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# EXISTENCE OF SOLUTION FOR SYSTEM OF INTEGRAL EQUATIONS AND FUNCTIONAL EQUATIONS THROUGH NON-NEGATIVE FUNCTION WITH PROPERTY OF ONENESS

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**Abstract.** In this article, we define the notion of a non-negative function satisfying property of oneness. Utilizing this concept, certain unified fixed point results are established. Furthermore generalized $\phi$-contractions mapping and some fixed point results in the framework of symmetric space are also presented. Some well-known results from the existence literature are obtained through our introduced setting. Results are substantiated by some innovative examples with the aid of visualization of surfaces. Moreover applications of obtained theory for the existence for the system of integral equations as well as for the system of functional equations occurred in dynamic programming are also presented.

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## 1. Introduction and Preliminaries

Taking the accounts its applications, fixed point theory has received considerable attention through the last ninety years in many different ways. Two of them are the notions of symmetric spaces and semi-metric spaces introduced and studied by Wilson [20].

Cicchese [6] proved the first fixed point theorem for contractions in semi-metric spaces. In [8], Hicks and Rhoades proved some common fixed point theorems in symmetric spaces and showed that a general probabilistic structure admits a compatible symmetric or semi-metric.

For more information on fixed point theory in symmetric spaces and semi-metric spaces, one can refer ([1],[9],[11],[14],[15]).

On the other hand Popa [16] initiated the concept of implicit function with a view to cover several contraction conditions of the existing literature in one go. In recent years, the idea of implicit function has been utilized by many authors, one can refer ([2],[13]) and references therein.

In this paper we prove some unified fixed point results utilizing the concept of non-negative function satisfying the property of oneness. Further more fixed point results for generalized $\Phi$-contraction are also established. Our results generalize earlier results obtained by Cho *et al.* [5], Imdad *et al.* [9] and Sahu *et al.*[17]. Finally, some the applications of our result are also provided.
For convenience we recall basic definitions and the properties from the theory of symmetric spaces used in the sequel.

A symmetric space is a pair $(X, d)$ consisting of a non-empty set $X$ and a function $d : X \times X \to [0, \infty)$ such that for all $x, y \in X$, the following conditions hold:
(W1) $d(x, y) = 0$ if and only if $x = y$;
(W2) $d(x, y) = d(y, x)$.

Recently, Sumati Kumari et al. [19] introduced quasi-symmetric space by relaxing condition (W2) from the aforesaid definition. Here we want to quote that the property which Sumati Kumari et al. [19] adopted is not only relaxing symmetric space but it is relaxing metric space, $b$- metric space, rectangular metric space, quasi metric spaces etc. So better to say it quasi-symmetric space, we redefine it as a non-negative function satisfying property of oneness.

**Definition 1.1.** Let X be a non-empty set. A function $d_o : X \times X \to [0, \infty)$ is said to be a non-negative function satisfying the property of oneness if

$$0 \leq d_o(x, y) \text{ for all } x, y \in X$$

and

$$d_o(x,y) = 0 \text{ if and only if } x = y.$$

The following functions meet such requirements.

**Example 1.2.** For $X = R$, define a function $d_o : X \times X \to [0, \infty)$ as

$$d_o(x,y) = \begin{cases} \sinh(x-y) & \text{if } x \geq y; \\ 1 + \sinh(y-x) & \text{if } y > x. \end{cases}$$

Then $d_o(x,y)$ is a non-negative function satisfying the property of oneness.

Many properties and notions in symmetric spaces are similar to those in metric spaces (but not all, because of the absence of the triangle inequality). For example, a sequence $\{x_n\} \subseteq X$ is said to be Cauchy sequence if given $\epsilon > 0$ there is $N \in \mathbf{N}$ such that $d(x_m, x_n) < \epsilon$, for all $m, n \geq N$.

In every symmetric space $(X,d)$ one may introduce the topology $\tau_d$ by defining the family of closed sets as follows:
A set $A \subseteq X$ is closed if and only if for each $x \in X, d(x,A) = 0$ implies $x \in A$, where

$$d(x,A) = \inf d(x,a) : a \in A. \tag{1.1}$$

The following conditions can be used as partial replacements for absence of triangle inequality in the symmetric space $(X,d)$.

(W3) $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$;

(W4) $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$;

(W) $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$ imply $\lim_{n\to\infty} d(x_n, z_n) = 0$;

(HE) Given $\{x_n\}, \{y_n\}$ and $x$ in $X$, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$;

(1C) Given $\{x_n\}$ and $y$ in $X$, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x,y)$;
(such symmetric is usually called 1-continuous)

(CC) Given $\{x_n\}, \{y_n\}$ and $x, y$ in $X$, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x,y)$; (such symmetric is usually called continuous)

The properties (W3) and (W4) were induced by Wilson [20], (W) by Mihet [14], (HE) by Aliouche [1], (CC) by Cho et al. [5].

A symmetric space $(X,d)$ is said to be $d$-Cauchy complete if every Cauchy sequence converges to some $x \in X$ in the topology $\tau_d$, and it is said to be weakly complete if every decreasing sequence $\{F_n\}$ of non-empty closed subsets, such that there exists a sequence $\{x_n\}, x_n \in F_n$ with $F_n \subseteq B(x_n, 2^{-n})$ has a non-empty intersection.

**Definition 1.3.** ([12]) Let $(X, d)$ be a non-negative function satisfying property of oneness (Or symmetric space) and let $A$ and $S$ be two self maps of $X$. A point $x$ in $X$ is called a coincidence point of $A$ and $S$ if and only if $Ax = Sx$. We shall call $w = Ax = Sx$ a point of coincidence of $A$ and $S$.

**Definition 1.4.** Let $A$ and $S$ be self mappings of a non-negative function satisfying property of oneness (Or symmetric space) $(X, d)$. Then the pair $(A, S)$ is said to be non-vacuously weakly compatible, if

(1) $C(A, S) \neq \phi$.
(2) Mappings $A$ and $S$ commute at coincidence point *i.e.* $ASu = SAu$, for every $u \in C(A, S)$, where $C(A, S)$ is the set of coincidence points of mappings $A$ and $S$.

**Definition 1.5.** Let $Y$ be an arbitrary set, $(X, d)$ be a non-negative function satisfying property of oneness (or symmetric space) and let $A, B, S, T$ be mappings from $Y$ into $X$. Then

(1) the pair $(A, S)$ is said to have the common limit range property with respect to the mapping $S$ (denoted by $CLR_S$) [18] if there exist a sequence $\{x_n\}$ in $Y$ such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u, \text{ for some } u \in S(Y).$$
(2) the pair $(A, S)$ and $(B, T)$ are said to have the common limit range property with respect to mappings $S$ and $T$ (denoted by $(CLR_{ST})$) [10] if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $Y$ such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = u,$$
for some $u \in S(Y) \cap T(Y)$.

