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CERTAIN FRACTIONAL INTEGRAL AND BETA TRANSFORM FORMULAS FOR THE EXTENDED APPELL'S AND LAURICELLA'S HYPERGEOMETRIC FUNCTIONS

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Abstract. Authors established some (presumably) new fractional integral and Beta transform formulas for the generalized extended Appell's and Lauricella's hypergeometric functions which have recently been introduced by Kim.

1. Introduction and Preliminaries

Generalizations and extensions of fractional operators (integral and differential operators) associated with special functions have recently been studied by many authors (see, for example, [3, 4, 7, 9])

Here, our aim is to develop some new fractional integral and Beta transform formulas for the generalized extended Appell's and Lauricella's hypergeometric functions which have recently been introduced by Kim [2].

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For our purpose, we begin by recalling the Riemann-Liouville fractional integral I^{θ}_{a+} (see, e.g., [10])

$$\left(I^{\theta}_{u+}f(\chi)\right)(\phi)=\frac{1}{\Gamma(\theta)}\int_{u}^{\phi}\frac{f(\chi)}{(\phi-\chi)^{1-\theta}}d\chi,\ (\theta\in\mathbb{C};\Re(\theta)>0,\phi>0) \tag{1.1}$$

where the function $f(\chi)$ is so constrained that the defining integral in (1.1) exists.

Very recently, Kim [2] introduced the following extended Appell's and Lauricella's hypergeometric functions type functions

$$\begin{aligned}&F_{1}^{(\alpha,\beta;\kappa,\mu)}\left[a,b,c;d;x,y;\gamma\right]\\&=\sum_{m,n}^{\infty}\frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(a+m+n,d-a\right)(b)_{m}(c)_{n}}{B\left(a,d-a\right)}\frac{x^{m}}{m!}\frac{y^{n}}{n!},\\&\qquad(|x|<1,|y|<1)\end{aligned} \tag{1.2}$$

$$\begin{aligned}&F_{2}^{(\alpha,\beta,\alpha',\beta';\kappa,\mu,\nu,\eta)}\left[a,b,c;d,e;x,y;\gamma\right]\\&=\sum_{m,n}^{\infty}\frac{(a)_{m+n}B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(b+n,d-b\right)B_{\gamma}^{(\alpha',\beta';\nu,\eta)}\left(c+m,e-c\right)}{B\left(a,d-a\right)B\left(c,e-c\right)}\frac{x^{m}}{m!}\frac{y^{n}}{n!},\\&\qquad(|x|+|y|<1)\end{aligned} \tag{1.3}$$

$$\begin{aligned}&F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a,b,c,d;e;x,y,z;\gamma\right]\\&=\sum_{m,n,r}^{\infty}\frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(a+m+n+r,e-a\right)}{B\left(a,e-a\right)}(b)_{m}(c)_{n}(d)_{r}\frac{x^{m}}{m!}\frac{y^{n}}{n!}\frac{z^{r}}{r!},\\&\qquad(\max\left\{|x|,|y|,|z|\right\}<1)\end{aligned} \tag{1.4}$$

where, $B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}(.,.)$ is the generalized beta function given by (see, for example, [5, 12])

$$B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(x,y\right)=\int_{0}^{1}t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{\gamma}{t^{\kappa}\left(1-t\right)^{\mu}}\right)dt, \tag{1.5}$$

$$(\min\left\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\kappa),\Re(\mu)\right\}>0)$$

For $\kappa=\mu=1$, it obviously reduces to the usual generalized Euler's beta function.

2. Fractional integral formulas

In this section, we present integral formulas of the generalized extended Appell's hypergeometric functions and the generalized extended Lauricella's hypergeometric function.

Theorem 2.1. *Let $u \in \mathbb{R}_+ = [0, \infty), \theta, \vartheta \in \mathbb{C}$, $\Re(\theta) > 0, \Re(\vartheta) > 0$. Then, for $\phi > u$ the following relation holds:*

$$\begin{aligned}&\left(I_{u+}^{\theta}\left\{(t-u)^{\vartheta-1}\, F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, \theta+\vartheta, c; d; \omega(t-u), y; \gamma\right]\right\}\right)(\phi)\\ &= (\phi-u)^{\theta+\vartheta-1}\,\frac{\Gamma(\vartheta)}{\Gamma(\theta+\vartheta)}\, F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, \vartheta, c; d; \omega\,(\phi-u), y; \gamma\right].\end{aligned} \tag{2.1}$$

Proof. Let $\mathfrak{F}$ be the left-hand side of the result (2.1). By making use of (1.1) and (1.2) and applying term-by-term fractional integration by virtue of the formula (see, e.g., [10])

$$\begin{aligned}&\left(I_{a+}^{\nu}\left\{(t-a)^{\mu-1}\right\}\right)(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}\,(x-a)^{\mu+\nu-1},\\ &\qquad (\mu, \nu \in \mathbb{C}; \Re(\mu) > 0, \Re(\nu) > 0).\end{aligned} \tag{2.2}$$

