ON THE LOCATION OF ZEROS OF A POLYNOMIAL WITH RESTRICTED COEFFICIENTS

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Abstract. If $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_j \geq a_{j-1}$, $a_0 > 0$, $j = 1, 2, \ldots, n$ is a polynomial of degree $n$, then according to a classical result of Eneström-Kakeya, all the zeros of $P(z)$ lie in $|z| \leq 1$. Joyal et al extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative. In this paper, I will prove some extensions and generalizations of this result by relaxing the hypothesis.

1. Introduction

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$. Then concerning the distribution of zeros of $P(z)$, Eneström and Kakeya [10, 11] proved the following interesting result.

**Theorem A.** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$  \hspace{1cm} (1.1)

Then $P(z)$ has all its zeros in $|z| \leq 1$.

In the literature [1-11], there exist several extensions and generalizations of this Theorem. Joyal et al. [9] extended Theorem A to the polynomials
whose coefficients are monotonic but not necessarily non-negative. In fact they proved the following result.

**Theorem B.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that
\[
a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0.
\]

Then \( P(z) \) has all its zeros in the disk
\[
|z| \leq \frac{1}{|a_n|} (|a_n| - a_0 + |a_0|).
\]

In this paper, we will prove some generalizations and extensions of Theorem B and of the Theorem A i.e., Eneström-Kakeya Theorem. In this direction we first present the following interesting result in which we relax the hypothesis and hence is a generalization of Theorem B. In fact, we prove the following:

2. **Main Results**

**Theorem 2.1.** Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0 \) be a polynomial of degree \( n \) satisfying
\[
a_n \geq a_{n-1} \geq \cdots \geq a_p, \quad 0 \leq p \leq n.
\]

Then all the zeros of \( P(z) \) lie in the disk
\[
|z| \leq \frac{a_n - a_p + M_p}{|a_n|},
\]
where
\[
M_p = \sum_{j=0}^{p} |a_j - a_{j-1}|.
\]

**Proof.** Consider the polynomial
\[
F(z) = (1 - z)P(z)
\]
\[
= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)
\]
\[
= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 - a_n z^{n+1} - a_{n-1} z^n - \cdots - a_0 z
\]
\[
= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_1 - a_0) z + a_0.
\]
This gives

\[ |F(z)| \geq |a_n z^{n+1}| - \left\{ |a_n - a_{n-1}| |z|^n + |a_{n-1} - a_{n-2}| |z|^{n-1} \right. \]

\[ + \cdots + |a_{p+1} - a_p| |z|^{p+1} + \cdots + |a_1 - a_0| |z| + |a_0| \right\} \]

\[ = |z|^n \left\{ |a_n| |z| - \left( |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \right. \right. \]

\[ + \cdots + |a_{p+1} - a_p| + \cdots + |a_1 - a_0| + |a_0| \right) \right\} \]

\[ = |z|^n \left\{ |a_n| |z| - \left( a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots \right. \right. \]

\[ + a_{p+1} - a_p + |a_p - a_{p-1}| + \cdots + |a_1 - a_0| + |a_0| \right) \right\} \]

\[ = |z|^n \left\{ |a_n| |z| - \left( a_n - a_p + |a_p - a_{p-1}| + \cdots + |a_1 - a_0| + |a_0| \right) \right\} \]

\[ = |z|^n \left\{ |a_n| |z| - \left( a_n - a_p + \sum_{j=0}^p |a_j - a_{j-1}| \right) \right\} \]

\[ > 0, \]

for \(|z||a_n| > (a_n - a_p + M_p)|\), where \(M_p = \sum_{j=0}^p |a_j - a_{j-1}|\), \(a_{-1} = 0\). Thus all the zeros of \(F(z)\) whose modulus is greater than 1 lie in the disk

\[ |z| \leq \frac{1}{|a_n|} \left( a_n - a_p + M_p \right). \]

But those zeros of \(F(z)\) whose modulus is less than or equal to 1 already satisfy the above inequality and all the zeros of \(P(z)\) are also the zeros of \(F(z)\). Hence
it follows that all the zeros of \( P(z) \) lie in the disk
\[
|z| \leq \frac{1}{|a_n|} (a_n - a_p + M_p).
\]
This completes the proof. \( \square \)

Remark 2.2. For \( p = 0 \), we get Theorem B.

Applying Theorem 2.1 to the polynomial \( P(tz) \), we get the following corollary.

