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ON THE LOCATION OF ZEROS OF A POLYNOMIAL WITH RESTRICTED COEFFICIENTS

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Abstract. If $P(z) = \sum_{j=0}^{n} a_j z^j, a_j \geq a_{j-1}, a_0 > 0, j = 1, 2, \cdots, n$ is a polynomial of degree n, then according to a classical result of Eneström-Kakeya, all the zeros of $P(z)$ lie in $|z| \leq 1$. Joyal et al extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative. In this paper, I will prove some extensions and generalizations of this result by relaxing the hypothesis.

1. Introduction

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Then concerning the distribution of zeros of $P(z)$, Eneström and Kakeya [10, 11] proved the following interesting result.

Theorem A. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0. \tag{1.1}$$

Then $P(z)$ has all its zeros in $|z| \leq 1$.

In the literature [1-11], there exist several extensions and generalizations of this Theorem. Joyal *et al.* [9] extended Theorem A to the polynomials

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whose coefficients are monotonic but not necessarily non-negative. In fact they proved the following result.

Theorem B. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0.$$

Then $P(z)$ *has all its zeros in the disk*

$$|z| \leq \frac{1}{|a_n|}\left(|a_n| - a_0 + |a_0|\right).$$

In this paper, we will prove some generalizations and extensions of Theorem B and of the Theorem A i,e., Eneström-Kakeya Theorem. In this direction we first present the following interesting result in which we relax the hypothesis and hence is a generalization of Theorem B. In fact, we prove the following:

2. Main Results

Theorem 2.1. *Let* $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0$ *be a polynomial of degree* n *satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_p, \qquad 0 \leq p \leq n.$$

Then all the zeros of $P(z)$ *lie in the disk*

$$|z| \leq \frac{a_n - a_p + M_p}{|a_n|},$$

where

$$M_p = \sum_{j=0}^{p} |a_j - a_{j-1}|.$$

Proof. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)\big(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\big) \\ &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 - a_n z^{n+1} - a_{n-1} z^n - \cdots - a_0 z \\ &= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_1 - a_0) z + a_0. \end{aligned}$$

This gives

$$|F(z)| \geq \left|a_n z^{n+1}\right| - \Big\{ \left|a_n - a_{n-1}\right| |z|^n + \left|a_{n-1} - a_{n-2}\right| |z|^{n-1} + \cdots + \left|a_{p+1} - a_p\right| |z|^{p+1} + \cdots + \left|a_1 - a_0\right| |z| + |a_0| \Big\}$$

$$= |z|^n \Big\{ |a_n||z| - \Big(|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_{p+1} - a_p|}{|z|^{n-p-1}} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Big) \Big\}.$$

Now, for $|z| > 1$, *i,e.,* $\frac{1}{|z|^{n-j}} < 1$, $0 \leq j \leq n$, we have

$$|F(z)| > |z|^n \Big\{ |a_n||z| - \Big(|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| + \cdots + |a_1 - a_0| + |a_0| \Big) \Big\}$$

$$= |z|^n \Big\{ |a_n||z| - \Big(a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_{p+1} - a_p + |a_p - a_{p-1}| + \cdots + |a_1 - a_0| + |a_0| \Big) \Big\}$$

$$= |z|^n \Big\{ |a_n||z| - \Big(a_n - a_p + |a_p - a_{p-1}| + \cdots + |a_1 - a_0| + |a_0| \Big) \Big\}$$

$$= |z|^n \Big\{ |a_n||z| - \Big(a_n - a_p + \sum_{j=0}^{p} \left|a_j - a_{j-1}\right| \Big) \Big\}$$

$$> 0,$$

for $|z||a_n| > \Big(a_n - a_p + M_p\Big)$, where $M_p = \sum_{j=0}^{p} \left|a_j - a_{j-1}\right|$, $a_{-1} = 0$. Thus all the zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$|z| \leq \frac{1}{|a_n|}\Big(a_n - a_p + M_p\Big).$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{1}{|a_n|}\Big(a_n - a_p + M_p\Big).$$

This completes the proof. □

Remark 2.2. For $p = 0$, we get Theorem B.

Applying Theorem 2.1 to the polynomial $P(tz)$, we get the following corollary.

Corollary 2.3. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0$ be a polynomial of degree n such that for any $t > 0$,*

$$t^n a_n \geq t^{n-1}a_{n-1} \geq \cdots \geq t^p a_p, \qquad 0 \leq p \leq n.$$

Then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \frac{a_n - t^{p-n}a_p}{|a_n|} + \sum_{j=0}^{p} \frac{|ta_j - a_{j-1}|}{t^{n-j+1}|a_n|}.$$

The following result follows from Corollary 2.3 by taking $p = n$.