We have for $\phi > u$

$$\begin{aligned}\mathfrak{F} =& \sum_{m,n}^{\infty} \frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\,(a+m+n, d-a)\,(\theta+\vartheta)_m (c)_n}{B\,(a, d-a)}\,\frac{\omega^m}{m!}\,\frac{y^n}{n!}\\ &\times \left(I_{u+}^{\theta}\left\{(t-u)^{\vartheta+m-1}\right\}\right)(\phi).\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathfrak{F} =&(\phi-u)^{\theta+\vartheta-1}\;\frac{\Gamma(\vartheta)}{\Gamma(\theta+\vartheta)}\\ &\times \sum_{m,n}^{\infty} \frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\,(a+m+n, d-a)\,(\vartheta)_m (c)_n}{B\,(a, d-a)}\,\frac{(\omega(\phi-u))^m}{m!}\,\frac{y^n}{n!}.\end{aligned} \tag{2.4}$$

By using (1.2), we arrive at the (2.1). This completes the proof. □

Theorem 2.2. *Let $u \in \mathbb{R}_+ = [0, \infty), \theta, \vartheta \in \mathbb{C}$, $\Re(\theta) > 0, \Re(\vartheta) > 0$. Then, for $\phi > u$ the following relation holds:*

$$\begin{aligned}&\left(I_{u+}^{\theta}\left\{(t-u)^{\vartheta-1}\, F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, b, \theta+\vartheta; d; x, \omega(t-u); \gamma\right]\right\}\right)(\phi)\\ &= (\phi-u)^{\theta+\vartheta-1}\,\frac{\Gamma(\vartheta)}{\Gamma(\theta+\vartheta)}\, F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, b, \vartheta; d; x, \omega\,(\phi-u); \gamma\right].\end{aligned} \tag{2.5}$$

Proof. The proof would run parallel to that of Theorem 2.1. We omit the details. □

Theorem 2.3. *Let $u \in \mathbb{R}_+ = [0,\infty), \theta, \vartheta \in \mathbb{C}$, $\Re(\theta) > 0, \Re(\vartheta) > 0$. Then, for $\phi > u$ the following relation holds:*

$$\left(I_{u+}^{\theta}\left\{(t-u)^{\vartheta-1} F_2^{(\alpha,\beta,\alpha',\beta';\kappa,\mu,\nu,\eta)}\left[\theta+\vartheta, b, c; d, e; \omega(t-u), \xi(t-u); \gamma\right]\right\}\right)(\phi)$$
$$= (\phi-u)^{\theta+\vartheta-1}\frac{\Gamma(\vartheta)^2\Gamma(d-\vartheta)}{\Gamma(\theta+\vartheta)^2\Gamma(d-(\theta+\vartheta))}$$
$$\times F_2^{(\alpha,\beta,\alpha',\beta';\kappa,\mu,\nu,\eta)}\left[\vartheta, b, c; d, e; \omega(\phi-u), \xi(\phi-u); \gamma\right]. \tag{2.6}$$

Proof. Here, using the (1.3) and (1.1) and a similar argument as in the proof of Theorem 2.1 and Theorem 2.2 will establish (2.6). □

By using the (1.4) and (1.1), and a similar argument as in the proof of Theorem 2.1 and 2.2 will establish the following theorems.

Theorem 2.4. *Let $u \in \mathbb{R}_+ = [0,\infty), \theta, \vartheta \in \mathbb{C}$, $\Re(\theta) > 0, \Re(\vartheta) > 0$. Then, for $\phi > u$ the following relation holds.*

$$\begin{aligned}&\left(I_{u+}^{\theta}\left\{(t-u)^{\vartheta-1} F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, \theta+\vartheta, c, d; e; \omega(t-u), y, z; \gamma\right]\right\}\right)(\phi)\\ &= (\phi-u)^{\theta+\vartheta-1}\frac{\Gamma(\vartheta)}{\Gamma(\theta+\vartheta)} F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, \vartheta, c, d; e; \omega(\phi-u), y, z; \gamma\right].\end{aligned} \tag{2.7}$$

Theorem 2.5. *Let $u \in \mathbb{R}_+ = [0,\infty), \theta, \vartheta \in \mathbb{C}$, $\Re(\theta) > 0, \Re(\vartheta) > 0$. Then, for $\phi > u$ the following relation holds:*

$$\begin{aligned}&\left(I_{u+}^{\theta}\left\{(t-u)^{\vartheta-1} F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, \theta+\vartheta, d; e; x, \omega(t-u), z; \gamma\right]\right\}\right)(\phi)\\ &= (\phi-u)^{\theta+\vartheta-1}\frac{\Gamma(\vartheta)}{\Gamma(\theta+\vartheta)} F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, \vartheta, d; e; x, \omega(\phi-u), z; \gamma\right].\end{aligned} \tag{2.8}$$