Corollary 2.3. Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \) be a polynomial of degree \( n \) such that for any \( t > 0 \),
\[
t^n a_n \geq t^{n-1} a_{n-1} \geq \cdots \geq t^p a_p, \quad 0 \leq p \leq n.
\]
Then all the zeros of \( P(z) \) lie in the disk
\[
|z| \leq \frac{a_n - t^{n-p} a_p}{|a_n|} + \sum_{j=0}^{p} \frac{|ta_j - a_{j-1}|}{t^{n-j+1}|a_n|}.
\]

The following result follows from Corollary 2.3 by taking \( p = n \).

Corollary 2.4. Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \) be a polynomial of degree \( n \). Then for any \( t > 0 \), all the zeros of \( P(z) \) lie in the disk
\[
|z| \leq \sum_{j=0}^{n} \frac{|ta_j - a_{j-1}|}{t^{n-j+1}|a_n|}.
\]

We also prove the following result which gives the lower bound for the moduli of zeros of a polynomial.

Theorem 2.5. If \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0 \) is a polynomial of degree \( n \) satisfying
\[
a_n \geq a_{n-1} \geq \cdots \geq a_p, \quad 0 \leq p \leq n.
\]
Then \( P(z) \) does not vanish in
\[
|z| < \min \left( 1, \frac{|a_0|}{|a_n| + a_n - a_p - |a_0| + M_p} \right),
\]
where
\[
M_p = \sum_{j=0}^{p} |a_j - a_{j-1}|.
\]
The bound is attained by the polynomial \( P(z) = z^n + z^{n-1} + \cdots + z + 1 \).
Proof. Consider the reciprocal polynomial
\[ R(z) = z^n P(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_p z^{n-p} + \cdots + a_n. \]
Let
\[ S(z) = (1 - z) R(z) \]
\[ = -a_0 z^{n+1} + (a_0 - a_1) z^n + \cdots + (a_p - a_{p+1}) z^{n-p} + \cdots + (a_{n-1} - a_n) z + a_n. \]
This gives
\[ |S(z)| \geq |a_0| |z|^{n+1} - \left| a_0 - a_1 \right| |z|^n + \cdots + \left| a_p - a_{p+1} \right| |z|^{n-p} + \cdots + \left| a_{n-1} - a_n \right| |z| + |a_n| \]
\[ = |z|^n \left\{ |a_0||z| - \left| a_0 - a_1 \right| + \cdots + \frac{|a_p - a_{p+1}|}{|z|^p} + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^{n}} \right\}. \]
Now, for \(|z| > 1\), that is \(\frac{1}{|z|^{n-j}} < 1, 0 \leq j \leq n\), we have
\[ |S(z)| \geq |z|^n \left\{ |a_0||z| - \left( |a_0 - a_1| + \cdots + |a_p - a_{p+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \right) \right\} \]
\[ = |z|^n \left\{ |a_0||z| - \left( |a_1 - a_0| + \cdots + |a_{p+1} - a_p| + |a_p - a_{p-1}| + \cdots + |a_n - a_{n-1}| + |a_n| \right) \right\} \]
\[ = |z|^n \left\{ |a_0||z| - \left( M_p - |a_0| + |a_p - a_{p-1}| + \cdots + |a_{n-1} - a_n| + |a_n| \right) \right\} \]
\[ = |z|^n \left\{ |a_0||z| - \left( M_p - |a_0| + a_n - a_p + |a_n| \right) \right\} \]
\[ > 0, \]
for \(|z| > \frac{1}{|a_0|} \left\{ |a_n| + a_n - a_p - |a_0| + M_p \right\}\), where \(M_p = \sum_{j=0}^{p} |a_j - a_{j-1}|, a_{-1} = 0\).
Thus all the zeros of \(S(z)\) whose modulus is greater than 1 lie in
\[ |z| \leq \frac{1}{|a_0|} \left\{ |a_n| + a_n - a_p - |a_0| + M_p \right\}. \]
Hence all the zeros of $S(z)$ and hence of $R(z)$ lie in
\[ |z| \leq \max \left\{ 1, \frac{1}{|a_0|} \left( |a_n| + a_n - a_p - |a_0| + M_p \right) \right\}. \]
Therefore, all the zeros of $P(z)$ lie in
\[ |z| \geq \min \left\{ 1, \frac{|a_0|}{|a_n| + a_n - a_p - |a_0| + M_p} \right\}. \]
Thus the polynomial $P(z)$ does not vanish in
\[ |z| < \min \left( 1, \frac{|a_0|}{|a_n| + a_n - a_p - |a_0| + M_p} \right). \]
This completes the proof.

For $p = 0$, Theorem 2.5 reduces to the following result.

**Corollary 2.6.** If $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree $n$ satisfying
\[ a_n \geq a_{n-1} \geq \cdots \geq a_0, \]
then $P(z)$ does not vanish in
\[ |z| < \frac{|a_0|}{|a_n| + a_n - a_0}. \]
The bound is attained by the polynomial $P(z) = z^n + z^{n-1} + \cdots + z + 1$.