## 2. An implicit relation

**Definition 2.1.** Let $\mathcal{F}_6$ be set of all functions $F(t_1, t_2, t_3, t_4, t_5, t_6) : R_+^6 \to R$ satisfying the condition
$$F(t, t, 0, t, 0, t) > 0, \qquad \forall\, t > 0. \tag{2.1}$$

**Example 2.2.** $F(t_1, t_2, ..., t_6) = t_1 - at_2 - b(t_3 + t_5) - c(t_4 + t_6)$,
where $a, b, c \geq 0$, $a + 2c < 1$.

**Example 2.3.** $F(t_1, t_2, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4, t_6\} - c\max\{t_2, t_5\}$, where $a, b, c \geq 0$, $a + b + c < 1$.

**Example 2.4.** $F(t_1, t_2, ..., t_6) = t_1 - k\max\{t_2, \min\{t_3, t_5\}, \min\{t_4, t_6\}\}$, $k \in [0, 1)$.

**Example 2.5.** $F(t_1, t_2, ..., t_6) = t_1 - k\max\{t_2, \frac{k}{2}(t_3 + t_5), \frac{k}{2}(t_4 + t_6)\}$, $k \in [0, 1)$.

**Example 2.6.** $F(t_1, t_2, ..., t_6) = t_1 - \phi(\max\{t_2, \frac{k}{2}(t_3 + t_5), \frac{k}{2}(t_4 + t_6)\})$, $k \in [0, 1)$, where $\phi : R^+ \to R^+$ such that $\phi$ is non-decreasing on $R^+$ and $0 < \phi(t) < t$, $\forall\, t \in (0, \infty)$.

**Example 2.7.** $F(t_1, t_2, ..., t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6)$, where $\phi : (R^+)^5 \to R^+$ satisfying the following conditions: $\phi$ is non-decreasing and upper semi-continuous and $\phi(t, t, t, t, t) = \psi(t) < t$ for each $t > 0$, where $\psi : R^+ \to R^+$ is a mapping with $\psi(0) = 0$.

**Example 2.8.** $F(t_1, t_2, ..., t_6) = t_1 - \phi(t_2, t_3, t_5, \sqrt{t_4.t_6})$ where $\phi$- is non- decreasing and upper semi-continuous and $\phi(t, t, t, t, t) = \psi(t) < t$ for each $t > 0$, where $\psi : R^+ \to R^+$ is a mapping with $\psi(0) = 0$.

**Example 2.9.** $F(t_1, t_2, ..., t_6) = t_1 - k\max\{t_2, t_3, ...t_6\}$, $k \in [0, 1)$.

**Example 2.10.** $F(t_1, t_2, ..., t_6) = t_1 - \phi(\max(t_2, t_3, ...t_6))$, where $\phi : R^+ \to R^+$ and $\phi(t) < t$, for $t > 0$ and $\phi(0) = 0$.

**Example 2.11.** $F(t_1, t_2, ..., t_6) = t_1 - k\min\{t_2, \max\{t_3, t_4, t_6\}, t_5\}$, $k \in [0, 1)$.

**Example 2.12.** $F(t_1, t_2, ..., t_6) = t_1 - k\max\{t_2, t_3, t_5, \frac{t_4 + t_6}{2}\}$, $k \in [0, 1)$.

**Example 2.13.** $F(t_1, t_2, ..., t_6) = t_1 - \lambda t_2 - \mu\frac{t_3.t_4}{1+t_2} - \gamma\frac{t_5.t_6}{1+t_2}$, where $\lambda, \mu, \gamma \geq 0$, $\lambda + \mu + \gamma < 1$.

**Example 2.14.**

$$F(t_1, t_2, ..., t_6) = \begin{cases} t_1 - \lambda t_2 - \mu\frac{t_2.t_5}{t_2 + t_4 + t_6}, & if\ t_2 + t_4 + t_6 \neq 0\ ; \\ t_1, & if\ t_3 + t_4 + t_6 = 0, \end{cases}$$

where $\lambda, \mu \geq 0$, $\lambda + \mu < 1$.

**Example 2.15.** $F(t_1, t_2, ..., t_6) = t_1 - k\max\{t_2, \sqrt{t_3.t_5}, \sqrt{t_4.t_6}\}$, $k \in [0, 1)$.

**Example 2.16.** $F(t_1, t_2, ..., t_6) = t_1^2 - k\max\{t_2.t_3, t_2.t_5, t_2.t_4, t_5.t_6\}$, $k \in [0, 1)$.

## 3. Fixed point theorem for the non-negative function satisfying property of oneness

In this section, some common fixed point theorems for two pairs of non-vacuously weakly compatible mappings on non-negative function $(X, d_o)$ having property of oneness.

**Theorem 3.1.** Let $X$ be a non-empty set and $d_o : X \times X \to X$ a non-negative function satisfying the property of oneness. If $P, Q, R$ and $S$ are four self mappings of $X$ such that

(1) for all $x, y \in X$ with $Px \neq Qy$ and $F \in \mathcal{F}_6$,

$$F(d_o(Px, Qy), d_o(Rx, Sy), d_o(Px, Rx), d_o(Px, Sy), d_o(Qy, Sy), d_o(Rx, Qy)) \leq 0. \tag{3.1}$$

(2) The pairs $(P, R)$ and $(Q, S)$ are non-vacuously weakly compatible, then $P, Q, R$ and $S$ have a unique common fixed point.

*Proof.* In view of non-vacuously weak compatibility of pairs of mappings $(P, R)$ and $(Q, S)$ for $x, y \in X$, we have

$$Px = Rx \quad \Rightarrow \quad PRx = RPx, \tag{3.2}$$

and

$$Qy = Sy \quad \Rightarrow \quad QSy = SQy. \tag{3.3}$$

Now we shall show that $Px = Qy$. On the contrary suppose $Px \neq Qy$, then utilizing inequality (3.1), we can get

$$F(d_o(Px, Qy), d_o(Rx, Sy), d_o(Px, Rx), d_o(Px, Sy), d_o(Qy, Sy), d_o(Rx, Qy)) \leq 0.$$

From (3.2) and (3.3), we have

$$F(d_o(Px, Qy), d_o(Px, Qy), 0, d_o(Px, Qy), 0, d_o(Px, Qy)) \leq 0$$

leads to a contradiction as $F \in \mathcal{F}_6$. Thus we get $Px = Qy$, this implies that $Px = Rx = Qy = Sy$ and

$$P^2x = PPx = PRx = RPx. \tag{3.4}$$

Next, we prove that $Px = P^2x$. If $P^2x \neq Px = Qy$. On utilizing inequality (3.1), with $x = Px$, $y = y$

$$F(d_o(P^2x, Qy), d_o(RPx, Sy), d_o(P^2x, RPx), d_o(P^2x, Sy), d_o(Qy, Sy), d_o(RPx, Qy)) \leq 0,$$

this implies that

$$F(d_o(P^2x, Px), d_o(P^2x, Px), 0, d_o(P^2x, Px), 0, d_o(P^2x, Px)) \leq 0,$$

which contradicts (2.1). Hence $P^2x = Px = Qy$, that is, $Px$ is a fixed point of $P$. Also $P^2x = RPx = Px$, this means that $Px$ is a fixed point of $R$.

