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FIXED POINT THEOREMS FOR A PAIR OF NON-SELF MAPPINGS UNDER WEAKLY CONTRACTIVE MAPS IN METRICALLY CONVEX SPACES

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Abstract. Using the idea of weak contractive condition due to Rhoades [17], we prove some coincidence and common fixed point theorems for a pair of non-self mappings which generalize earlier results due to Khan and Imdad [15], Rhoades [17], Alber and Guerre-Delabriere [2], Beg and Abbas [5], Pant [16] and others. An illustrative example is discussed besides an application.

1. Introduction

In 1997, Alber and Guerre-Delabriere [2] initiated the study of weakly contractive map and utilized the same to certain fixed point results in Hilbert spaces. Rhoades [17] extended some of their results in Banach spaces under similar setting.

Recently, Khan and Imdad [15] utilized this new concept of weakly contractive map in non-self setting of Rhoades type (e.g. [17]) and proved a fixed point theorem in complete metric spaces.

The aim of this paper is to extend and generalize a fixed point theorem due to Khan and Imdad [15] to a pair of coincidentally commuting mappings as well as weakly compatible mappings which either partially or completely generalized the corresponding results due to Khan and Imdad [15], Rhoades [17], Alber and Guerre-Delabriere [2], Beg and Abbas [5], Pant [16] and others.

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Here for the sake of completeness, we state the result due to Khan and Imdad [15] which runs as follows:

Theorem 1.1. *Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T : K \to X$ satisfying:*

(i) *for each $x \in \delta K$ (The boundary of K), $Tx \in K$, and*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function with $\phi(t) = 0$ for $t = 0$. Then T has a unique fixed point in K.

Before proving our results, we collect the relevant definitions for our future use.

Definition 1.2. Let (X, d) be a metric space and K be a nonempty subset of X. Let $F, T : K \to X$ mappings, F is said to be generalized T weakly contractive on K, if $Fx, Tx \in K$ and

$$d(Fx, Fy) \leq d(Tx, Ty) - \phi(d(Tx, Ty))$$

for all $x, y \in K$ and $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function with $\phi(t) = 0$ for $t = 0$.

Definition 1.3. A pair of non-self mapping (F, T) defined on a nonempty subset K of a metric space (X, d) is said to be coincidentally commuting if $Tx, Fx \in K$ and $Tx = Fx \Rightarrow FTx = TFx$.

Definition 1.4. A pair of non-self mapping (F, T) defined on a nonempty subset K of a metric space (X, d) is said to be weakly compatible if for every sequence $\{x_n\}$ in K and from the relation $\lim_{n\to\infty} d(Fx_n, Tx_n) = 0$ and $Tx_n \in K$ $(n \in N)$, it follows that $\lim_{n\to\infty} d(Ty_n, FTx_n) = 0$ for every $y_n \in K$ with $y_n = Fx_n \in K, n \in N$.

Definition 1.5. A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Definition 1.6. Let K be a non-empty subset of a Banach Space X. Let F be a weakly contractive mapping with respect to T, $TK \subset FK$, and FK is a convex subset of X. Define a sequence $\{y_n\}$ in FK as:

$$y_n = T(x_{n+1}) = (1 - \alpha_n)Tx_n + \alpha_n Fx_n, n \geq 0,$$

where $0 \leq \alpha_n \leq 1$ for each n. The sequence is called modified Mann iterative scheme for non-self mappings.

2. Main Result

Our main result runs as follows:

Theorem 2.1. *Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $F, T : K \to X$ such that F is a generalized T weakly contractive mapping of K into X and*

(ii) $\delta K \subset TK, FK \cap K \subset TK$,
(iii) $Tx \in \delta K \Rightarrow Fx \in K$ *and*
(iv) TK *is closed in* X.

Then there exists coincidence point in K. Moreover, if (F, T) is coincidentally commuting then the coincidence point of K remains a unique common fixed point of F and T.