Next we prove the following more general result which is also a generalization of Theorem B.

**Theorem 2.7.** Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0$ be a polynomial of degree $n$ satisfying
\[ a_n \geq a_{n-1} \geq \cdots \geq a_p, \quad 0 \leq p \leq n \]
and
\[ \max_{|z|=1} \left| \sum_{j=0}^p (a_j - a_{j-1}) z^j \right| \leq M, \quad (a_{-1} = 0). \]
Then all the zeros of $P(z)$ lie in
\[ |z| \leq \max \left( 1, \frac{a_n - a_p + M}{|a_n|} \right). \]
Proof. Consider the polynomial

\[ F(z) = (1 - z)P(z) \]

\[ = (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0) \]

\[ = a_n z^n + \cdots + a_1 z + a_0 - a_n z^{n+1} + a_0 z \]

\[ = -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \cdots + (a_p - a_{p-1}) z^{p+1} \]

\[ + (a_p - a_{p-1}) z^p + \cdots + (a_2 - a_1) z^2 + (a_1 - a_0) z + a_0 \]

\[ = R(z) - a_n z^{n+1}, \]

where

\[ R(z) = (a_n - a_{n-1}) z^n + \cdots + (a_{p+1} - a_p) z^{p+1} + (a_p - a_{p-1}) z^p + \cdots + (a_1 - a_0) z + a_0. \]

Let

\[ R^*(z) = z^n R(1/z) = a_0 z^n + (a_1 - a_0) z^{n-1} + \cdots + (a_p - a_{p-1}) z^{n-p} \]

\[ + (a_p - a_{p-1}) z^{n-p} + (a_{p+1} - a_p) z^{n-p-1} + \cdots + (a_n - a_{n-1}). \]

Then, we have

\[ |R^*(z)| \leq |a_0 z^n + (a_1 - a_0) z^{n-1} + \cdots + (a_p - a_{p-1}) z^{n-p}| \]

\[ + |(a_{p+1} - a_p) z^{n-p-1} + \cdots + (a_n - a_{n-1})| \]

\[ \leq \sum_{j=0}^{p} |a_j - a_{j-1}| z^{n-j} + |(a_{p+1} - a_p) z^{n-p-1} + \cdots + (a_n - a_{n-1})| \]

\[ \leq M + a_n - a_p, \]

for \(|z| = 1\), where \(M\) is defined as above. Hence by maximum modulus principle, it follows that

\[ |R^*(z)| \leq M + a_n - a_p, \quad \text{for } |z| \leq 1. \]

Therefore

\[ |R(z)| \leq |z|^n (M + a_n - a_p), \quad \text{for } |z| \geq 1. \]

This gives for \(|z| > 1\),

\[ |F(z)| \geq |a_n z^{n+1}| - |R(z)| \]

\[ \geq |a_n z^{n+1}| - z^n (M + a_n - a_p) \]

\[ \geq |a_n||z|^n \left\{ |z| - \frac{M + a_n - a_p}{|a_n|} \right\} \]

\[ > 0, \]
for $|z| > \frac{M + a_n - a_p}{|a_n|}$. Thus all zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$|z| \leq \frac{M + a_n - a_p}{|a_n|}.$$ 

Therefore all zeros of $F(z)$ lie in the disk

$$|z| \leq \max\left\{1, \frac{M + a_n - a_p}{|a_n|}\right\}.$$ 

But all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leq \max\left\{1, \frac{M + a_n - a_p}{|a_n|}\right\}.$$ 

This completes the proof of Theorem 2.7.

\[\Box\]

**Remark 2.8.** Let $\max_{|z|=1} \left| \sum_{j=0}^{p} (a_j - a_{j-1})z^j \right|$ is attained at $z = e^{i\alpha}$. Then

$$M = \left| \sum_{j=0}^{p} (a_j - a_{j-1})e^{i\alpha} \right|$$

$$\leq \sum_{j=0}^{p} |a_j - a_{j-1}|$$

$$= M_p, \ 0 \leq p \leq n,$$

where $M_p$ is defined as in Theorem 2.1. Thus

$$M \leq M_p, \ 0 \leq p \leq n.$$ 

From this, we conclude that Theorem 2.7 is a refinement of Theorem 2.1.

The following result is an immediate consequence of the Theorem 2.7.

**Corollary 2.9.** Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ be a polynomial of degree $n$. Then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{M}{|a_n|}.$$
where

\[ M = \max_{|z|=1} \left| \sum_{j=0}^{n} (a_j - a_{j-1})z^j \right|. \]

References