Proceeding similarly, we prove that $Q^2y = Qy$. Therefore $Px = P^2x = Qy = Q^2y = QQy = QPx$ i.e. $QPx = Px$. This impliea that $Px$ is a fixed point of $Q$.

Moreover, $Px = P^2x = Qy = Q^2y = QQy = QSy = SQy = Px$, that is, $SPx = Px$. Therefore, $Px$ is a fixed point of $S$. Hence $Px$ is a common fixed point of $P, Q, R$ and $S$.

In order to prove the uniqueness, suppose that $z(= Px)$ and $w$ are two distinct fixed points of $P, Q, R$ and $S$. Then with inequality (3.1), we have

$$F(d_o(Pz, Qw), d_o(Rz, Sw), d_o(Pz, Rz), d_o(Pz, Sw), \\ d_o(Qw, Sw), d_o(Rz, Qw)) \leq 0.$$

Hence we have

$$F(d_o(z, w), d_o(z, w), d_o(z, z), d_o(z, w), d_o(w, w), d_o(z, w)) \leq 0,$$

this implies that

$$F(d_o(z, w), d_o(z, w), 0, d_o(z, w), 0, d_o(z, w)) \leq 0,$$

which is a contradiction as $F \in \mathcal{F}_6$. Thus $z = w$. Hence $z = Px$ is the unique common fixed point of $P, Q, R$ and $S$. This completes the proof. □

Following example substantiates the validity of Theorem 3.1.

**Example 3.2.** Let $X = [0, 1]$. Define a non-negative function $d_o : X \times X \to R^+$ by

$$d_o(x, y) = \begin{cases} x - y & if\ x \geq y \\ y - x + 1 & if\ x < y; \end{cases}$$

Then $d_o(x, y)$ is a non-negative function satisfying the property of oneness. $d_o(x, y) = 0$ when $x = y$. Define mappings $P, Q, R$ and $S$ by

$$Px = \frac{x}{6}, \qquad Qx = \frac{x}{8}, \qquad Rx = \frac{x}{2} \quad \text{and} \quad Sx = \frac{x}{3}.$$

Then, clearly pairs $(P, R)$ and $(Q, S)$ are non-vacuously weakly compatible pairs for coincidence point $x = 0$ in $X$.

Now invoking inequality (3.1) to

$$F(t_1, t_2, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $k \in (0,1)$.

$$F(d_o(Px,Qy),d_o(Rx,Sy),d_o(Px,Rx),d_o(Px,Sy),d_o(Qy,Sy),d_o(Rx,Qy))$$
$$= d_o(Px,Qy)$$
$$- k\max\{d_o(Rx,Sy),d_o(Px,Rx),d_o(Px,Sy),d_o(Qy,Sy),d_o(Rx,Qy)\}.$$

Thus we have to show that

$$d_o(Px,Qy)$$
$$\leq k\max\{d_o(Rx,Sy),d_o(Px,Rx),d_o(Px,Sy),d_o(Qy,Sy),d_o(Rx,Qy)\}. \tag{3.5}$$

By using routine calculation it is easy to verify (3.5), which is also demonstrated by the following figure, where in purple surface representing right hand side of (3.5) is dominating blue surface representing left hand side function $d_o(Px,Qy)$. Thus inequality (3.1) of Theorem 3.1 is satisfied for all $x, y \in [0,1]$.

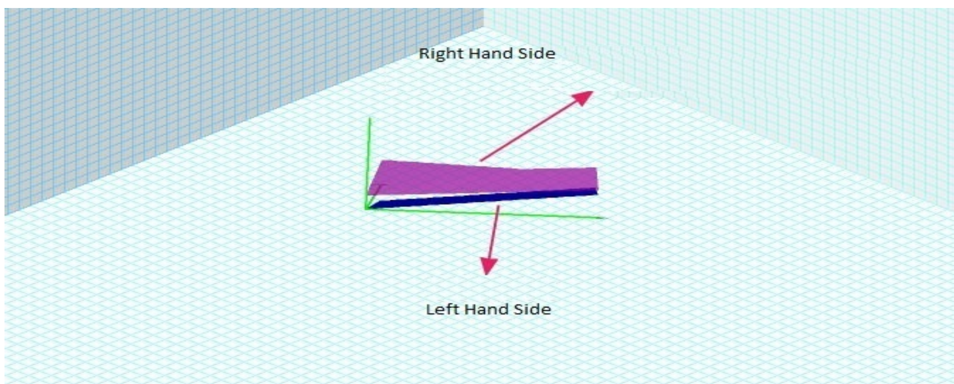


FIGURE 1

Therefore, all the conditions of Theorem 3.1 are satisfied. Notice that $x = 0$ remains fixed under $P, Q, R, S$ and which is indeed unique.

**Remark 3.3.** Notice that Theorem 3.1 never requires continuity and closedness of mappings and any condition on suitable containment between ranges of involved mappings. Also Theorem 3.1 remains true for metric spaces, quasi metric spaces as well as b-metric spaces.

By setting $P = Q$ and $R = S$ in Theorem 3.1, we deduce the following corollary involving a pair of mappings.

**Corollary 3.4.** *Let $X$ be a non-empty set and $d_o : X \times X \to X$ a non-negative function satisfying the property of oneness and $P$ and $R$ be two self mappings of $X$ such that*

$$F(d_o(Px,Py), d_o(Rx,Ry), d_o(Px,Rx), d_o(Px,Ry), \\ d_o(Py,Ry), d_o(Rx,Py)) \leq 0,$$

*for all $x, y \in X$ with $Px \neq Py$ and $F \in \mathcal{F}_6$. If pair $(P, R)$ is non-vacuously weakly compatible, then $P$ and $R$ have a unique common fixed point.*

Employing to Examples [2.2-2.16] of implicit function, following corollary involving several known as well as unknown results, are obtained.

**Corollary 3.5.** Let $X$ be a non-empty set and $d_o : X \times X \to R^+$ a non-negative function satisfying the property of oneness. If $P, Q, R$ and $S$ are four self mappings of $X$ such that any one of the following inequality is satisfied (for all $x, y \in X$ with $Px \neq Qy$).

I:

$$d_o(Px,Qy) \leq a.d_o(Rx,Sy) + b(d_o(Px,Rx) + d_o(Qy,Sy)) \\ + c(d_o(Px,Sy) + d_o(Rx,Qy)),$$

where $a, b, c \geq 0, a + 2c < 1$.

II:

$$d_o(Px,Qy) \leq a.d_o(Rx,Sy) \\ + b\max\{d_o(Px,Rx), d_o(Px,Sy), d_o(Rx,Qy)\} \\ + c\max\{d_o(Rx,Sy), d_o(Qy,Sy)\},$$

where $a, b, c \geq 0, a + b + c < 1$.