Corollary 2.4. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n. Then for any $t > 0$, all the zeros of $P(z)$ lie in the disk*

$$|z| \leq \sum_{j=0}^{n} \frac{|ta_j - a_{j-1}|}{t^{n-j+1}|a_n|}.$$

We also prove the following result which gives the lower bound for the moduli of zeros of a polynomial.

Theorem 2.5. *If $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0$ is a polynomial of degree n satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_p, , \qquad 0 \leq p \leq n.$$

Then $P(z)$ does not vanish in

$$|z| < \min\left(1, \frac{|a_0|}{|a_n| + a_n - a_p - |a_0| + M_p}\right),$$

where

$$M_p = \sum_{j=0}^{p} |a_j - a_{j-1}|.$$

The bound is attained by the polynomial $P(z) = z^n + z^{n-1} + \cdots + z + 1$.

Proof. Consider the reciprocal polynomial

$$R(z) = z^n P(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_p z^{n-p} + \cdots + a_n.$$

Let

$$\begin{aligned} S(z) &= (1-z)R(z) \\ &= -a_0 z^{n+1} + (a_0 - a_1) z^n + \cdots + (a_p - a_{p+1}) z^{n-p} + \cdots + (a_{n-1} - a_n) z + a_n. \end{aligned}$$

This gives

$$\begin{aligned} &\left|S(z)\right| \\ &\geq |a_0||z|^{n+1} - \left\{ |a_0 - a_1||z|^n + \cdots + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n||z| + |a_n| \right\} \\ &= |z|^n \left\{ |a_0||z| - \left(|a_0 - a_1| + \cdots + \frac{|a_p - a_{p+1}|}{|z|^p} + \cdots + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \right) \right\}. \end{aligned}$$

Now, for $|z| > 1$, that is $\dfrac{1}{|z|^{n-j}} < 1, 0 \leq j \leq n$, we have

$$\begin{aligned} &\left|S(z)\right| \\ &\geq |z|^n \left\{ |a_0||z| - \left(|a_0 - a_1| + \cdots + |a_p - a_{p+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \right) \right\} \\ &= |z|^n \left\{ |a_0||z| - \left(|a_1 - a_0| + \cdots + |a_{p+1} - a_p| + |a_p - a_{p-1}| \right.\right. \\ &\qquad \left.\left. + \cdots + |a_n - a_{n-1}| + |a_n| \right) \right\} \\ &= |z|^n \left\{ |a_0||z| - \left(M_p - |a_0| + |a_p - a_{p-1}| + \cdots + |a_n - a_{n-1}| + |a_n| \right) \right\} \\ &= |z|^n \left\{ |a_0||z| - \left(M_p - |a_0| + a_n - a_p + |a_n| \right) \right\} \\ &> 0, \end{aligned}$$

for $|z| > \dfrac{1}{|a_0|}\left\{ |a_n| + a_n - a_p - |a_0| + M_p \right\}$, where $M_p = \sum_{j=0}^{p} \left|a_j - a_{j-1}\right|, \ \ a_{-1} = 0.$
Thus all the zeros of $S(z)$ whose modulus is greater than 1 lie in

$$|z| \leq \frac{1}{|a_0|}\left\{ |a_n| + a_n - a_p - |a_0| + M_p \right\}.$$

Hence all the zeros of $S(z)$ and hence of $R(z)$ lie in

$$|z| \leq \max\left\{1, \frac{1}{|a_0|}\left(|a_n| + a_n - a_p - |a_0| + M_p\right)\right\}.$$

Therefore, all the zeros of $P(z)$ lie in

$$|z| \geq \min\left\{1, \frac{|a_0|}{|a_n| + a_n - a_p - |a_0| + M_p}\right\}.$$

Thus the polynomial $P(z)$ does not vanish in

$$|z| < \min\left(1, \frac{|a_0|}{|a_n| + a_n - a_p - |a_0| + M_p}\right).$$

This completes the proof. □

For $p = 0$, Theorem 2.5 reduces to the following result.

Corollary 2.6. *If $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree n satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_0,$$

then $P(z)$ does not vanish in

$$|z| < \frac{|a_0|}{|a_n| + a_n - a_0}.$$

The bound is attained by the polynomial $P(z) = z^n + z^{n-1} + \cdots + z + 1$.

