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FIXED POINT THEOREMS FOR MULTIVALUED NONLINEAR CONTRACTION MAPPING IN G-METRIC SPACE

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Abstract. In this paper, we prove existence and uniqueness of fixed points for a multivalued mapping in complete G-metric space which satisfies some nonlinear contractive conditions. We show that the multivalued nonlinear contraction mapping reduces to multivalued linear contraction mapping by establishing the boundedness of orbit of the same. Our results generalize existing results in the literature.

1. Introduction

Over the past few decades, the metric fixed point theory has attracted considerable attention and become an important field of research in both pure and applied sciences. The well-known Banach contraction principle, formulated and proved by Banach in 1922, enunciates that any contractive self-mappings

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on a complete metric space has a unique fixed point. Because of its significance, Banach Contraction Principle has been extended and generalized in various directions (see [5],[6],[7],[12],[21],[49] and references therein).

Later, in 1969, Nadler [31] initiated the study of fixed points for multivalued mappings and generalized the Banach contraction principle. Inspired by this, many authors proved fixed point results for multivalued mappings satisfying different contractive conditions in metric space (see [8],[13],[14], [20],[23],[39]). Some of the important generalizations among these are given by Reich[39], obtained by extending the theorem of Boyd and Wong[4], to multivalued mappings whose range is compact sets, Mizoguzhi-Takahashi[23] and Kaneko[20].

On the other hand, several generalizations of standard metric spaces have appeared. Pseudo metric space, ultra metric space, partial metric space, cone metric space, b-metric space are few among such generalizations of metric space. Many authors have studied fixed and common fixed point theorems of various mappings in such spaces. We refer the reader to ([1],[17],[22],[32]-[38],[47],[48]). In sixties Gahler introduced the notion of 2-metric claiming that its a proper generalization of usual notion of metric spaces([15],[16]). But, different authors proved that the results obtained by Gahler are independent, rather than generalizations, of the corresponding results in metric spaces. Then Dhage, in 1993, introduced D-metric and attempted to develop topological structures in such spaces ([11], [10]). Later on, Mustafa and Sims demonstrated the flaws in the topological properties of this space[26] and consequently, they introduced the concept of G-metric space[30]. In recent years, various fixed point theorems for single valued and multivalued maps have been proved in G-metric space setting (refer [2],[3],[18],[19],[24],[25],[27],[28],[29],[41]-[46]). The wide application of such fixed point theorems attracted many researchers to study about G-metric space.

Motivated by above mentioned works, in this paper, we prove fixed point theorems for multivalued mappings, which generalize theorems of Nadler and Kaneko, in the setting of G-metric space. Then, we introduce the orbit of a multivalued mapping and prove the boundedness of orbit of multivalued mapping satisfying certain contractive condition in G-metric space. Further, we prove fixed point theorem equivalent to the theorem analogue to Nadler's from which we can easily deduce that, the multivalued nonlinear contractions can be reduced to multivalued linear contraction mapping. In the next section, we present the necessary definitions and results in G-metric spaces, which will be used for the rest of the paper.

2. Preliminaries

We give some of the basic concepts and results in G-metric spaces that will be needed in the sequel.

Definition 2.1. ([30]) Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

(P1) $G(x, y, z) = 0 \; if \; x = y = z, \; for \; x, y, z \in X$.
(P2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.
(P3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
(P4) $G(x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetry in all three variables).
(P5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$(rectangle inequality).

Then, the pair (X,G) is called G-metric space and G is called generalized metric or G-metric on X. A G-metric space is said to be symmetric if $G(x, y, y) = G(x, x, y)$.

We can always define a metric d_G from a given G-metric on X by

$$d_G(x, y) = G(x, y, y) + G(x, x, y).$$

If X is symmetric, then $d_G(x, y) = 2G(x, y, y) = 2G(x, x, y)$.

Example 2.2. ([30]) Let (X, d) be a metric space. Then the function $G : X \times X \times X \to [0, \infty)$ defined by

$$G(x, y, z) = \frac{1}{3}(d(x, y) + d(y, z) + d(x, z))$$

and

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

are G-metrics on X.

Example 2.3. The metric defined on $\mathbb{R}$ by $G(x, y, z) = |x - y| + |y - z|$ is not a G-metric. Take $x = 1, \; y = 5$ and $z = 3$ to see that, it does not have symmetry in all three variables.

Definition 2.4. ([30]) Let (X, G) be a G-metric space. A sequence $\{x_n\} \in X$ is said to be G-convergent to $x \in X$ if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x) < \epsilon$, for all $n, m \geq N$. Then x is called the limit of the sequence and we denote it by $x_n \to x \; or \; \lim_{n\to\infty} x_n = x$.

Proposition 2.5. ([30]) *Let (X, G) be a G-metric space. The following statements are equivalent:*

(i) $\{x_n\}$ *is G-convergent to* x.
(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.
(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$.
(iv) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.
(v) $d_G(x_n, x) \to 0$ *as* $n \to \infty$.

Example 2.6. Consider $\mathbb{R}$ with the G-metric $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$. Then the sequence $x_n = \dfrac{1}{n}$ is G-convergent in $\mathbb{R}$(for given $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$).

Definition 2.7. ([30]) Let (X, G) be a G-metric space. A sequence $\{x_n\} \in X$ is said to be G-Cauchy sequence if, for any $\epsilon > 0$ there exists a positive integer N such that, $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$. If every G-Cauchy sequence in X is G-convergent in X, then (X, G) is G-complete.

Example 2.8. Consider $X = [0, \infty]$ with $G(x, y, z) = |x-y|+|y-z|+|x-z|$. The sequence $x_n = \dfrac{1}{n^2}$ is G-Cauchy for, given $\epsilon > 0$ and $m > n > l$, choose $N > \dfrac{1}{\sqrt{\epsilon}}$, then $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$.

Kaewcharoen *et al.* [19] introduced the concept of Hausdorff G-distance and proved some properties of the same.