III:

$$d_o(Px,Qy) \leq k\max\{d_o(Rx,Sy), \\ \min\{d_o(Px,Rx), d_o(Qy,Sy)\}, \\ \min\{d_o(Px,Sy), d_o(Rx,Qy)\}\}, \quad k \in (0,1).$$

IV:

$$d_o(Px,Qy) \leq k\max\Big\{d_o(Rx,Sy), \frac{k}{2}\Big(d_o(Px,Rx) + d_o(Qy,Sy)\Big), \\ \frac{k}{2}\Big(d_o(Px,Sy) + d_o(Rx,Qy)\Big)\Big\}, \quad k \in (0,1).$$

V:

$$d_o(Px,Qy) \leq \phi\Big(\max\Big\{d_o(Rx,Sy), \frac{k}{2}\Big(d_o(Px,Rx) + d_o(Qy,Sy)\Big), \\ \frac{k}{2}\Big(d_o(Px,Sy) + d_o(Rx,Qy)\Big)\Big\}\Big),$$

where $k \in (0,1)$, $\phi : R^+ \to R^+$ such that $\phi$ is non-decreasing on $R^+$ and $0 < \phi(t) < t$, $\forall\, t \in (0, \infty)$.

VI:

$$d_o(Px, Qy) \leq \phi(d_o(Rx, Sy), d_o(Px, Rx), d_o(Px, Sy), d_o(Qy, Sy), d_o(Rx, Qy)),$$

where $\phi : {R^+}^5 \to R^+$ such that $\phi$ is non-decreasing and upper semi-continuous and $\phi(t,t,t,t,t) = \psi(t) < t$ for each $t > 0$, where $\psi : R^+ \to R^+$ is a mapping with $\psi(0) = 0$.

VII:

$$d_o(Px, Qy) \leq \phi(d_o(Rx, Sy), d_o(Px, Rx), d_o(Px, Sy), \sqrt{d_o(Qy, Sy) \cdot d_o(Rx, Qy)}),$$

where $\phi$ is non-decreasing and upper semi-continuous and $\phi(t,t,t,t) = \psi(t) < t$ for each $t > 0$ where $\psi : R^+ \to R^+$ is a mapping with $\psi(0) = 0$.

VIII:

$$d_o(Px, Qy) \leq k \max\{d_o(Rx, Sy), d_o(Px, Rx), d_o(Px, Sy), d_o(Qy, Sy), d_o(Rx, Qy)\},$$

where $k \in (0,1)$.

IX:

$$d_o(Px, Qy) \leq \phi(\max\{d_o(Rx, Sy), d_o(Px, Rx), d_o(Px, Sy), d_o(Qy, Sy), d_o(Rx, Qy)\}),$$

where $\phi : R^+ \to R^+$ and $\phi(t) < t$, for $t > 0$ and $\phi(0) = 0$.

X:

$$d_o(Px, Qy) \leq k \min\{d_o(Rx, Sy), \max\{d_o(Px, Rx), d_o(Px, Sy), d_o(Rx, Qy)\}, d_o(Qy, Sy)\},$$

where $k \in (0,1)$.

XI:

$$d_o(Px,Qy) \leq k \max\Big\{d_o(Rx,Sy), d_o(Px,Rx), d_o(Qy,Sy), \frac{d_o(Px,Sy)+d_o(Rx,Qy)}{2}\Big\},$$

where $k \in (0,1)$.

XII:

$$d_o(Px,Qy) \leq \lambda d_o(Rx,Sy) + \mu \frac{d_o(Px,Rx)\cdot d_o(Px,Sy)}{1+d_o(Rx,Sy)} + \gamma \frac{d_o(Qy,Sy)\cdot d_o(Rx,Qy)}{1+d_o(Rx,Sy)},$$

where $\lambda, \mu, \gamma \geq 0,\ \lambda+\mu+\gamma < 1$.

XIII:

$$d_o(Px,Qy) \begin{cases} \leq & \lambda . d_o(Rx,Sy) + \mu \frac{d_o(Rx,Sy)\cdot d_o(Qy,Sy)}{d_o(Rx,Sy)+d_o(Px,Sy)+d_o(Rx,Qy)}, \\ & \text{if } d_o(Rx,Sy)+d_o(Px,Sy)+d_o(Rx,Qy) \neq 0; \\ = 0, & \text{if } d_o(Rx,Sy)+d_o(Px,Sy)+d_o(Rx,Qy) = 0, \end{cases}$$

where $\lambda, \mu \geq 0,\ \lambda+\mu < 1$.

XIV:

$$d_o(Px,Qy) \leq k \max\{d_o(Rx,Sy), \sqrt{d_o(Px,Rx)\cdot d_o(Qy,Sy)}, \sqrt{d_o(Px,Sy)\cdot d_o(Rx,Qy)}\},$$

where $k \in (0,1)$.

XV:

$$d_o(Px,Qy)]^2 \leq k \max\{d_o(Rx,Sy)\cdot d_o(Px,Rx), d_o(Rx,Sy)\cdot d_o(Qy,Sy), d_o(Rx,Sy)\cdot d_o(Px,Sy), d_o(Qy,Sy)\cdot d_o(Rx,Qy)\},$$

where $k \in (0,1)$.

If $(P,R)$ and $(Q,S)$ are pairs of non-vacuously weakly compatible mappings, then $P,Q,R$ and $S$ have a unique common fixed point.

*Proof.* Proof follows from Theorem 3.1 and Examples [2.2-2.16]. □

**Remark 3.6.** Host of the corollaries corresponding to Condition $(I)$ to $(XV)$ are new results as these never require any conditions on containment of ranges amongst involved mappings. Some conditions listed in above corollary are well known and generalize certain relevant results of existing literature from symmetric space as well as metric spaces. In fact contraction Condition $(III)$, $(V)$, $(VI)$ and $(VII)$ of Corollary 3.5 are respectively Theorem 3.1 of Cho, Lee and Bae [5], Theorem 2.4 of Imdad, Ali and Khan [9], Theorem 3.3 of Sahu, Imdad and Kumar[17] and Theorem 3.4 of Sahu, Imdad and Kumar [17]. Consequently results proved in [5],[9] and in [17] can also be extended to our setting.

Restricting Corollary 3.5 $(V)$ to a pair of mappings $(P, R)$, following corollary is deduced.

**Corollary 3.7.** Let $X$ be a non-empty set and $d_o : X \times X \to R^+$ be a non-negative function satisfying the property of oneness. If $P, R$ are two self mappings of $X$ such that any one of the following inequality is satisfied (for all $x, y \in X$ with $Px \neq Ry$).

$$d_o(Px, Py) \leq \phi\Big( \max \Big\{ d_o(Rx, Ry), \frac{k}{2}\Big(d_o(Px, Rx) + d_o(Py, Ry)\Big), \frac{k}{2}\Big(d_o(Px, Ry) + d_o(Rx, Py)\Big) \Big\} \Big),$$

where $k \in (0, 1)$, $\phi : R^+ \to R^+$ such that $\phi$ is non-decreasing on $R^+$ and $0 < \phi(t) < t$, for all $t \in (0, \infty)$. If the pair $(P, R)$ is non-vacuously weakly compatible mappings then $P$ and $R$ have a unique common fixed point.

