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CERTAIN INTEGRAL FORMULAS INVOLVING LOGARITHM FUNCTION

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Abstract. A remarkably large number of integral formulas involving logarithm function have been presented. Here, with the aid of a known technique, we aim to show how certain integral formulas involving logarithm function can be nicely established by choosing to use some known integral formulas which are expressed, mainly, in terms of gamma function and its related functions.

1. Introduction and preliminaries

Shen [17] clarified the connections between Stirling numbers of the first kind $s(n, k)$ and Riemann zeta function $\zeta(n)$, with certain series and integrals evaluated in terms of $\zeta(n)$ and $s(n, k)$, by mainly analyzing the well-known Gauss summation formula

$$ {}_2F_1(a,\, b,\, ;c\,;\,1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad (\Re(c-a-b)>0), \tag{1.1}$$

where ${}_2F_1$ is the hypergeometric series (see, e.g., [20, Section 1.5]; see also [16, 18]).

Since then, by essentially using the Shen's method (see Lemma 2.3), a large number of series involving harmonic and generalized harmonic numbers and some integral formulas have been established (see, e.g., [3, 4, 5, 6, 7, 8, 9]; see also [10, 11, 12, 13, 21, 22]).

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In this sequel, with the aid of the Shen's method, we aim to show how certain integral formulas involving logarithm function can be nicely established by choosing to use some known integral formulas which are expressed, mainly, in terms of gamma function and its related functions.

For our purpose, we recall some functions and notations. The Hurwitz (or generalized) zeta function $\zeta(s,a)$ is defined by (see, e.g., [20, Sections 2.2 and 2.3])

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad \left(\Re(s) > 1,\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{1.2}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{Z}^-$ be the sets of complex numbers, positive integers, and negative integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$. A special case of $\zeta(s,a)$ when $a = 1$ is called the Riemann zeta function $\zeta(s)$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1). \tag{1.3}$$

The polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) = \frac{d^n}{dz^n} \psi(z) \quad \left(n \in \mathbb{N}_0,\ z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{1.4}$$

where $\Gamma(z)$ is the familiar gamma function and the psi-function ψ is defined by $\psi(z) := \frac{d}{dz} \log \Gamma(z) = \psi^{(0)}(z)$ (see, e.g., [19, 20, Chapter 1]). We recall a well-known (useful) relationship between the polygamma functions $\psi^{(n)}(z)$ and the generalized zeta function $\zeta(z,a)$:

$$\psi^{(k)}(z) = (-1)^{k+1}\, k!\, \zeta(k+1, z) \quad \left(k \in \mathbb{N},\ z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{1.5}$$

The generalized harmonic numbers $H_n^{(s)}$ of order s are defined by (*cf.* [15]; see also [1] and [19, p. 156])

$$H_n^{(s)} := \sum_{j=1}^{n} \frac{1}{j^s} \quad (n \in \mathbb{N};\ s \in \mathbb{C}) \tag{1.6}$$

and

$$H_n := H_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}) \tag{1.7}$$

are the harmonic numbers. Here we assume $H_0 := 0$, $H_0^{(s)} := 0$ $(s \in \mathbb{C} \setminus \{0\})$, and $H_0^{(0)} := 1$. We also recall the generalized harmonic functions $H_n^{(s)}(z)$

defined by

$$H_n^{(s)}(z) := \sum_{j=1}^{n} \frac{1}{(j+z)^s} \quad \left(n \in \mathbb{N},\ s \in \mathbb{C},\ z \in \mathbb{C} \setminus \mathbb{Z}^-\right). \tag{1.8}$$

which reduces to $H_n^{(s)}(0) = H_n^{(s)}$. It is easy to derive the following expression (*cf.* [19, Eq. 1.2(54)]):

$$\begin{aligned}\psi^{(m)}(z+n) - \psi^{(m)}(z) &= (-1)^m\, m! \sum_{k=1}^{n} \frac{1}{(z+k-1)^{m+1}} \\ &= (-1)^m\, m!\, H_n^{(m+1)}(z-1) \quad (m,\, n \in \mathbb{N}_0)\,,\end{aligned} \tag{1.9}$$

which immediately gives $H_n^{(s)}(a)$ another expression:

$$\psi(z+n) - \psi(z) = H_n(z-1) \quad (n \in \mathbb{N}_0) \tag{1.10}$$

and

$$\zeta(m+1,z) - \zeta(m+1,z+n) = H_n^{(m+1)}(z-1) \quad (m \in \mathbb{N},\ n \in \mathbb{N}_0)\,. \tag{1.11}$$

Obviously

$$H_n^{(s)} = \zeta(s) - \zeta(s,n+1) \qquad (\Re(s) > 1,\ n \in \mathbb{N})\,. \tag{1.12}$$

2. Certain integral formulas deducible from a beta type integral

We begin by recalling a known integral formula (see, e.g., [14, p. 316, Entry 3.196-3])

$$\int_a^b (x-a)^\mu\, (b-x)^\nu\, dx = (b-a)^{\mu+\nu+1}\, B(\mu+1,\nu+1) \tag{2.1}$$

$$(b > a,\, \Re(\mu) > -1,\, \Re(\nu) > -1),$$

where $B(\alpha,\beta)$ is the beta function defined by (see, e.g., [20, Section 1.1])

$$B(\alpha,\, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \end{cases} \tag{2.2}$$

We first claim that the integral in (2.1) as a function of the parameter μ or ν can be differentiated under the integral, asserted in the following lemma.