Definition 2.9. Let X be a G-metric space and $CB(X)$ denotes the family of all non-empty closed bounded subsets of X. Let $A, B, C \in CB(X)$, the Hausdorff G-distance on $CB(X)$ is defined by

$$H_G(A, B, C) = \max\left\{\sup_{x\in A} G(x, B, C),\ \sup_{y\in B} G(A, y, C),\ \sup_{z\in C} G(A, B, z)\right\},$$

where

$$\begin{aligned} G(x, B, C) &= d_G(x, B) + d_G(B, C) + d_G(x, C), \\ d_G(x, B) &= \inf\{d_G(x, y),\ y \in B\}, \\ d_G(A, B) &= \inf\{d_G(x, y),\ x \in A\ ,\ y \in B\}. \end{aligned}$$

Lemma 2.10. ([19]) *Let* (X, G) *be a G-metric space and* $A, B \in CB(X)$. *Then for each* $a \in A$, *we have*

$$G(a, B, B) \leq H_G(A, B, B).$$

Lemma 2.11. ([19]) *Let* (X, G) *be a G-metric space. If* $A, B \in CB(X)$ *and* $a \in A$, *then for each* $\epsilon > 0$, *there exists* $b \in B$ *such that*

$$G(a, b, b) \leq H_G(A, B, B) + \epsilon.$$

3. Main results

First we state two lemmas which will be useful in the sequel. The results follow directly from the definition of Hausdorff G-distance.

Lemma 3.1. *Let (X, G) be a G-metric space. Let $x \in X$ and $K \in CB(X)$. Then there exists $k \in K$ such that $G(x, k, k) \leq G(x, K, K) + \epsilon$.*

Lemma 3.2. *Let (X, G) be a G-metric space. Let $x \in X$ and $K \in CB(X)$. Then $H_G(\bar{A}, \bar{B}, \bar{C}) = H_G(A, B, C)$, where $\bar{A}$ denotes the closure of A.*

In the next theorem, we prove the existence of fixed points for a multivalued mapping in a G-metric space which is an analogous to Nadler's theorem in G-metric space setting.

Theorem 3.3. *Let (X, G) be a complete G-metric space. Let $T : X \to CB(X)$ be a multivalued mapping satisfying*

$$H_G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{3.1}$$

where $0 < k < 1$, for all $x, y, z \in X$. Then T has a fixed point in X. Further, if we assume $x \in Tx$, $y \in Ty$ and $G(x, y, y) \leq H_G(Tx, Ty, Ty)$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. If $x_0 \in Tx_0$, there is nothing to prove. Suppose $x_0 \notin Tx_0$ and since Tx_0 is non-empty, choose $x_1 \in Tx_0$. Since $Tx_0, Tx_1 \in CB(X)$ and $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$ such that

$$G(x_1, x_2, x_2) \leq H_G(Tx_0, Tx_1, Tx_1) + k.$$

Since $Tx_1, Tx_2 \in CB(X)$ and $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that

$$G(x_2, x_3, x_3) \leq H_G(Tx_1, Tx_2, Tx_2) + k^2.$$

Continuing this process, we obtain a sequence $\{x_n\}_{n=1}^{\infty}$ of points of X such that, $x_{n+1} \in Tx_n$ and

$$G(x_n, x_{n+1}, x_{n+1}) \leq H_G(Tx_{n-1}, Tx_n, Tx_n) + k^n, \ \forall n \geq 1. \tag{3.2}$$

Now, using (3.1) and (3.2),

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &\leq H_G(Tx_{n-1}, Tx_n, Tx_n) + k^n \\
&\leq kG(x_{n-1}, x_n, x_n) + k^n \\
&\leq k[H_G(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) + k^{n-1}] + k^n \\
&\leq k^2[G(x_{n-2}, x_{n-1}, x_{n-1}) + k^{n-2}] + 2k^n \\
&\vdots \\
&\leq k^n G(x_0, x_1, x_1) + nk^n.
\end{aligned}$$

Thus

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1) + nk^n, \ \forall n \geq 1, \tag{3.3}$$

which implies that for any n and m,

$$\begin{aligned}
&G(x_n, x_{n+m}, x_{n+m}) \\
&\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+m}, x_{n+m}) \\
&\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) \\
&\quad + \cdots + G(x_{n+m-2}, x_{n+m-1}, x_{n+m-1}) + G(x_{n+m-1}, x_{n+m}, x_{n+m}) \\
&\leq k^n G(x_0, x_1, x_1) + nk^n + k^{n+1} G(x_0, x_1, x_1) + (n+1)k^{n+1} \\
&\quad + \cdots + k^{n+m-1} G(x_0, x_1, x_1) + (n+m-1)k^{n+m-1} \\
&= \sum_{i=n}^{n+m-1} k^i G(x_0, x_1, x_1) + \sum_{i=n}^{n+m-1} ik^i.
\end{aligned}$$

Since $0 < k < 1$, each term on RHS converges and thus, $G(x_n, x_{n+m}, x_{n+m}) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a G-Cauchy sequence. So, the sequence converges to some $x \in X$.

Now, we shall prove that $x \in Tx$. Consider

$$G(x_{n+1}, Tx, Tx) \leq H_G(Tx_n, Tx, Tx) \leq kG(x_n, x, x).$$

Since $x_n \to x$, RHS goes to 0 as $n \to \infty$. Thus

$$G(x, Tx, Tx) \to 0 \ as \ n \to \infty.$$

For Tx is closed, it follows that $x \in Tx$. Now, assume $x \in Tx, \ y \in Ty$ and $G(x, y, y) \leq H_G(Tx, Ty, Ty)$. Suppose $x \neq y$, consider

$$G(x, y, y) \leq H_G(Tx, Ty, Ty) \leq kG(x, y, y) < G(x, y, y),$$

which leads to a contradiction. So we have $x = y$. This completes the proof. □

Example 3.4. Let $X = [0, 1]$ with the G-metric $G(x, y, z) = |x - y| + |y - z| + |x - z|$. The mapping $T : X \to CB(X)$ is defined by $Tx = \left[0, \frac{x}{10}\right]$ and T has a unique fixed point.