*Proof.* This corollary follows immediately in view of Example 2.5. □

**Remark 3.8.** Result mention in Corollary 3.7 generalizes Imdad, Ali and Khan [9] [ Theorem 2.1].

## 4. Fixed point results via generalized Φ-contraction in symmetric spaces

In this section, generalize Φ-contraction in the frame work of symmetric spaces is defined and utilized it to prove some fixed point results in which involved mappings satisfy $CLR_{ST}$ property. We denote Φ, the collection of all functions $\varphi : [0, \infty) \to [0, \infty)$, which are upper semi-continuous from the right, non-decreasing and satisfy

$$\limsup_{s \to t^+} \varphi(s) < t, \varphi(t) < t, \quad for\ all\ t > 0.$$

**Definition 4.1.** Let $(X,d)$ be a symmetric space and let $Y$ be a non-empty set. Let four mappings $P,Q,R,S : Y \to X$ satisfying the following condition

$$(d(Px,Qy))^2 \le a\varphi_1((d(Rx,Sy))^2)+b\varphi_2\Big(\frac{d(Px,Rx)d(Qy,Sy)}{1+d(Px,Sy)+d(Rx,Qy)}\Big), \quad (4.1)$$

for all $x,y \in X$ and some $\varphi_i \in \Phi (i = 1,2), a,b \ge 0, a < 1$. Condition (4.1) is called generalized $\Phi$-contraction.

Now we state and prove the main result of this section.

**Theorem 4.2.** Let $(X,d)$ be a symmetric space where d satisfies conditions $(1C)$ and $(HE)$. Let $Y$ be an arbitrary non-empty set and let $P,Q,R,S :$ $Y \to X$. Suppose that Condition (4.1) holds. If the pairs $(P,R)$ and $(Q,S)$ share the $(CLR_{RS})$ property, then $(P,R)$ and $(Q,S)$ have a coincidence point each. If, moreover, $Y = X$ and both pairs $(P,R)$ and $(Q,S)$ are non-vacuously weakly compatible, then $P,Q,R$ and $S$ have a unique common fixed point.

*Proof.* Since the pairs $(P,R)$ and $(Q,S)$ satisfy the $(CLR_{RS})$ property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $Y$ such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Sy_n = \eta,$$

where $\eta \in R(Y) \cap S(Y)$. As $\eta \in R(Y)$, there exists a point $v \in Y$ such that $Rv = \eta$. Now we show that $Pv = \eta$, suppose the contrary. Putting $x = v$ and $y = y_n$ in Condition (4.1), we get

$$(d(Pv,Qy_n))^2 \le a\varphi_1((d(Rv,Sy_n))^2)+b\varphi_2\Big(\frac{d(Pv,Rv)d(Qy_n,Sy_n)}{1+d(Pv,Sy_n)+d(Rv,Qy_n)}\Big),$$

this implis that

$$(d(Pv,Qy_n))^2 < a(d(Rv,Sy_n))^2+b\Big(\frac{d(Pv,Rv)d(Qy_n,Sy_n)}{1+d(Pv,Sy_n)+d(Rv,Qy_n)}\Big). \quad (4.2)$$

Passing to the upper limit as $n \to \infty$ in Condition (4.2) and using properties $(1C)$ and $(HE)$, we have

$$(d(Pv,\eta))^2 < a(d(Rv,\eta))^2+b\Big(\frac{d(Pv,\eta)d(\eta,\eta)}{1+d(Pv,\eta)+d(\eta,\eta)}\Big).$$

This implies that

$$(d(Pv,\eta))^2 < a(d(Rv,\eta))^2,$$

hence we have

$$(d(Pv,\eta))^2 < 0,$$

which is a contradiction. Therefore $Pv = Rv = \eta$, which shows that $v$ is a coincidence point of the pair $(P,R)$.

As $\eta \in S(Y)$, there exists a point $\nu \in Y$ such that $S\nu = \eta$. In order to prove that also $Q\nu = \eta$, suppose the contrary. Putting $x = v$ and $y = \nu$ in Condition (4.1), we have

$$(d(Pv, Q\nu))^2 \leq a\varphi_1((d(Rv, S\nu))^2) + b\varphi_2\Big(\frac{d(Pv, Rv)d(Q\nu, S\nu)}{1 + d(Pv, S\nu) + d(Rv, Q\nu)}\Big).$$

Hence we have

$$(d(Pv, Q\nu))^2 < a(d(Rv, S\nu))^2 + b\Big(\frac{d(Pv, Rv)d(Q\nu, S\nu)}{1 + d(Pv, S\nu) + d(Rv, Q\nu)}\Big).$$

This implies that

$$(d(\eta, Q\nu))^2 < a(d(\eta, \eta))^2 + b\Big(\frac{d(\eta, \eta)d(Q\nu, \eta)}{1 + d(\eta, \eta) + d(\eta, Q\nu)}\Big).$$

Therefore we have

$$(d(\eta, Q\nu))^2 < 0,$$

which is a contradiction. Thus $Q\nu = S\nu = \eta$, showing that $\nu$ is a coincidence point of the pair $(Q, S)$.

Assuming that $Y = X$. If both pairs $(P, R)$ and $(Q, S)$ are vacuously weakly compatible, $Pv = Rv$ and $Q\nu = S\nu$, imply that $P\eta = PRv = RPv = R\eta$ and $Q\eta = QS\nu = SQ\nu = S\eta$.

To prove that $\eta = P\eta$, suppose the contrary. Utilizing condition (4.1) with $x = \eta$ and $y = \nu$, we have

$$(d(P\eta, Q\nu))^2 \leq a\varphi_1((d(R\eta, S\nu))^2) + b\varphi_2\Big(\frac{d(P\eta, R\eta)d(Q\nu, S\nu)}{1 + d(P\eta, S\nu) + d(R\eta, Q\nu)}\Big).$$

Hence we have

$$(d(P\eta, Q\nu))^2 < a(d(R\eta, S\nu))^2 + b\Big(\frac{d(P\eta, R\eta)d(Q\nu, S\nu)}{1 + d(P\eta, S\nu) + d(R\eta, Q\nu)}\Big).$$

This implies that

$$(d(P\eta, \eta))^2 < ad(P\eta, \eta))^2 + b\Big(\frac{d(P\eta, P\eta)d(\eta, \eta)}{1 + d(P\eta, \eta) + d(P\eta, \eta)}\Big).$$