Proof. First, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x \in \delta K$. Then (due to $\delta K \subseteq TK$) there exists a point $x_0 \in K$ such that $x = Tx_0$. Since $Tx_0 \in \delta K$ and $Tx \in \delta K \Rightarrow Fx \in K$, we conclude that $Fx_0 \in FK \cap K \subset TK$. Let $x_1 \in K$ be such that $y_1 = Tx_1 = Fx_0 \in K$. Let $y_2 = Fx_1$. Suppose $y_2 \in K$, then $y_2 \in FK \cap K \subseteq TK$, which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Suppose $y_2 \notin K$. Then there exists a point $p \in \delta K$ such that

$$d(Tx_1, p) + d(p, y_2) = d(Tx_1, y_2). \tag{2.1}$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ with $p = Tx_2$ so that the equation (2.1) becomes

$$d(Tx_1, Tx_2) + d(Tx_2, y_2) = d(Tx_1, y_2).$$

Let $y_3 = Fx_2$. Thus, repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(v) $y_{n+1} = Fx_n$,
(vi) $y_n \in K \Rightarrow y_n = Tx_n$,
(vii) $y_n \notin K \Rightarrow Tx_n \in \delta K$, and

$$d(Tx_{n-1}, Tx_n) + d(Tx_n, y_n) = d(Tx_{n-1}, y_n).$$

We denote

$$P = \{Tx_i \in \{Tx_n\} : Tx_i = y_i\}$$

and

$$Q = \{Tx_i \in \{Tx_n\} : Tx_i \neq y_i\}.$$

Obviously, two consecutive terms cannot lie in Q. Now, we distinguish the following three cases:

Case 1. If Tx_n and $Tx_{n+1} \in P$, then, by using monotone property of ϕ,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(y_n, y_{n+1}) \\ &= d(Fx_{n-1}, Fx_n) \\ &\leq d(Tx_{n-1}, Tx_n) - \phi(d(Tx_{n-1}, Tx_n)) \\ &\leq d(Tx_{n-1}, Tx_n). \end{aligned}$$

Case 2. If $Tx_n \in P$ and $Tx_{n+1} \in Q$, then

$$d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, y_{n+1}) = d(Tx_n, y_{n+1}).$$

Therefore, we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Tx_n, y_{n+1}) \\ &= d(Fx_{n-1}, Fx_n) \\ &\leq d(Tx_{n-1}, Tx_n) - \phi(d(Tx_{n-1}, Tx_n)) \\ &\leq d(Tx_{n-1}, Tx_n). \end{aligned}$$

Case 3. Let $Tx_n \in Q$ and $Tx_{n+1} \in P$. Since Tx_n is a convex linear combination of Tx_{n-1} and y_n, it follows that

$$d(Tx_n, Tx_{n+1}) \leq \max\{d(Tx_{n-1}, Tx_{n+1}), d(y_n, Tx_{n+1})\}.$$

If $d(Tx_{n-1}, Tx_{n+1}) \leq d(Tx_{n+1}, y_n)$, then, by using monotone property of ϕ,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Tx_{n+1}, y_n) \\ &= d(Fx_{n-1}, Fx_n) \\ &\leq d(Tx_{n-1}, Tx_n) - \phi(d(Tx_{n-1}, Tx_n)) \\ &\leq d(Tx_{n-1}, Tx_n). \end{aligned}$$

Otherwise if $d(Tx_{n+1}, y_n) \leq d(Tx_{n-1}, Tx_{n+1})$, then

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_{n+1}) = d(Fx_{n-2}, Fx_n).$$

Therefore

$$\begin{aligned} d(Fx_{n-2}, Fx_n) &\leq d(Tx_{n-2}, Tx_n) - \phi(d(Tx_{n-2}, Tx_n)) \\ &\leq d(Tx_{n-2}, Tx_n). \end{aligned} \tag{2.2}$$

Notice that

$$\begin{aligned} d(Tx_{n-2}, Tx_n) &\leq d(Tx_{n-2}, Tx_{n-1}) \ + \ d(Tx_{n-1}, Tx_n) \\ &\leq \max\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-1}, Tx_n)\}. \end{aligned}$$