Without loss of generality, assume that $x < y < z$, then $Tx \subset Ty \subset Tz$.

$$\begin{aligned}
&H_G(Tx, Ty, Tz) \\
&= \max\left\{ \sup_{a \in Tx} G(a, Ty, Tz), \ \sup_{b \in Ty} G(Tx, b, Tz), \ \sup_{c \in Tz} G(Tx, Ty, c) \right\}
\end{aligned}$$

and

$$d_G(x,y) = d(x,y,y) + d(x,x,y) = 4|x-y|.$$

Now, we calculate

$$\begin{array}{rcl} d_G(Ty,Tz) & = & \inf\{4|b-c|,\ b \in [0, \frac{y}{10}],\ c \in [0, \frac{z}{10}]\} = 0, \\ d_G(a,Ty) & = & \inf\{4|a-b|,\ b \in [0, \frac{y}{10}]\} = 0, \\ d_G(a,Tz) & = & \inf\{4|a-c|,\ c \in [0, \frac{z}{10}]\} = 0. \end{array}$$

Thus, $\sup\limits_{a \in Tx} G(a,Ty,Tz) = 0$, and

$$\begin{array}{rcl} d_G(Tx,Tz) & = & \inf\{4|a-c|,\ a \in [0, \frac{x}{10}],\ c \in [0, \frac{z}{10}]\} = 0, \\ d_G(b,Tz) & = & \inf\{4|b-c|,\ c \in [0, \frac{z}{10}]\} = 0, \\ d_G(b,Tx) & = & \inf\{4|b-a|,\ a \in [0, \frac{x}{10}]\} = \begin{cases} 0\ if\ \ b \in [0, \frac{x}{10}] \\ 4b - \frac{4x}{10}\ if\ \ b \geq \frac{x}{10}. \end{cases} \end{array}$$

So, we get

$$\sup_{b \in Ty} G(Tx,b,Tz) = \frac{4y-4x}{10}.$$

Similarly,

$$G(Tx,Ty,c) = \begin{cases} 0\ ,\ if c \in [0, \frac{x}{10}] \\ 4c - \frac{4x}{10}\ ,\ if c \in [\frac{x}{10}, \frac{y}{10}] \\ 8c - \frac{4y}{10} - \frac{4x}{10}\ ,\ if c \in [\frac{y}{10}, \frac{z}{10}], \end{cases}$$

which gives

$$\sup_{c \in Tz} G(Tx,Ty,c) = \frac{8z-4y-4x}{10}.$$

Hence we have

$$\begin{array}{rcl} H_G(Tx,Ty,Tz) & = & \max\left\{0, \frac{4y-4x}{10}, \frac{8z-4y-4x}{10}\right\} \\ & = & \frac{8z-4y-4x}{10} \\ & \leq & \frac{8z-8x}{10} \\ & = & \frac{8}{10}|z-x| \\ & \leq & \frac{8}{10}\Big(|x-y| + |y-z| + |x-z|\Big) \\ & = & \frac{8}{10}G(x,y,z). \end{array}$$

Therefore,

$$H_G(Tx, Ty, Tz) \leq \frac{8}{10} G(x, y, z),$$

and T has a unique fixed point which is zero.

In a G-metric space, a subset $K \subseteq X$ is said to be proximinal if, for each $x \in X$, there exists an element $k \in K$ such that, $G(x, k, k) = G(x, K, K)$. We shall denote this family by $P(X)$. A mapping $\phi : X \times X \times X \to [0, \infty)$ is called compactly positive if,

$$\inf\{\phi(x, y, z) : \ a \leq G(x, y, z) \leq b\} > 0,$$

for each finite interval $[a, b] \subseteq (0, \infty)$. Again, we say $T : X \to P(X)$ weakly contractive if, there exists a mapping ϕ, compactly positive, such that

$$H_G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(x, y, z),$$

for each $x, y \in X$.

Next, we give an equivalent definition of weakly contractive multivalued mapping.

Proposition 3.5. *Let (X, G) be a G-metric space and $T : X \to CB(X)$. The following statements are equivalent:*

(a) *T is weakly contractive.*
(b) *$H_G(Tx, Ty, Tz) \leq h(x, y, z)G(x, y, z)$ for some non negative function $h(x, y, z)$ satisfying*

$$\sup\{h(x, y, z) \ : \ a \leq G(x, y, z) \leq b\} < 1,$$

for each closed interval $[a, b] \subset (0, \infty)$.

Proof. $(a) \implies (b)$: Since T is weakly contractive, there exists compactly positive mapping ϕ such that, $H_G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(x, y, z)$. Define

$$h(x, y, z) = \begin{cases} 1 - \dfrac{\phi(x, y, z)}{G(x, y, z)}, & G(x, y, z) \neq 0 \\ 0 & , \ G(x, y, z) = 0. \end{cases} .$$

Then, $H_G(Tx, Ty, Tz) \leq h(x, y, z)G(x, y, z)$ and it can be easily verified that, $h(., ., .)$ satisfies the required conditions.
$(b) \implies (a)$: To prove the reverse implication, define $\phi(., ., .)$ using $h(., ., .)$, then we have desired result. □

The following theorem generalizes Theorem 2 of [20] proved by Kaneko.

Theorem 3.6. *Let (X, G) be a complete G-metric space and let $T : X \to P(X)$ be such that*

$$H_G(Tx, Ty, Tz) \leq h(x, y, z)G(x, y, z), \ \forall \ x, y, z \in X, \tag{3.4}$$

and for some non-negative function $h(x, y, z)$ satisfying

$$\sup\{h(x, y, z) \ : \ a \leq G(x, y, z) \leq b\} < 1,$$

for each closed interval $[a, b] \subset (0, \infty)$. Assume also that if, $(x_n, y_n, z_n) \in X \times X \times X$ is such that

$$\lim_{n\to\infty} G(x_n, y_n, z_n) = 0 \ \text{ then, } \lim_{n\to\infty} h(x_n, y_n, z_n) = k, \text{ for some } k \in [0, \infty). \tag{3.5}$$

Then T has a fixed point in X. Also, if we assume $x \in Tx$, $y \in Ty$ and $G(x, y, y) \leq H_G(Tx, Ty, Ty)$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $Tx_0 \in P(X)$. If $x_0 \in Tx_0$, the proof is done. If not, there is $x_1 \in Tx_0$ such that

$$G(x_0, x_1, x_1) = G(x_0, Tx_0, Tx_0).$$

Continuing this process to generate the sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that $x_n \in Tx_{n-1}$,

$$G(x_n, x_{n+1}, x_{n+1}) = G(x_n, Tx_n, Tx_n).$$

Now

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(x_n, Tx_n, Tx_n) \\ &\leq H_G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq h(x_{n-1}, x_n, x_n)G(x_{n-1}, x_n, x_n) \\ &< G(x_{n-1}, x_n, x_n), \end{aligned}$$

which implies that, $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a monotone decreasing sequence which is bounded below by 0 and hence it converges.