Therefore we have

$$(d(P\eta, \eta))^2 < a(d(P\eta, \eta))^2,$$

which is a contradiction. Thus $\eta = P\eta = R\eta$. Therefore, $\eta$ is a common fixed point of the pair $(P, R)$. Further, we shall show that $Q\eta = \eta$, suppose the contrary, putting $x = v$ and $y = \eta$ in Condition (4.1), we have

$$(d(P\eta, Q\eta))^2 \leq a\varphi_1((d(Rv, S\eta))^2) + b\varphi_2\Big(\frac{d(Pv, Rv)d(Q\eta, S\eta)}{1 + d(Pv, S\eta) + d(Rv, Q\eta)}\Big).$$

Hence we have

$$(d(P\eta, Q\eta))^2 < a(d(Rv, S\eta))^2 + b\Big(\frac{d(Pv, Rv)d(Q\eta, S\eta)}{1 + d(Pv, S\eta) + d(Rv, Q\eta)}\Big).$$

This implies that

$$(d(\eta, Q\eta))^2 < a(d(\eta, Q\eta))^2 + b\Big(\frac{d(\eta, \eta)d(Q\eta, Q\eta)}{1 + d(\eta, \eta) + d(\eta, Q\eta)}\Big).$$

Therefore we have

$$(d(\eta, Q\eta))^2 < a(d(\eta, Q\eta))^2,$$

which is a contradiction. Thus $\eta = Q\eta = T\eta$ and we can conclude that $\eta$ is a common fixed point of $P, Q, R$ and $S$.

For uniqueness, let $\rho$ be any other common fixed point of $P, Q, R$ and $S$. That is $Q\rho = T\rho = S\rho = P\rho = \rho$.

Putting $x = \eta$ and $y = \rho$ in Condition (4.1), we have

$$(d(P\eta, Q\rho))^2 \leq a\varphi_1((d(R\eta, S\rho))^2) + b\varphi_2\Big(\frac{d(P\eta, R\eta)d(Q\rho, S\rho)}{1 + d(P\eta, S\rho) + d(R\eta, Q\rho)}\Big).$$

Hence we have

$$(d(P\eta, Q\rho))^2 < a(d(R\eta, S\rho))^2 + b\Big(\frac{d(P\eta, R\eta)d(Q\rho, S\rho)}{1 + d(P\eta, S\rho) + d(R\eta, Q\rho)}\Big).$$

This implies that

$$(d(P\eta, Q\rho))^2 < a(d(\eta, \rho))^2 + b\Big(\frac{d(\eta, \eta)d(\rho, \rho)}{1 + d(\eta, \rho) + d(\eta, \rho)}\Big).$$

Therefore we have

$$(d(\eta, \rho))^2 < a(d(\eta, \rho))^2.$$

Hence $\eta = \rho$. Consequently $P, Q, R$ and $S$ have unique common fixed point. □

Restricting Theorem 4.2 to two mappings with $P = Q$ and $R = S$, we get the following corollary.

**Corollary 4.3.** Let $(X, d)$ be a symmetric space where $d$ satisfies conditions $(1C)$ and $(HE)$. Let $Y$ be an arbitrary non-empty set and let $P, R : Y \to X$. Suppose that following conditions hold.

$$(d(Px, Py))^2 \leq a\varphi_1((d(Rx, Ry))^2) + b\varphi_2\Big(\frac{d(Px, Rx)d(Py, Ry)}{1 + d(Px, Ry) + d(Rx, Py)}\Big), \quad (4.3)$$

for all $x, y \in X$ and some $\varphi_i \in \Phi (i = 1, 2), a, b \geq 0, a < 1$. If the mappings $P$ and $R$ share the $(CLR_R)$ property, then $P$ and $R$ have a coincidence point each. If, moreover, $Y = X$ and mappings $P$ and $R$ are non-vacuously weakly compatible, then $P$ and $R$ have a unique common fixed point.

**Example 4.4.** Let $Y = [2,20) \subset [2,+\infty) = X$ and let $X, Y$ be equipped with the symmetric $d(x,y) = |x-y|$ for all $x, y \in X$, which obviously satisfies $(1C)$ and $(HE)$. Consider the mappings $P, R : Y \to X$ given by

$$Px = \begin{cases} 2, & \text{i}f\ x \in \{2\} \cup (6,20) \\ 10, & \text{i}f\ x \in (2,6]. \end{cases} \quad and \quad Rx = \begin{cases} 2, & \text{i}f\ x = 2 \\ 20, & \text{i}f\ x \in (2,6] \\ \frac{x+8}{7}, & \text{i}f\ x \in (6,20). \end{cases}$$

Then we have $P(Y) = \{2,10\} \subsetneq \{2\} \cup \{20\} = R(Y)$. Then $P$ and $R$ satisfy the $CLR_R$ property and also satisfy the non-vacuously weakly compatibility. We consider two sequences, $\{x_n\} = \{2\}$ and $\{y_n\} = \{6 + \frac{1}{n}\}$, where $n \in N$. Then we have

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = 2,$$

where $2 \in P(Y) \cap R(Y)$, we note that $P(Y)$ and $R(Y)$ are not closed subset of $X$.

Now, define functions $\varphi_i : [0,+\infty) \to [0,+\infty)$ by

$$\varphi_i(t) = kt, \quad \text{with } \frac{2}{3} < k < 1, \text{ for } i \in \{1,2\} \text{ and all } t \geq 0.$$

It is clear that $\frac{d(Px,Rx)d(Py,Ry)}{1+d(Px,Ry)+d(Rx,Py)} \geq 0$, so in order to verify inequality (4.3) it is enough to show that

$$(d(Px,Py))^2 \leq a\varphi_1((d(Rx,Ry))^2).$$

Now following cases are discussed.

Case I: When $x, y \in \{2\} \cup (6,20)$. Then by Figure 2, we conclude that

$$(d(Px,Py))^2 \leq a\varphi_1((d(Rx,Ry))^2)$$

as red surface showing the function $a\varphi_1((d(Rx,Ry))^2)$ with $a = \frac{1}{2}$ and $\varphi_1(t) = kt$, $\frac{2}{3} < k < 1$ is dominating the purple surface representing $(d(Rx,Ry))^2$.

Case II: When $x \in \{2\} \cup (6,20)$ and $y \in (2,6]$. Figure 3 shows that right hand side (Red surface) is superimposing left hand side (purple surface), then inequality (4.3) is satisfied in this case.

On the similar pattern, one can verify inequality (4.3) for other possible cases. Thus all the conditions of Corollary 4.3 is satisfied for all $x, y \in Y$ and 2 is a unique common fixed point of $P$ and $Q$.

