) is a complete metric space. Let $L = \{\eta, \omega, \tau, \kappa\}$ with $\eta \preceq_L \omega \preceq_L \kappa$ and $\eta \preceq_L \tau \preceq_L \kappa$, where ω and τ are not comparable. Then $(L, \preceq_L)$ is a complete distributive lattice. Define a pair of mappings $S, T : X \to \Im_L(X)$ as follows:

$$S(x)(t) = \begin{cases} \kappa \text{ if } 0 \le t \le \frac{x}{6} \\ \omega \text{ if } \frac{x}{6} < t \le \frac{x}{3} \\ \tau \text{ if } \frac{x}{3} < t \le \frac{x}{2} \\ \eta \text{ if } \frac{x}{2} < t \le 1 \end{cases},$$

$$T(x)(t) = \begin{cases} \kappa \text{ if } 0 \le t \le \frac{x}{12} \\ \eta \text{ if } \frac{x}{12} < t \le \frac{x}{8} \\ \omega \text{ if } \frac{x}{8} < t \le \frac{x}{4} \\ \tau \text{ if } \frac{x}{4} < t \le 1 \end{cases}.$$

Let $\Theta(t) = e^{\sqrt{t}} \in \Omega$ for $t > 0$. And for all $x \in X$, there exists $\alpha_L(x) = \kappa$, such that

$$[Sx]_{\alpha_L(x)} = \left[0, \frac{x}{6}\right], \qquad [Tx]_{\alpha_L(x)} = \left[0, \frac{x}{12}\right].$$

and all conditions of Theorem 2.1 are satisfied. And 0 is a common fixed point of S and T.

Acknowledgments: This article was funded by the Deanship of Scientific Research (DSR), Khalifa University of Science and Technology, UAE. Therefore, the first author acknowledges with thanks DSR, Khalifa University of Science and Technology for financial support.

References

[1] M.S. Abdullahi and A. Azam, *L-fuzzy Fixed Point Theorems for L-fuzzy Mappings via β_{F_L}-admissible with Applications*, J. of Uncertainty Anal. and Appl., 2017 **5**:2

[2] H. Adibi, Y.J. Cho, D. O'Regan and R. Saadati, *Common fixed point theorems in L-fuzzy metric spaces*, Appl. Math. Comput. **182** (2006), 820–828.

[3] A. Ahmad, A. Al-Rawashdeh and A. Azam, *Fixed point results for $\{\alpha, \xi\}$-expansive locally contractive mappings*, J. of Inequ. and Appl., 2014:**364** (2014).

[4] A. Al-Rawashdeh and J. Ahmad, *Common Fixed Point Theorems for JS- Contractions*, Bull. of Math. Anal. and Appl., **8**(4) (2016), 12–22.

[5] SC. Arora and V. Sharma, *Fixed points for fuzzy mappings*, Fuzzy Sets Syst., **110** (2000), 127–130.

[6] A. Azam and I. Beg, *Common fixed points of fuzzy maps*, Math. Comput. Model., **49** (2009), 1331–1336.

[7] A. Azam, M. Arshad and P. Vetro, *On a pair of fuzzy φ-contractive mappings*, Math. Comput. Model. **52** (2010), 207–214.

[8] A. Azam, *Fuzzy Fixed Points of Fuzzy Mappings via a Rational Inequality*, Hacettepe J. of Math. and Statis., **40**(3) (2011), 421–431.

[9] A. Azam, N. Mahmood, M. Rashid and M. Pavlović, *L-fuzzy fixed points in cone metric spaces*, J. Adv. Math. Stud., **9**(10) (2016), 121–131.

[10] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fundam. Math. **3** (1922), 133–181.

[11] D. Butnariu, *Fixed point for fuzzy mapping*, Fuzzy Sets and Syst., **7** (1982), 191–207.

[12] R.K. Bose and D. Sahani, *Fuzzy mappings and fixed point theorems*, Fuzzy Sets and Syst., **21** (1987), 53–58.

[13] S.S. Chang, Y.J. Cho, B.S. Lee, J.S. Jung and S.M. Kang, *Coincidence point and minimization theorems in fuzzy metric spaces*, Fuzzy Sets Syst., **88** (1997) 119–128.

[14] Y.J. Cho and N. Petrot, *Existence theorems for fixed fuzzy points with closed α-cut sets in complete metric spaces*, Commun. Korean Math. Soc., **26**(1) (2011), 115–124.

[15] G. Durmaz, *Some theorems for new type multivalued contractive maps on metric space*, Turkish J. of Math. DOI: 10.3906/mat-1510-75.

[16] J.A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl., **18** (1967), 145–174.

[17] HA. Hancer, G. Minak and I. Altun, *On a broad category of multivalued weakly Picard operators*, Fixed Point Theory, **18**(1) (2017), 229–236.

[18] S. Heilpern, *Fuzzy mappings and fixed point theorem*, J. Math. Anal. Appl., **83**(2) (1981), 566–569.

[19] N. Hussain, V. Parvaneh, B. Samet and C. Vetro, *Some fixed point theorems for generalized contractive mappings in complete metric spaces*, Fixed Point Theory and Appl., 2015:**185** (2015).

[20] M. Jleli and B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl. 2014: **38** (2014).

[21] Z. Li and S. Jiang, *Fixed point theorems of JS-quasi-contractions,* Fixed Point Theory and Appl., 2016:**40** (2016).

[22] Jr. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–478.

[23] W. Onsod , T. Saleewong, J. Ahmad, A. E. Al-Mazrooei and P. Kumam, *Fixed points of a Θ-contraction on metric spaces with a graph*, Commun. Nonlinear Anal., **2** (2016), 139–149.

[24] D. Qiu and L. Shu, *Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings*, Information Sciences **178** (2008), 3595–3604.

[25] M. Rashid, A. Azam and N. Mehmood, *L-fuzzy fixed points theorems for L-fuzzy mappings via β_{F_L}-admissible pair*, Sci. World J., **2014** (2014), 1–8.

[26] M. Rashid, M.A. Kutbi and A. Azam, *Coincidence theorems via alpha cuts of L-fuzzy sets with applications*, Fixed Point Theory Appl., **212** (2014), 1–16.

[27] R.A. Rashwan and M.A. Ahmad, *Common fixed point theorems for fuzzy mappings*, Arch. Math. (Brno) **38**(3) (2002), 219–226.

[28] R. Saadati, S.M. Vaezpour and Y.J. Cho, *Quicksort algorithm: Application of a fixed point theorem in intuitionistic fuzzy quasi-metric spaces at a domain of words*, J. Comput. Appl. Math., **228** (2009), 219–225.

[29] Z. Shi-sheng, *Fixed point theorems for fuzzy mappings (II)*, Appl. Math. Mech., **7**(2) (1986), 147–152.

[30] F. Vetro, *A generalization of Nadler fixed point theorem*, Carpathian J. Math., **31** (2015), 403–410.

[31] M.D. Weiss, *Fixed points and induced fuzzy topologies for fuzzy sets*, J. Math. Anal. Appl., **50** (1975), 142–150.

[32] LA. Zadeh, *Fuzzy sets*, Inf. Control., **8**(3) (1965), 338–353.