Claim: $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0$. Suppose that $\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = \epsilon$. Consider

$$\begin{aligned} &G(x_n, x_{n+1}, x_{n+1}) \\ &\leq h(x_{n-1}, x_n, x_n)G(x_{n-1}, x_n, x_n) \\ &\leq h(x_{n-1}, x_n, x_n)h(x_{n-2}, x_{n-1}, x_{n-1})G(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\vdots \\ &\leq h(x_{n-1}, x_n, x_n)h(x_{n-2}, x_{n-1}, x_{n-1}) \cdots h(x_0, x_1, x_1)G(x_0, x_1, x_1). \end{aligned}$$

So,

$$\epsilon < G(x_n, x_{n+1}, x_{n+1}) < G(x_0, x_1, x_1).$$

Let $k = \sup\{h(x, y, z) \ : \ \epsilon \leq G(x, y, z) \leq G(x_0, x_1, x_1)\} < 1$. Then

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1),$$

which implies, $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

Let $k_0 < 1$, choose $N \in \mathbb{N}$ such that, $h(x_n, x_{n+1}, x_{n+1}) \leq k_0, \ \forall\ n \geq N$. Thus

$$\begin{aligned}
&G(x_n, x_{n+1}, x_{n+1})\\
&\leq h(x_{n-1}, x_n, x_n)h(x_{n-2}, x_{n-1}, x_{n-1}) \cdots h(x_0, x_1, x_1)G(x_0, x_1, x_1)\\
&\leq k_0^{n-N} r^N G(x_0, x_1, x_1)\\
&\leq R^n G(x_0, x_1, x_1),
\end{aligned}$$

where $R = \max\{k_0, r\}$ and $r = \max\{h(x_i, x_{i+1}, x_{i+1}) \ : \ i = 0, 1, \cdots N-1\} < 1$. So we have for $m > n$,

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq \sum_{i=n}^{m-1} G(x_i, x_{i+1}, x_{i+1})\\
&\leq \sum_{i=n}^{m-1} R^i G(x_0, x_1, x_1)\\
&\leq \frac{R^n}{1-R} G(x_0, x_1, x_1).
\end{aligned}$$

Hence $\{x_n\}$ is a G-Cauchy sequence and so $x_n \to x$ in X.

Now, we prove that x is a fixed point of T. We have

$$G(x_n, Tx, Tx) \leq H_G(Tx_{n-1}, Tx, Tx) \leq h(x_{n-1}, x, x)G(x_{n-1}, x, x).$$

As $n \to \infty$, $G(x, Tx, Tx) = 0$ and thus $x \in Tx$. Now, assume $x \in Tx, \ y \in Ty$ and $G(x, y, y) \leq H_G(Tx, Ty, Ty)$. Suppose $x \neq y$, consider

$$G(x, y, y) \leq H_G(Tx, Ty, Ty) \leq h(x, y, y)G(x, y, y) < G(x, y, y),$$

which is a contradiction. So we have $x = y$. □

In the next theorem, we have used the condition of weakly contractivity along with a weaker condition and extended Theorem 3.6 to multivalued mappings whose range is closed bounded sets.

Theorem 3.7. *Let (X, G) be a complete G-metric space and $T : X \to CB(X)$ be weakly contractive. Also assume that*

$$\liminf_{b\to 0} \frac{\lambda(a, b)}{b} > 0, \tag{3.6}$$

where $\lambda(a, b) = \inf\{\phi(x, y, z) \mid x, y, z \in X, \ a \leq G(x, y, z) \leq b\}$, for each finite interval $[a, b] \subset (0, \infty)$. Then T has a fixed point. If we assume $x \in Tx, \ y \in Ty$ and $G(x, y, y) \leq H_G(Tx, Ty, Ty)$, then T has a unique fixed point in X.

Proof. Let $x_1 \in X$ and $r_1 = G(x_1, Tx_1, Tx_1)$. If $r_1 > 0$, take $R_1 = \dfrac{\lambda(r_1, 2r_1)}{2r_1}$. It can be easily seen that, $R_1 < 1$ and since ϕ is compactly positive, $R_1 > 0$. Now, ϵ_1 be such that $0 < \epsilon_1 < \min\left\{\dfrac{R_1}{1-R_1}, 1\right\}$. Choose $x_2 \in Tx_1$ such that

$$G(x_1, x_2, x_2) < (1+\epsilon_1)G(x_1, Tx_1, Tx_1).$$

We have

$$G(x_2, Tx_2, Tx_2) \leq H_G(Tx_1, Tx_2, Tx_2) \leq G(x_1, x_2, x_2) - \phi(x_1, x_2, x_2).$$

Now,

$$\begin{aligned}
&G(x_1, Tx_1, Tx_1) - G(x_2, Tx_2, Tx_2)\\
&\geq \frac{1}{1+\epsilon_1}G(x_1, x_2, x_2) - [G(x_1, x_2, x_2) - \phi(x_1, x_2, x_2)]\\
&= \frac{1}{1+\epsilon_1}G(x_1, x_2, x_2) - \left[1 - \frac{\phi(x_1, x_2, x_2)}{G(x_1, x_2, x_2)}\right]G(x_1, x_2, x_2)\\
&\geq \frac{1}{1+\epsilon_1}G(x_1, x_2, x_2) - \left[1 - \frac{\lambda(r_1, 2r_1)}{2r_1}\right]G(x_1, x_2, x_2)\\
&= \left[\frac{1}{1+\epsilon_1} - (1-R_1)\right]G(x_1, x_2, x_2).
\end{aligned}$$

Hence we have

$$G(x_1, Tx_1, Tx_1) - G(x_2, Tx_2, Tx_2) \geq \left[\frac{1}{1+\epsilon_1} - (1-R_1)\right]G(x_1, x_2, x_2). \quad (3.7)$$

Note that, $\left[\dfrac{1}{1+\epsilon_1} - (1-R_1)\right] > 0$. Now let $r_2 = G(x_2, Tx_2, Tx_2)$ and $R_2 = \dfrac{\lambda(r_2, \frac{3}{2}r_2)}{\frac{3}{2}r_2}$ if $r_2 > 0$. Using (3.7), $r_1 - r_2 > 0$ which implies, $\dfrac{r_1}{r_2} > 1$. Now, choose ϵ_2 to be, $0 < \epsilon_2 < \min\left\{\dfrac{R_2}{1-R_2}, \dfrac{r_1}{r_2} - 1, \dfrac{1}{2}\right\}$, and $x_3 \in Tx_2$ such that

$$G(x_2, x_3, x_3) < (1+\epsilon_2)G(x_2, Tx_2, Tx_2).$$

Proceeding as above, we obtain

$$G(x_2, Tx_2, Tx_2) - G(x_3, Tx_3, Tx_3) \geq \left[\frac{1}{1+\epsilon_2} - (1-R_2)\right]G(x_2, x_3, x_3).$$