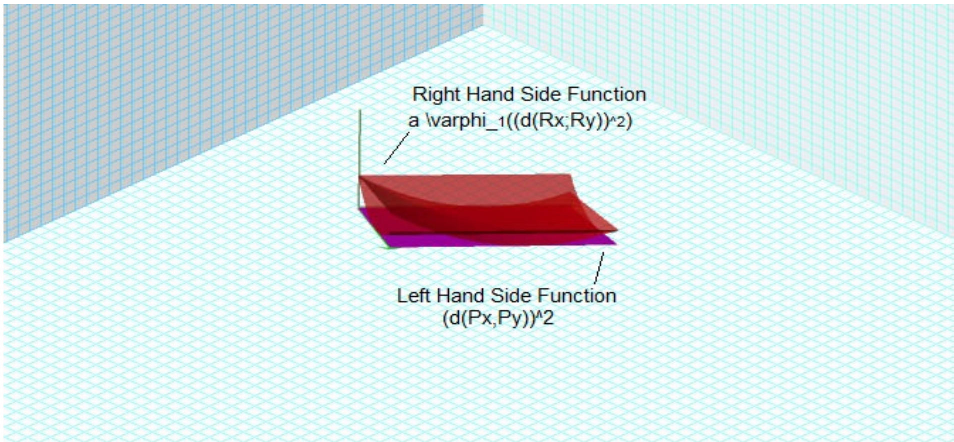


Figure 2

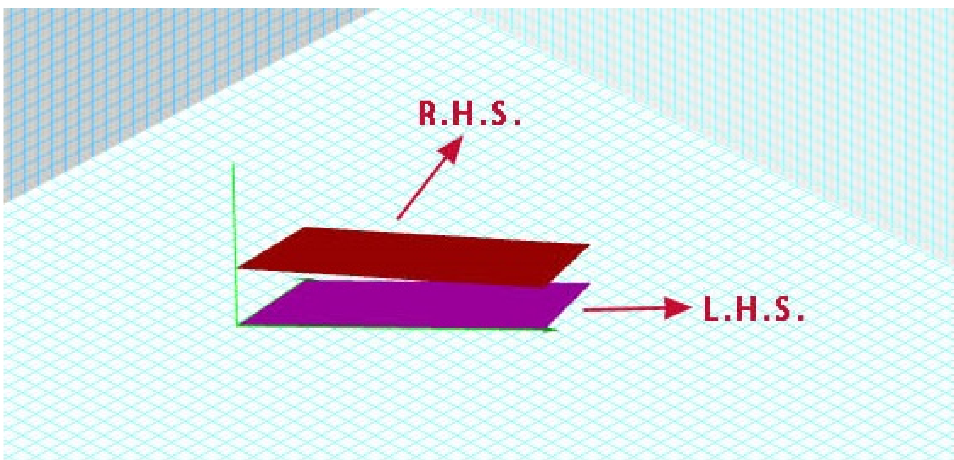


Figure 3

## 5. Application to system of integral equations

Consider the following system of integral equations:

$$\begin{aligned} u(t) &= \int_0^\lambda K_1(t,s,u(s))ds + g(t); \\ u(t) &= \int_0^\lambda K_2(t,s,u(s))ds + g(t); \\ u(t) &= \int_0^\lambda K_3(t,s,u(s))ds + g(t); \\ u(t) &= \int_0^\lambda K_4(t,s,u(s))ds + g(t), \end{aligned} \tag{5.1}$$

where $t \in I = [0, \lambda], \lambda > 0$.

In this section, we present a theorem which shows the existence and uniqueness of solution of the system (5.1).

Consider

$$C(I) = \{u : I \to R | u \text{ is continuous on } I\}.$$

Define $d_0 : C(I) \times C(I) \to R$ by

$$d_0(u, v) = \max_{t \in I} |u(t) - v(t)|, \quad \forall u, v \in C(I).$$

Then clearly $(C(I), d)$ is a non-negative function satisfying property of oneness.

Now, we define mapping $\lambda_i : C(I) \to C(I), \quad i = 1, 2, 3, 4$ by

$$\lambda_i x(t) = \int_0^\lambda K_i(t, s, x(s))ds + g(t), \quad t \in I,\ i \in \{1, 2, 3, 4\}.$$

Consider the following conditions:

(i) $K_i : I \times I \times R \to R, i \in \{1, 2, 3, 4\}$ and $g : I \to R$ are continuous;

(ii) there exists a continuous function $G : I \times I \to R^+$ such that

$$\begin{aligned} &|K_1(t, s, u(t)) - K_2(t, s, u(t))| \\ &\le G(t, s) \max\{|\lambda_3 u(t) - \lambda_4 v(t)|, \\ &|\lambda_1 u(t) - \lambda_3 u(t)|, |\lambda_1 u(t) - \lambda_4 v(t)|, \\ &|\lambda_2 v(t) - \lambda_4 v(t)|, |\lambda_3 u(t) - \lambda_2 v(t)|\}, \end{aligned} \tag{5.2}$$

for all $u, v \in C(I)$ and $t, s \in I$;

(iii) $\max_{t \in I} \int_0^\lambda G(t, s)ds = \alpha < \lambda$;

(iv) $\lambda_1\lambda_3 u = \lambda_3\lambda_1 u$ whenever $\lambda_1 u = \lambda_3 u$, for some $u \in C(I)$ and $\lambda_2\lambda_4 v = \lambda_4\lambda_2 v$ whenever $\lambda_2 v = \lambda_3 v$, for some $v \in C(I)$.

Now we prove the subsequent theorem which shows the existence and uniqueness solution of system of integral equations (5.1).

**Theorem 5.1.** Suppose that hypothesis [(i)-(iv)] hold. Then system of integral equations (5.1) has a unique solution in $C(I)$.

*Proof.* For all $u, v \in C(I)$ and by hypothesis (ii) and (iii), one can get

$$\begin{aligned} |\lambda_1 u(t) - \lambda_2 v(t)| &\le \int_0^\lambda |K_1(t, s, u(s)) - K_2(t, s, v(s))|ds \\ &\le \int_0^\lambda G(t, s) \max\{|\lambda_3 u(t) - \lambda_4 v(t)|, |\lambda_1 u(t) - \lambda_3 u(t)|, \\ &\quad |\lambda_1 u(t) - \lambda_4 v(t)|, |\lambda_2 v(t) - \lambda_4 v(t)|, |\lambda_3 u(t) - \lambda_2 v(t)|\}ds \\ &\le \alpha \max\{|\lambda_3 u(t) - \lambda_4 v(t)|, |\lambda_1 u(t) - \lambda_3 u(t)|, \\ &\quad |\lambda_1 u(t) - \lambda_4 v(t)|, |\lambda_2 v(t) - \lambda_4 v(t)|, |\lambda_3 u(t) - \lambda_2 v(t)|\}. \end{aligned}$$

By the routine calculation, it is easy to get

$$\begin{aligned} d_0(\lambda_1 u, \lambda_2 v) \leq & \alpha \max\{d_0(\lambda_3 u, \lambda_4 v), d_0(\lambda_1 u, \lambda_3 u), \\ & d_0(\lambda_1 u, \lambda_4 v), d_0(\lambda_2 v, \lambda_4 v), d_0(\lambda_3 u, \lambda_2 v)\}. \end{aligned}$$

Now, we notice that Corollary 3.5 (inequality VIII) applies to the operator $\lambda_i, i = 1, 2, 3, 4$ with $\lambda_1 = P, \lambda_2 = Q, \lambda_3 = R$ and $\lambda_4 = S$ and $\alpha = k \in (0, 1)$. Furthermore, hypothesis (iv) shows that the pairs $(\lambda_1, \lambda_3)$ and $(\lambda_2, \lambda_4)$ are non-vacuously weakly compatible and so $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ have a unique common fixed point. Then there exist a unique $h^* \in C(I)$, a common fixed point of $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$. That is $h^*$ is a unique solution to system of integral equations (5.1). □

## 6. Application to dynamic programmings

In this section, we assume that $X$ and $Y$ are Banach spaces, $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $R = (\infty, \infty)$ and $B(S)$ denote the set of all bounded real valued functions on $S$.