Theorem 2.6. *Let $u \in \mathbb{R}_+ = [0,\infty), \theta, \vartheta \in \mathbb{C}$, $\Re(\theta) > 0, \Re(\vartheta) > 0$. Then, for $\phi > u$ the following relation holds:*

$$\begin{aligned}&\left(I_{u+}^{\theta}\left\{(t-u)^{\vartheta-1} F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, c, \theta+\vartheta; e; x, y, \omega(t-u); \gamma\right]\right\}\right)(\phi)\\ &= (\phi-u)^{\theta+\vartheta-1}\frac{\Gamma(\vartheta)}{\Gamma(\theta+\vartheta)} F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, c, \vartheta; e; x, y, \omega(\phi-u); \gamma\right].\end{aligned} \tag{2.9}$$

3. Beta transform formulas

In this section, we present some beta transform formulas of the generalized extended Appell's hypergeometric function and the generalized extended Lauricella's hypergeometric function.

For our purpose, we recall the beta transform of $f(z)$ defined by (see [11])

$$\mathcal{B}\{f(z):a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}\, f(z)\, dz. \tag{3.1}$$

Theorem 3.1. *The beta transform of the generalized extended Appell's hypergeometric function is*

$$\mathcal{B}\left\{ F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, l+s, c; d; x\,z, y; \gamma\right] : l, s\right\} = B(l,s)\, F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, l, c; d; x, y; \gamma\right]. \tag{3.2}$$

Proof. Using the definition of beta transform and (1.2), we have

$$\begin{aligned} &\mathcal{B}\left\{ F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, l+s, c; d; x\,z, y; \gamma\right] : l, s\right\} \\ &= \int_0^1 z^{l-1}(1-z)^{s-1} \\ &\quad \times \left\{ \sum_{m,n}^{\infty} \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}\left(a+m+n, d-a\right)(l+s)_m (c)_n}{B\left(a, d-a\right)} \frac{(x\,z)^m}{m!}\frac{y^n}{n!} \right\} dz, \end{aligned} \tag{3.3}$$

and changing the order of integration and summation, yields

$$\begin{aligned} &\mathcal{B}\left\{ F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, l+s, c; d; x\,z, y; \gamma\right] : l, s\right\} \\ &= \sum_{m,n}^{\infty} \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}\left(a+m+n, d-a\right)(l+s)_m (c)_n}{B\left(a, d-a\right)} \frac{(x\,)^m}{m!}\frac{y^n}{n!} \\ &\quad \times \left\{ \int_0^1 z^{l+m-1}(1-z)^{s-1} dz \right\}, \\ &= \sum_{m,n}^{\infty} \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}\left(a+m+n, d-a\right)\Gamma(l+s+m)(c)_n}{\Gamma(l+s) B\left(a, d-a\right)} \frac{\Gamma(l+m)\,\Gamma(s)}{\Gamma(l+m+s)}\frac{x^m}{m!}\frac{y^n}{n!}, \\ &= \sum_{m,n}^{\infty} \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}\left(a+m+n, d-a\right)(c)_n}{B\left(a, d-a\right)} \frac{\Gamma(l+m)\,\Gamma(s)}{\Gamma(l+s)}\frac{x^m}{m!}\frac{y^n}{n!}. \end{aligned}$$

Multiplying by $\frac{\Gamma(l)}{\Gamma(l)}$ and using Definition of classical beta function and (1.2), we get right-hand side of (3.2). □

Theorem 3.2. *The beta transform of the generalized extended Appell's hypergeometric function is*

$$\mathcal{B}\left\{ F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, b, l+s; d; x, y\,z; \gamma\right]\right\} = B(l,s)\, F_1^{(\alpha,\beta;\kappa,\mu)}\left[a, b, l; d; x, y; \gamma\right]. \tag{3.4}$$

Proof. The proof of Theorem 3.2 is on same parallel lines of Theorem 3.1. □

Theorem 3.3. *The beta transform of the generalized extended Lauricella's hypergeometric function is*

$$\begin{aligned}&\mathcal{B}\left\{ F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, s+l, c, d; e; x\,z, y, z; \gamma\right] : l, s\right\}\\&\quad = B(l,s)\, F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, l, c, d; e; x, y, z; \gamma\right];\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\mathcal{B}\left\{ F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, s+l, d; e; x, y\,z, z; \gamma\right] : l, s\right\}\\&\quad = B(l,s)\, F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, l, d; e; x, y, z; \gamma\right];\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\mathcal{B}\left\{ F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, c, s+l; e; x, y, z\,z; \gamma\right] ; l, s\right\}\\&\quad = B(l,s)\, F_{D,\gamma}^{(3;\alpha,\beta;\kappa,\mu)}\left[a, b, c, l; e; x, y, z; \gamma\right].\end{aligned} \tag{3.7}$$

Proof. Using the (1.4) and Definition of beta transform, and a similar argument as in the proof of Theorem 3.1 and 3.2 will establish the theorem 3.3. □

Concluding remarks: Here, it is important to mention that fractional integral and transform formulas involving multiple variables hypergeometric functions are very important to solving boundary value problems.

Our results may be useful in the study of boundary value problems, statistic theory and computer science.

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