Iteratively, we get $r_n = G(x_n, Tx_n, Tx_n)$ and $R_n = \dfrac{\lambda(r_n, \frac{n+1}{n}r_n)}{\frac{n+1}{n}r_n}$. Now, choose ϵ_n such that, $0 < \epsilon_n < \min\left\{\dfrac{R_n}{1-R_n}, \dfrac{r_{n-1}}{r_n} - 1, \dfrac{1}{n}\right\}$ and find $x_{n+1} \in Tx_n$

satisfying

$$G(x_n, x_{n+1}, x_{n+1}) < (1+\epsilon_n)G(x_n, Tx_n, Tx_n),$$

and

$$\begin{aligned} &G(x_n, Tx_n, Tx_n) - G(x_{n+1}, Tx_{n+1}, Tx_{n+1}) \\ &\geq \left[\frac{1}{1+\epsilon_n} - (1-R_n)\right] G(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

If $r_n = 0$ for any n, then the proof is done. If $r_n > 0$, for all n, then by (3.6) $\liminf R_n > 0$ and $\limsup(1-R_n) < 1$. Thus we have, for some $k > 0$,

$$G(x_n, Tx_n, Tx_n) - G(x_{n+1}, Tx_{n+1}, Tx_{n+1}) \geq kG(x_n, x_{n+1}, x_{n+1}),$$

for sufficiently large n. The sequence $\{r_n\}$ converges as it is monotonically decreasing sequence and bounded below. Consider for $m > n$,

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \sum_{i=n}^{m-1} G(x_i, x_{i+1}, x_{i+1}) \\ &\leq \frac{1}{k}\sum_{i=n}^{m-1}[G(x_i, Tx_i, Tx_i) - G(x_{i+1}, Tx_{i+1}, Tx_{i+1})] \\ &= \frac{1}{k}[G(x_n, Tx_n, Tx_n) - G(x_m, Tx_m, Tx_m)] \\ &= \frac{1}{k}(t_n - t_m) \to 0 \ as \ n \to \infty. \end{aligned}$$

Thus $\{x_n\}$ is G-Cauchy sequence in X. Since X is G-complete, $x_n \to x \in X$.

Now it remains to prove that x is a fixed point for T. Consider

$$\begin{aligned} G(x_n, Tx, Tx) &\leq H_G(Tx_{n-1}, Tx, Tx) \\ &\leq G(x_{n-1}, x, x) - \phi(x_{n-1}, x, x) \\ &\leq G(x_{n-1}, x, x). \end{aligned}$$

As $n \to \infty$, we get $G(x, Tx, Tx) \to 0$ and thus $x \in Tx$.

Next, assume $x \in Tx,\ y \in Ty$ are two fixed points of T and $G(x, y, y) \leq H_G(Tx, Ty, Ty)$. Suppose $x \neq y$, consider

$$G(x, y, y) \leq H_G(Tx, Ty, Ty) \leq G(x, y, y) - \phi(x, y, y) < G(x, y, y),$$

which leads to a contradiction and thus $x = y$. □

Example 3.8. Let $X = [0, \infty)$ with the G-metric $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$. Define $T : X \to CB(X)$ by $Tx = \left[0, \frac{x}{5(1+x)}\right]$. For $x \leq y \leq z$, by carrying out the calculation as in Example 3.4, we can see that

$H_G(Tx, Ty, Tz) \leq h(x, y, z)G(x, y, z)$, where $h(x, y, z) : X \times X \times X \to [0, \infty)$ given by $h(x, y, z) = \dfrac{4}{1 + G(x, y, z)}$ and T has a fixed point $x = 0$.

In [46], the following theorem concerning the common fixed points for a single valued mapping and a multivalued mapping in a G-metric space have proved.

Theorem 3.9. *Let (X, G) be a complete G-metric space. Let $g : X \to X$ and $T : X \to CB(X)$ be two mappings. Assume that there is a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t+} \alpha(r) < 1$, for every $t \geq 0$ such that*

$$H_G(Tx, Ty, Tz) \leq \alpha(G(gx, gy, gz))G(gx, gy, gz), \tag{3.8}$$

for all $x, y, z \in X$. If for any $x \in X$, $Tx \subseteq g(X)$ and $g(X)$ is a G-complete subspace of X, then g and T have a point of coincidence in X.

If we let g to be the identity mapping, we get the following result.

Theorem 3.10. *Let (X, G) be a complete G-metric space and $T : X \to CB(X)$ be a multivalued mapping. Assume that, there is a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t+} \alpha(r) < 1$ for every $t \geq 0$ such that*

$$H_G(Tx, Ty, Tz) \leq \alpha(G(x, y, z))G(x, y, z) \tag{3.9}$$

for all $x, y, z \in X$. Then T has a fixed point in X.

Theorem 3.10 is a generalized version of Theorem 3.3 and it also generalizes theorem of Mizoguzhi-Takahashi in [23]. Now we define the orbit of a multivalued mapping and then will be proving the boundedness of orbit of certain multivalued mappings to get a fixed point result equivalent to Theorem 3.3.

Definition 3.11. Let $T : X \to CB(X)$ be a multivalued mapping and let $x \in X$, we define the orbit of T at x as

$$O_T(x) = \{x\} \bigcup Tx \bigcup T^2x \bigcup \cdots,$$

$$O_T(x) = \bigcup_{n=0}^{\infty} T^n x \quad \text{where} \quad T^n x = \bigcup_{w \in T^{n-1}} Tw.$$

Example 3.12. Consider the multivalued mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = [0, x]$, the orbit of T is bounded for each $x \in \mathbb{R}$. Whereas, the orbit of T given by $Tx = [0, x^4]$ is not bounded for $|x| > 1$.