The basic form of the functional equation of dynamic programming is given by Bellman and Lee [3] as follows:

$$f(x) = opt_y H(x, y, f(T(x, y))),$$

where $x$ and $y$ represent the state and decision vectors respectively, $T$ represents the transformation of the process and $f(x)$ represents the optimal return function with initial state $x$ (here opt denotes max or min).

In this section, we study the existence and uniqueness of a common solution of the following functional equations arising in the dynamic programmings

$$f(x) = \sup_{y \in D} G_1(x, y, f(T(x, y))), \quad x \in S, \tag{6.1}$$

$$g(x) = \sup_{y \in D} G_2(x, y, g(T(x, y))), \quad x \in S, \tag{6.2}$$

$$h(x) = \sup_{y \in D} F_1(x, y, h(T(x, y))), \quad x \in S, \tag{6.3}$$

$$k(x) = \sup_{y \in D} G_2(x, y, k(T(x, y))), \quad x \in S, \tag{6.4}$$

where $T : S \times D \to S$ and $G_i, F_i : S \times D \times R \to R, i = 1, 2$.

Assume that the mapping $A_i$ and $T_i (i = 1, 2)$ are given by

$$A_i p(x) = \sup_{y \in D} G_i(x, y, p(T(x, y))), \tag{6.5}$$

$$T_i q(x) = \sup_{y \in D} F_i(x, y, q(T(x, y))), \tag{6.6}$$

for all $x \in S$; $p, q \in B(S), i = 1, 2$.

**Theorem 6.1.** Assume that the following conditions hold:

(i) $G_i$ and $F_i$ are bounded for $i = 1, 2$.

(ii)
$$|G_1(x, y, p(t)) - G_2(x, y, q(t))| \leq a|T_1p(t) - T_2q(t)| + b|A_1s(t) - T_1q(t)| + c|A_1p(t) - T_2q(t)| + d|A_2q(t) - T_2q(t)| + e|T_1p(t) - A_2q(t)|,$$
for all $(x, y) \in S \times D,\ p, q \in B(S),\ and\ t \in S$, where $a, b, c, d, e \geq 0, a + c + e < 1$ and the mappings $A_i$ and $T_i (i = 1, 2)$ are given as in (6.5) and (6.6).

(iii) For any $p, q \in B(S)$ with $A_1p(x) = T_1p(x)$ we have $A_1T_1p(x) = T_1A_1p(x)$ and with $A_2q(x) = T_2q(x)$ we have $A_2T_2q(x) = T_2A_2q(x)$.

Then the system of functional equations (6.1) - (6.4) have a unique common solution in $B(S)$.

*Proof.* For any $p, q \in B(S)$, let
$$d_o(p, q) = \sup_{x \in D} |p(x) - q(x)|;\ \ x \in S.$$
Then $(B(S), d_o)$ is a complete non-negative function satisfying property of oneness. From condition (iii) of Theorem 6.1, it is easy to conclude that the pair $(A_1, T_1)$ and $(A_2, T_2)$ are non-vacuously weakly compatible.

Let $p_1, p_2$ be any two points of $B(S)$, let $x \in S$ and $\lambda$ be any positive number, there exist $y_1, y_2 \in D$ such that
$$A_ip_i(x) < G_i(x, y_i, p_i(x_i)) + \lambda, \tag{6.7}$$
where $x_i = T(x, y_i),\ \ i = 1, 2$. Further,
$$A_1p_1(x) \geq G_1(x, y_2, p_1(x_2)), \tag{6.8}$$
$$A_2p_2(x) \geq G_2(x, y_1, p_2(x_1)), \tag{6.9}$$
from (6.7),(6.8) and (6.9), we get
$$\begin{aligned} A_1p_1(x) - A_2p_2(x) &< G_1(x, y_1, p_1(x_1)) - G_2(x, y_1, p_2(x_1)) + \lambda \\ &< |G_1(x, y_1, p_1(x_1)) - G_2(x, y_1, p_2(x_1))| + \lambda \\ &< a|T_1p_1(x_1) - T_2p_2(x_1)| + b|A_1p_1(x_1) - T_1p_2(x_1)| \\ &\quad + c|A_1p_1(x_1) - T_2p_2(x_1)| + d|A_2p_2(x_1) - T_2p_2(x_1)| \\ &\quad + e|T_1p_1(x_1) - A_2p_2(x_1)| + \lambda, \end{aligned}$$
$$\begin{aligned} A_1p_1(x) - A_2p_2(x) &< ad_o(T_1p_1, T_2p_2) + bd_o(A_1p_1, T_1p_1) + cd_o(A_1p_1, T_2p_2) \\ &\quad + dd_o(A_2p_2, T_2p_2) + ed_o(T_1p_1, A_2p_2) + \lambda, \end{aligned} \tag{6.10}$$

$$A_1p_1(x) - A_2p_2(x) \geq -\left[ad_o(T_1p_1, T_2p_2) + bd_o(A_1p_1, T_1p_1) + cd_o(A_1p_1, T_2p_2) + dd_o(A_2p_2, T_2p_2) + ed_o(T_1p_1, A_2p_2)\right]. \tag{6.11}$$

It follows from (6.10) and (6.11) that

$$|A_1p_1(x) - A_2p_2(x)| \leq ad_o(T_1p_1, T_2p_2) + bd_o(A_1p_1, T_1p_1) + cd_o(A_1p_1, T_2p_2) + dd_o(A_2p_2, T_2p_2) + ed_o(T_1p_1, A_2p_2) + \lambda. \tag{6.12}$$

As inequality (6.12) is true for any $x \in S$ and $\lambda > 0$ is any positive number, we get

$$d(A_1p_1(x) - A_2p_2(x)) \leq ad_o(T_1p_1, T_2p_2) + bd_o(A_1p_1, T_1p_1) + cd_o(A_1p_1, T_2p_2) + dd_o(A_2p_2, T_2p_2) + ed_o(T_1p_1, A_2p_2).$$

Therefore by Corollary 3.5 (inequality IX), $A_1, A_2, T_1, T_2$ have a unique common fixed point $h^* \in B(S)$. □

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