Here, if $d(Tx_{n-2}, Tx_{n-1}) \leq d(Tx_{n-1}, Tx_n)$, then from (2.2) we have

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n).$$

Otherwise, if $d(Tx_{n-1}, Tx_n) \leq d(Tx_{n-2}, Tx_{n-1})$, then

$$(Tx_n, Tx_{n+1}) \leq d(Tx_{n-2}, Tx_{n-1}).$$

Thus in all the cases, we have

$$d(Tx_n, Tx_{n+1}) \leq \max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-2}, Tx_{n-1})\}.$$

It follows that the sequence $\{d(Tx_n, Tx_{n+1})\}$ is monotonically decreasing. Therefore it leads to a limit $l \geq 0$. If $l > 0$ then we have

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n) - \phi(l)$$

or

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-2}, Tx_{n-1}) - \phi(l).$$

Also

$$d(Tx_n, Tx_{n+N}) \leq d(Tx_{n-1}, Tx_n) - N\phi(l)$$

or

$$d(Tx_n, Tx_{n+N}) \leq d(Tx_{n-2}, Tx_{n-1}) - N\phi(l)$$

which is a contradiction for N large enough. Therefore

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0.$$

Now, for $m, n > N$ with $n < m$, we have

$$\begin{aligned} d(Tx_n, Tx_m) \leq\ & d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) \\ & + d(Tx_{n+2}, Tx_{n+3}) + \cdots + d(Tx_{m-1}, Tx_m). \end{aligned} \tag{2.3}$$

Using (2.3) and $\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0$, along with weak contractivity of T with respect to F, we obtain $d(Tx_n, Tx_m) \to 0$ as $m, n \to \infty$, which shows that $\{Tx_n\}$ is Cauchy. First suppose that there exists a subsequence $\{Tx_{n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}(u)$. Then $u = Tv$. Here, one also needs to note that $\{Fx_{n_k-1}\}$ also converges to u. Using contraction condition, we can write

$$\begin{aligned} d(Fv, Fx_{n_k-1}) &\leq d(Tv, Tx_{n_k-1}) - \phi(d(Tv, Tx_{n_k-1})) \\ &\leq d(Tv, Tx_{n_k-1}). \end{aligned}$$

Hence, we have

$$d(Fv, u) \leq d(Tv, u),$$

it implies that $Fv = Tv$. This means that v is a coincidence point of (F, T). Since the pair (F, T) is coincidentally commuting, therefore

$$u = Fv = Tv \Rightarrow Fu = FTv = TFv = Tu.$$

To prove that u is a fixed point of F, let on contrary that $Fu \neq u$. Then

$$\begin{aligned} d(Fu, u) &= d(Fu, Fv) \\ &\leq d(Tu, Tv) - \phi(d(Tu, Tv)) \\ &\leq d(Tu, Tv) = d(Fu, u). \end{aligned}$$

Hence we have $u = Fu$, which shows that u is a fixed point of F. Also, we can show that u is a fixed point of T. Thus u is a common fixed point of F and T. The uniqueness of common fixed point follows easily. This completes the proof. □

Remark 2.2. Theorem 2.1 remains true if closedness of TK is replaced by closedness of FK.

Remark 2.3. By setting $F = I_K$ in Theorem 2.1, we obtain a result due to Khan and Imdad [15].

Remark 2.4. By setting $F = I_K$ and $K = X$ in Theorem 2.1, we obtain a result due to Rhoades [17].

Remark 2.5. By setting $F = I_K$ and $K = X$ in Theorem 2.1, one deduces a partial generalization of theorem due to Alber and Guerre-Delabriere [2].

Remark 2.6. By setting $K = X$ in Theorem 2.1, we obtain a result due to Beg and Abbas [5].

Remark 2.7. By setting $K = X$ and $\phi(t) = t - r(t)$ in Theorem 2.1, we obtain a result due to Pant [16].

In the next theorem, we utilize the idea of weakly compatible mappings in place of closedness of TK or FK.