Definition 3.13. *Let $T : X \to CB(X)$ be a multivalued mapping. T is said to be invariant under $S \subseteq X$ if, $Tx \subseteq S,$ whenever $x \in S$.*

Proposition 3.14. *Let (X, G) be a complete G-metric space and $T : X \to CB(X)$ be a multivalued Lipschitz mapping. Let $S = \overline{O_T(x_0)},\ x_0 \in X$. Then T is invariant under S.*

Proof. Let $x \in S$. Then, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \in O_T(x)$ such that $x_n \to x$. Since T is multivalued Lipschitz mapping, there exists $\alpha > 0$ such that

$$H_G(Tx_n, Tx_n, Tx) \leq \alpha G(x_n, x_n, x),$$

which implies that $H_G(Tx_n, Tx_n, Tx) \to 0,$ since $G(x_n, x_n, x) \to 0$. We need to prove that $Tx \subseteq S$. Let $y \in Tx$. Then by Lemma 2.10,

$$G(y, Tx_n, Tx_n) \leq H_G(Tx, Tx_n, Tx_n).$$

Now, choose $z_n \in Tx_n$ such that, $G(y, z_n, z_n) \leq G(y, Tx_n, Tx_n) + \dfrac{1}{n}$, then there is a sequence of points of $O_T(x)$ which converges to y. *i.e.*, $G(y, z_n, z_n) \to 0$, which implies that $y \in Tx \subset S$. Hence T is invariant under $S = \overline{O_T(x)}$. □

Lemma 3.15. *Suppose that $h : X \times X \times X \to [0, \infty)$ satisfies*

$$\sup\{h(x, y, z) \ : \ a \leq G(x, y, z) \leq b\} < 1, \tag{3.10}$$

for each closed interval $[a, b] \subset (0, \infty)$. Assume also that if, $(x_n, y_n, z_n) \in X \times X \times X$ is such that

$$\lim_{n\to\infty} G(x_n, y_n, z_n) = 0, \text{ then } \lim_{n\to\infty} h(x_n, y_n, z_n) = k, \text{ for some } k \in [0, \infty). \tag{3.11}$$

Then, $\sup\{h(x, y, z) : \ 0 \leq G(x, y, z) \leq b\} < 1$.

Proof. Let $M = \sup\{h(x, y, z) : \ 0 \leq G(x, y, z) \leq b\}$ and suppose that $M = 1$. Then, there exists $(x_n, y_n, z_n) \in X \times X \times X$ such that $\lim\limits_{n\to\infty} h(x_n, y_n, z_n) = 1$. But using (3.11), $G(x_n, y_n, z_n)$ must converge to 0, which is a contradiction to the condition (3.10). Hence the conclusion follows. □

Next, we prove that under certain contractive condition the orbit of a multivalued mapping is bounded.

Theorem 3.16. *Let (X, G) be a complete G-metric space and $T : X \to CB(X)$ be such that*

$$H_G(Tx, Ty, Tz) \leq h(x, y, z)G(x, y, z), \ \forall \ x, y, z \in X, \tag{3.12}$$

where $h : X \times X \times X \to (0, \infty)$ *is such that*

$$\sup\{h(x,y,z) \ : \ 0 \leq G(x,y,z) \leq b\} < 1.$$

Then the orbit of T *is bounded.*

Proof. Let $x \in X$ and T satisfies (3.12). First we prove that $H_G(T^n x, T^{n+1}x, T^{n+1}x)$ converges to some nonnegative number.

Let $u \in T^n x = \bigcup_{z \in T^{n-1}x} Tz$ and $T^{n+1}x = \bigcup_{w \in T^n x} Tw$.

Now,

$$\begin{aligned} G(u, T^{n+1}x, T^{n+1}x) &= d_G(u, T^{n+1}x) + d_G(T^{n+1}x, T^{n+1}x) + d_G(u, T^{n+1}x) \\ &= 2d_G(u, T^{n+1}x) \\ &= \inf\{d_G(u,t), t \in T^{n+1}x\} \\ &\leq d_G(u, Tw) \text{ for each } \ w \in T^n x \\ &\leq H_G(Tz, Tw, Tw) \text{ for some } z \in T^{n-1}x \\ &\leq h(z,w,w)G(z,w,w) \\ &\leq G(z,w,w) \\ &\leq 2d_G(z,w). \end{aligned}$$

Taking infimum over $w \in T^n x$,

$$G(u, T^{n+1}x, T^{n+1}x) \leq d_G(z, T^n x) \leq G(z, T^n x, T^n x)$$

for each $u \in T^n x$. Similarly,

$$G(v, T^n x, T^{n+1}x) \leq G(z, T^n x, T^{n+1}x)$$

for each $v \in T^{n+1}x$. Hence

$$H_G(T^n x, T^{n+1}x, T^{n+1}x) \leq H_G(T^{n-1}x, T^n x, T^n x).$$

Thus, $H_G(T^n x, T^{n+1}x, T^{n+1}x)$ is a decreasing sequence of non-negative real numbers and it follows that, $\lim_{n\to\infty} H_G(T^n x, T^{n+1}x, T^{n+1}x) = r$.

Now we prove that $\overline{\{T^n x\}}$ is a G-Cauchy sequence in $CB(X)$. Let $u \in T^n x$. Then for each $w \in T^n x$

$$G(u, T^{n+1}x, T^{n+1}x) \leq h(z,w,w)G(z,w,w)$$

for some $z \in T^{n-1}x$.

Now, $\epsilon > 0$ be given. For each $n \in \mathbb{N}$ and $z_n \in T^{n-1}x$, there is $w_n \in T^n x$ satisfying

$$G(z_n, w_n, w_n) \leq G(z_n T^n x, T^n x) + \epsilon, \tag{3.13}$$

which implies that

$$G(z_n, w_n, w_n) \leq H_G(T^{n-1}x, T^n x, T^n x) + \epsilon.$$

Since $\{H_G(T^{n-1}x, T^nx, T^nx)\}$ converges to r, $G(z_n, w_n, w_n)$ is bounded for all n. We have, $\sup\{h(x,y,z) \ : \ 0 \leq G(x,y,z) \leq r+\epsilon\} < 1$ and so, take $\sup\{h(z_n, w_n, w_n) = k\ ,\ k \in [0,1)$. Now

$$\begin{aligned} G(u, T^{n+1}x, T^{n+1}x) &\leq h(z_n, w_n, w_n)\{G(z_n, T^nx, T^nx)+\epsilon\}, \\ \sup_{u\in T^nx} G(u, T^{n+1}x, T^{n+1}x) &\leq \sup_{z_n \in T^{n-1}x} h(z_n, w_n, w_n) H_G(T^{n-1}x, T^nx, T^nx) \\ &\leq kH_G(T^{n-1}x, T^nx, T^nx) \text{ (arbitrary } \epsilon). \end{aligned}$$