Next we prove the following more general result which is also a generalization of Theorem B.

Theorem 2.7. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_p z^p + \cdots + a_1 z + a_0$ be a polynomial of degree n satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_p, \qquad 0 \leq p \leq n$$

and

$$\max_{|z|=1}\left|\sum_{j=0}^{p}(a_j - a_{j-1})z^j\right| \leq M, \ \ (a_{-1} = 0).$$

Then all the zeros of $P(z)$ lie in

$$|z| \leq \max\left(1, \frac{a_n - a_p + M}{|a_n|}\right).$$

Proof. Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z)\\ &= (1-z)\big(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\big)\\ &= a_n z^n + \cdots + a_1 z + a_0 - a_n z^{n+1} - a_{n-1}z^n - \cdots - a_0 z\\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_{p+1} - a_p)z^{p+1}\\ &\quad + (a_p - a_{p-1})z^p + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0\\ &= R(z) - a_n z^{n+1},\end{aligned}$$

where

$$R(z) = (a_n - a_{n-1})z^n + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \cdots + (a_1 - a_0)z + a_0.$$

Let

$$\begin{aligned}R^*(z) = z^n R(1/z) &= a_0 z^n + (a_1 - a_0)z^{n-1} + \cdots + (a_p - a_{p-1})z^{n-p}\\ &\quad + (a_p - a_{p-1})z^{n-p} + (a_{p+1} - a_p)z^{n-p-1} + \cdots + (a_n - a_{n-1}).\end{aligned}$$

Then, we have

$$\begin{aligned}|R^*(z)| &\le \big|a_0 z^n + (a_1 - a_0)z^{n-1} \cdots + (a_p - a_{p-1})z^{n-p}\big|\\ &\qquad + \big|(a_{p+1} - a_p)z^{n-p-1} + \cdots + (a_n - a_{n-1})\big|\\ &\le \Big|\sum_{j=0}^{p}(a_j - a_{j-1})z^{n-j}\Big| + |(a_{p+1} - a_p)|\,|z|^{n-p-1}\\ &\qquad + \cdots + |(a_n - a_{n-1})|\\ &\le M + a_n - a_p,\end{aligned}$$

for $|z| = 1$, where M is defined as above. Hence by maximum modulus principle, it follows that

$$|R^*(z)| \le M + a_n - a_p, \quad \text{for } |z| \le 1.$$

Therefore

$$|R(z)| \le |z|^n (M + a_n - a_p), \quad \text{for } |z| \ge 1.$$

This gives for $|z| > 1$,

$$\begin{aligned}|F(z)| &\ge \big|a_n z^{n+1}\big| - \big|R(z)\big|\\ &\ge \big|a_n z^{n+1}\big| - z^n (M + a_n - a_p)\\ &\ge \big|a_n\big||z|^n \left\{|z| - \frac{M + a_n - a_p}{\big|a_n\big|}\right\}\\ &> 0,\end{aligned}$$

for $|z| > \dfrac{M + a_n - a_p}{|a_n|}$. Thus all zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$|z| \leq \frac{M + a_n - a_p}{|a_n|}.$$

Therefore all zeros of $F(z)$ lie in the disk

$$|z| \leq Max\left\{1, \frac{M + a_n - a_p}{|a_n|}\right\}.$$

But all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leq Max\left\{1, \frac{M + a_n - a_p}{|a_n|}\right\}.$$

This completes the proof of Theorem 2.7. □

Remark 2.8. Let $\max\limits_{|z|=1}\left|\sum\limits_{j=0}^{p}(a_j - a_{j-1})z^j\right|$ is attained at $z = e^{i\alpha}$. Then

$$\begin{aligned} M &= \left|\sum_{j=0}^{p}(a_j - a_{j-1})e^{i\alpha}\right| \\ &\leq \sum_{j=0}^{p}|a_j - a_{j-1}| \\ &= M_p, \ \ 0 \leq p \leq n, \end{aligned}$$

where M_p is defined as in Theorem 2.1. Thus

$$M \leq M_p, \ \ 0 \leq p \leq n.$$

From this, we conclude that Theorem 2.7 is a refinement of Theorem 2.1.

The following result is an immediate consequence of the Theorem 2.7.

Corollary 2.9. *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n. Then all the zeros of $P(z)$ lie in*

$$|z| \leq \frac{M}{|a_n|},$$

where

$$M = \max_{|z|=1}\left|\sum_{j=0}^{n}(a_j - a_{j-1})z^j\right|.$$

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