Theorem 2.8. *Let K be a non-empty closed subset of a complete metrically convex metric space X. Let T be a weakly contractive mapping with respect to F. If T and F are weakly compatible and $TK \subset FK$ with the conditions (ii) and (iii) are holds, then F and T have a common fixed point.*

Proof. On the lines of the proof of Theorem 2.1, we obtain a point $v \in K$ such that $Tv = Fv = u$ which further implies that $TFv = FTv$. Obviously $Tu = Fu$. Now, we have to show that $Fu = u$. If it is not so, then consider

$$d(Tu, u) = d(Fu, Fv) \leq d(Tu, Tv) - \phi(d(Tu, Tv) < d(Tu, Tv) < d(Tu, u),$$

whis is a contradiction. Therefore $Tu = u$. This implies that $Fu = u$. Hence u is a common fixed point of F and T. This completes the proof. □

3. An Application

As an application of Theorem 2.8, we prove the following theorem.

Theorem 3.1. *Let K be a non-empty closed subset of a normed space X. Let T be a weakly contractive mapping with respect to F. Let T and F be weakly compatible with $TK \subset FK$ and FK be a complete subspace of X. If $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the modified Mann iterative scheme is convergent to a common fixed point of F and T.*

Proof. On the lines of the Theorem 2.1 and Theorem 2.8, we obtain a common fixed point u of the mappings F and T. Now, we consider

$$\begin{aligned}\|y_n - u\| &= \|(1-\alpha_n)Fx_n + \alpha_n Tx_n - Fu\|\\ &= \|(1-\alpha_n)(Fx_n - Fu) + \alpha_n(Tx_n - Tu)\|\\ &\le (1-\alpha_n)\|(Fx_n - Fu)\| + \alpha_n\|(Tx_n - Tu)\|\\ &\le \|(Fx_n - Fu)\| - \alpha_n\phi(\|(Fx_n - Fu)\|)\\ &\le \|y_{n-1} - u\|.\end{aligned}$$

On letting $n \to \infty$, then $\lim_{n\to\infty} \|y_{n-1} - u\| = p \ge 0$. If $p > 0$, then for any positive integer N, we have

$$\begin{aligned}\sum_{n=N}^{\infty} \alpha_n\phi(p) &\le \sum_{n=N}^{\infty} \alpha_n\phi(\|y_n - u\|)\\ &\le \sum_{n=N}^{\infty}(\|y_{n-1} - u\| - \|y_n - u\|)\\ &< \|y_N - u\|,\end{aligned}$$

which is a contradiction for the selection of α_n. Hence modified Mann iterative scheme is convergent to a common fixed point of the mappings F and T. This completes the proof. □

4. An Illustrative Example

Finally, we furnish an example to establish the utility of our result.

Example 4.1. Let $X = R$ with Euclidean metric and $K = [0,1]$. Define $F, T : K \to X$ and $\phi : [0,\infty) \to [0,\infty)$ as:

$$Tx = (2x-1),\ 0 \le x \le 1,\ Fx = x,\ 0 \le x \le 1 \text{ and } \phi(t) = \frac{t}{3}.$$

Since $\delta K = \{0,1\}$ and $TK = [-1,1]$, $\delta K \subset TK$. Further $FK = [0,1] \cap K = [0,1] \subset TK$. Also $T(\frac{1}{2}) = 0 \in \delta K$. Hence $F(\frac{1}{2}) = \frac{1}{2} \in K$ and $T(1) = 1 \in \delta K$

implies $F(1) = 1 \in K$, whereas the pair (F, T) is coincidentally commuting as $FT1 = 1 = TF1$. Moreover, for the verification of contraction condition, if $0 \leq x, y \leq 1$, then

$$\begin{aligned} d(Fx, Fy) &= |x - y| \\ &\leq \frac{4}{3}|x - y| \\ &= d(Tx, Ty) - \phi(d(Tx, Ty)). \end{aligned}$$

Thus the contraction condition and all other conditions of Theorem 2.1 are satisfied. Notice that 1 is a common fixed point of F and T.

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