Repeating the same arguments, we can prove that

$$\sup_{v\in T^{n+1}x} G(v, T^nx, T^{n+1}x) \leq kH_G(T^{n-1}x, T^nx, T^nx).$$

Thus, $H_G(T^nx, T^{n+1}x, T^{n+1}x) \leq kH_G(T^{n-1}x, T^nx, T^nx)$. This is true for each $n \in \mathbb{N}$. So iteratively we get

$$H_G(T^nx, T^{n+1}x, T^{n+1}x) \leq k^n G(x, Tx, Tx).$$

Since $k \in (0,1)$, the sequence $\{T^n(x)\}$ is G-Cauchy. Let $A_n = \overline{\{T^n(x)\}}$. We have by Lemma 3.2, $H_G(\bar{A}, \bar{B}, \bar{B}) = H_G(A, B, B)$ and $CB(X)$ is complete in Hausdorff G-distance, there is an $A \in CB(X)$ such that $A_n \to A$.

Now, choose $\delta > 0$. Let $p, q \in O_T(x)$. So for $p \in T^nx$ there exists $p_1 \in A$ so that $G(p, p_1, p_1) \leq H_G(T^nx, T^nx, A) + \delta$ and for $q \in T^mx$ there exists $q_1 \in A$ so that $G(q, q, q_1) \leq H_G(T^mx, T^mx, A) + \delta$.
So,

$$\begin{aligned} G(p,q,q) &\leq G(p, p_1, p_1) + G(p_1, q_1, q_1) + G(q_1, q, q) \\ &\leq H_G(T^nx, T^nx, A) + H_G(T^mx, T^mx, A) + 2\delta + G(p_1, q_1, q_1). \end{aligned}$$

Since δ is arbitrary constant, $A_n \to A$ and $A \in CB(X)$, $G(p,q,q) \leq M$ for some $M > 0$. Thus orbit of T is bounded. □

Now, we give a fixed point theorem for multivalued mappings which is equivalent to 3.3.

Theorem 3.17. *Let S be a bounded complete G-metric space and $T : S \to CB(S)$ be such that*

$$H_G(Tx, Ty, Tz) \leq kG(x,y,z)$$

for all $x, y \in S$, where, $0 < k < 1$. Then T has a fixed point. Further, if we assume $x \in Tx$, $y \in Ty$ and $G(x,y,y) \leq H_G(Tx, Ty, Ty)$, then T has a unique fixed point in X.

It is obvious that Theorem 3.3 implies Theorem 3.17. For the reverse implication, we have from Proposition 3.14, T is invariant under $\overline{O_T(x)}$ and T satisfies (3.12). So by Theorem 3.16, $O_T(x)$ is bounded. Thus by taking $S = \overline{O_T(x)}$ in Theorem 3.17, T has a fixed point.

Using the extra bounded condition on G-metric space, we can see that Theorem 3.6 and 3.10 follows as corollaries of Theorem 3.17.

Corollary 3.18. *Let (X, G) be a complete G- metric space and let $T : X \to CB(X)$ be such that*

$$H_G(Tx, Ty, Tz) \leq h(x, y, z)G(x, y, z), \ \forall \ x, y, z \in X,$$

and for some non negative function $h(x, y, z)$ satisfying $\sup\{h(x, y, z) \ : \ a \leq G(x, y, z) \leq b\} < 1$, for each closed interval $[a, b] \subset (0, \infty)$. Assume also that if, $(x_n, y_n, z_n) \in X \times X \times X$ is such that $\lim_{n\to\infty} G(x_n, y_n, z_n) = 0$, then $\lim_{n\to\infty} h(x_n, y_n, z_n) = k$ for some $k \in [0, \infty)$. Then T has a fixed point in X.

Proof. We have from Theorem 3.16, $O_T(x)$ is bounded. So, we can define $b = \sup\{G(x, y, y) \ : \ x, y \in O_T(x)\}$. Now by Lemma 3.15, $\sup\{h(x, y, z) \ : \ 0 \leq G(x, y, z) \leq b\} < 1$. Thus T becomes a multi valued contraction mapping when it restrict to $O_T(x)$. Taking $S = O_T(x)$, it follows by Theorem 3.17 that T has a fixed point in X. □

Corollary 3.19. *Let (X, G) be a complete G- metric space. Let $T : X \to CB(X)$ be a multi valued mapping. Assume that there is a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\lim\sup_{r\to t+} \alpha(r) < 1$ for every $t \geq 0$ such that*

$$H_G(Tx, Ty, Tz) \leq \alpha(G(x, y, z))G(x, y, z),$$

for all $x, y, z \in X$. Then T has a fixed point in X.

Proof. Take $h(x, y, z) = \alpha(G(x, y, z))$. It can easily be seen that the $h(., ., .)$ thus defined satisfies all the condition of Theorem 3.16 and so repeating the same arguments as above, it follows that T has a fixed point in X. □

Remark 3.20. In the proof of Corollary 3.18 and Corollary 3.19, It can be seen that the multivalued mappings reduced to multivalued contraction mappings. We have used the boundedness of the orbit of multivalued mappings to obtain the results easily.

References

[1] M. Abbas, B. Ali and C. Vetro, *A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces*, Topol. Appl., **160** (2013), 553-563.

[2] T.V. An, N.V. Dung and V.L. Hang, *A new approach to fixed point theorems on G-metric spaces*, Topol. Appl., **160** (2013), 1486-1493.
[3] M. Asadi and P. Salimi, *Some fixed point and common fixed point theorems on G-metric spaces,* Nonlinear Funct. Anal. and Appl., **21**(3) (2016), 523-530.
[4] D.W. Boyd and J.S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **89** (1968), 458-464.
[5] S.K. Chatterjea, *Fixed-point theorems*, Comptes Rendus de lAcademie Bulgare des Sciences. **25** (1972), 727-730.
[6] Lj.B. Cirić, *Generalized contractions and fixed point theorems*, Publ. Inst.Math. (Beograd), **12**(26) (1971), 19-26.
[7] Lj.B. Cirić, *A generalization of Banachs contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267-273.
[8] P.Z. Daffer and H. Kaneko, *Fixed points of generalized contractive multi-valued mappings*, J. Math. Anal. Appl., **192** (1995), 655-666.
[9] P.Z. Daffer, H. Kaneko and Wu Li, *On a conjecture of S. Reich*, Proc. Amer. Math. Soc., **124** (1996), 3159-3162.
[10] B.C. Dhage, *Generalized metric space and mapping with fixed point*, Bull. Calcutta Math. Soc., **84** (1992), 329-336.
[11] B.C. Dhage, *Generalized metric space and topological structure I*, Analele Stiintifice ale Universitătii "Al. I. Cuza" din Iasi. Serie Nouă. Mathematică. **46**(1) (2000), 3-24.
[12] W-S Du, *On coincidence point and fixed point theorems for nonlinear multivalued maps*, Topol. Appl., **159** (2012), 49-56.
[13] A.A. Eldred, J. Anuradha and P. Veeramani, *On equivalence of generalized multi-valued contractions and Nadlers fixed point theorem*, J. Math. Anal. Appl., **336** (2007), 751-757.
[14] G.M. Eshaghi, H. Baghani, H. Khodaei and M. Ramezani, *A generalization of Nadlers fixed point theorem*, J. Nonlinear Sci. Appl., **3**(2) (2010), 148-151.
[15] S. Gahler, *2-metriche raume und ihre topologische strukture*, Math. Nachr., **26** (1963), 115-148.
[16] S. Gahler, *Zur geometric 2-metriche raume*, Reevue Roumaine de Math.Pures et Appl., **XI** (1966), 664-669.
[17] L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332**(2) (2007), 1468-1476.
[18] A. Kaewcharoen, *Common fixed point theorems for contractive mappings satisfying ϕ mapps in G-metric spaces*, Banach J. Math. Anal., **6**(1) (2012), 101-111.
[19] A. Kaewcharoen and A. Kaewkhao, *Common fixed points for single-valued and multi-valued mappings in G-metric spaces*, Int. J. Math. Anal. (Ruse). **5** (2011), 1775-1790.
[20] H. Kaneko, *Generalized contractive multi-valued mappings and their fixed points*, Math. Japon., **33** (1988), 57-64.
[21] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., **60** (1968), 71-76.
[22] W.A. Kirk and N. Shahzad, *Some fixed point results in ultrametric spaces*. Topol. Appl., **159** (2012), 3327-3334.
[23] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math.Anal. Appl., **141** (1989), 177-188.
[24] Z. Mustafa, M. Khandaqji and W. Shatanawi, *Fixed point results on complete G-metric spaces*, Studia Sci. Math. Hungar., **48** (2011), 304-319.
[25] Z. Mustafa and B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory Appl., Article ID 917175 (2009).

[26] Z. Mustafa and B. Sims, *Some remarks concerning D-metric spaces*, Proceedings of the Internatinal Conference on Fixed Point Theory and Appl., Valencia (Spain), July (2003), 189-198.

[27] Z. Mustafa, M. Arshad, K.U. Khan, J. Ahmad and M. M. Jaradat, *Common fixed points for multivalued mappings in G-metric spaces with applications*, J. Nonlinear Sci. Appl., **10** (2017), 2550-2564.

[28] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory Appl. Article ID 189870 (2008).

[29] Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of fixed point results in G-metric spaces*, Int. J. Math. Math. Sci. Article ID 283028 (2009).

[30] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7**(2) (2006), 289-297.

[31] S.B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475-488.

[32] M. Pitchaimani and D. Ramesh Kumar, *Some common fixed point theorems using implicit relation in 2-Banach spaces*, Surv. Math. Appl., **10** (2015), 159-168.

[33] M. Pitchaimani and D. Ramesh Kumar, *Common and coincidence fixed point theorems for asymptotically regular mappings in 2-Banach Space*, Nonlinear Funct. Anal. Appl., **21**(1) (2016), 131-144.

[34] M. Pitchaimani and D. Ramesh Kumar, *On construction of fixed point theory under implicit relation in Hilbert spaces*, Nonlinear Funct. Anal. Appl. **21**(3) (2016), 513-522.

[35] M. Pitchaimani and D. Ramesh Kumar, *On Nadler type results in ultrametric spaces with application to well-posedness*, Asian-European J. of Math., **10**(4) (2017), 1750073(1-15) DOI: 10.1142/S1793557117500735.

[36] M. Pitchaimani and D. Ramesh Kumar, *Generalized Nadler type results in ultrametric spaces with application to well-posedness*, Afri. Mat., **28** (2017), 957-970.

[37] D. Ramesh Kumar and M. Pitchaimani, *Set-valued contraction mappings of Prešić-Reich type in ultrametric spaces*, Asian-European J. of Math., **10**(4) (2017), 1750065(1-15), DOI: 10.1142/S1793557117500656.

[38] D. Ramesh Kumar and M. Pitchaimani, *A generalization of set-valued Prešić-Reich type contractions in ultrametric spaces with applications*, J. Fixed Point Theory Appl., (2016), DOI: 10.1007/s11784-016-0338-4.

[39] S. Reich, *Fixed points of contractive functions*, Boll. Unione Mat. Ital., **4** (1972), 26-42.

[40] S. Reich, *Some remarks concerning contraction mappings*, Can. Math. Bull., **14** (1971), 121-124.

[41] R. Saadati, S.M. Vaezpour, P. Vetro and B.E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Model., **52** (2010), 797-801.

[42] S. Sedghi, N. Shobkolaei and S.H. Sadati, *A generalization of Caristi Kirk's theorem for common fixed points on G-metric spaces*, Nonlinear Funct. Anal. and Appl., **20**(4) (2015), 551-559.

[43] W. Shatanawi, *Fixed point theory for contractive mappings satisfying ϕ -maps in G-metric spaces*, Fixed Point Theory Appl., Article ID 181650 (2010).

[44] W. Shatanawi, *Coupled fixed point theorems in generalized metric spaces*, Hacet. J. Math.Stat., **40** (2011), 441-447.

[45] W. Shatanawi, *Some fixed point theorems in ordered G-metric spaces and applications*, Abstr. Appl. Anal. Article ID 126205 (2011).

[46] N. Tahat, H. Aydi, E. Karapnar and W. Shatanawi, *Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces*, Fixed Point Theory Appl., (2012), DOI:10.1186/1687-1812.2012-48.

[47] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo. **56**(2) (2007), 464-468.
[48] D. Wardowski, *On set-valued contractions of Nadler type in cone metric spaces*, Appl. Math. Lett., **24** (2011), 275-278.
[49] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., Article ID 94 (2012).