Lemma 2.1. *Let* $b > a$ *and* $\min\{\Re(\mu) >, \Re(\nu)\} > -1$. *Then*

$$\begin{aligned} \frac{\partial}{\partial\mu}\int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,dx &= \int_a^b \frac{\partial}{\partial\mu}(x-a)^{\mu}\,(b-x)^{\nu}\,dx \\ &= \int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,\ln(x-a)\,dx. \end{aligned} \tag{2.3}$$

Proof. Let

$$f(\mu) := \int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,dx \quad (b > a,\, \Re(\mu) > -1,\, \Re(\nu) > -1).$$

For any fixed $\mu_0 \in \mathbb{C}$ with $\Re(\mu_0) > -1$ and $\triangle\,\mu \in \mathbb{C}$ with $0 < |\triangle\,\mu|$ and $\Re(\mu_0 + \triangle\mu) > -1$, it suffices to show that

$$\lim_{\triangle\mu\to 0}\frac{f(\mu_0+\triangle\mu) - f(\mu_0)}{\triangle\mu} = \int_a^b (x-a)^{\mu_0}\,(b-x)^{\nu}\,\ln(x-a)\,dx. \tag{2.4}$$

Indeed, let

$$M := \int_a^b (x-a)^{\Re(\mu_0)}\,(b-x)^{\Re(\nu)}\,dx = (b-a)^{\Re(\mu_0+\nu+1)}\,|B(\mu_0+1, \nu+1)| > 0.$$

Let $\epsilon > 0$ be given. Since

$$\lim_{\triangle\mu\to 0}\left\{\frac{(x-a)^{\triangle\mu}-1}{\triangle\mu} - \ln(x-a)\right\} = 0 \quad (a < x < b),$$

there exists δ such that $\Re(\mu_0 + \triangle\mu) > -1$ and $0 < |\triangle\mu| < \delta$ implies

$$\left|\frac{(x-a)^{\triangle\mu}-1}{\triangle\mu} - \ln(x-a)\right| < \frac{\epsilon}{M+1}. \tag{2.5}$$

For such a chosen δ with $0 < |\triangle\mu| < \delta$, in view of (2.5), we find

$$\begin{aligned} &\left|\frac{f(\mu_0+\triangle\mu) - f(\mu_0)}{\triangle\mu} - \int_a^b (x-a)^{\mu_0}\,(b-x)^{\nu}\,\ln(x-a)\,dx\right| \\ &\leq \int_a^b \left|\frac{(x-a)^{\triangle\mu}-1}{\triangle\mu} - \ln(x-a)\right| (x-a)^{\Re(\mu_0)}\,(b-x)^{\Re(\nu)}\,dx \\ &\leq \frac{\epsilon}{M+1}\int_a^b (x-a)^{\Re(\mu_0)}\,(b-x)^{\Re(\nu)}\,dx \\ &\leq \frac{\epsilon}{M+1}\,M < \epsilon. \end{aligned}$$

Hence we complete to prove (2.4). □

Remark 2.2. Similarly as in the proof of Lemma 2.1, we can differentiate the integral (2.1) as a function of ν under the integral sign. So can higher-order partial derivatives of the integral (2.1) with respect to the parameters μ and ν.

Higher-order partial differential recursive formulas for $B(\mu+1,\nu+1)$ are given in the following lemma.

Lemma 2.3. *Let* $b > a$, $\min\{\Re(\mu), \Re(\nu)\} > -1$, *and* $k \in \mathbb{N}_0$. *Then*

$$\begin{aligned}&\frac{\partial^{k+1}}{\partial\mu^{k+1}}B(\mu+1,\nu+1)\\&=\sum_{j=0}^{k}\binom{k}{j}\left\{\frac{\partial^j}{\partial\mu^j}B(\mu+1,\nu+1)\right\}\left\{\psi^{(k-j)}(\mu+1)-\psi^{(k-j)}(\mu+\nu+2)\right\}\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}&\frac{\partial^{k+1}}{\partial\nu^{k+1}}B(\mu+1,\nu+1)\\&=\sum_{j=0}^{k}\binom{k}{j}\left\{\frac{\partial^j}{\partial\nu^j}B(\mu+1,\nu+1)\right\}\left\{\psi^{(k-j)}(\nu+1)-\psi^{(k-j)}(\mu+\nu+2)\right\}\end{aligned}\tag{2.7}$$

Proof. Taking the logarithmic derivative of

$$B(\mu+1,\nu+1)=\frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}\tag{2.8}$$

with respect to the variable μ, we have

$$\frac{\partial}{\partial\mu}B(\mu+1,\nu+1)=B(\mu+1,\nu+1)\{\psi(\mu+1)-\psi(\mu+\nu+2)\}.\tag{2.9}$$

Applying Leibniz's higher-order differential formula for product of two functions to the right side of (2.9), we obtain (2.6).

In view of (2.8), $B(\mu+1,\nu+1)$ is symmetric with respect to both variables μ and ν. By interchanging the role of μ and ν in (2.6), we get (2.7). □

We note that the technique in the proof of Lemma 2.3 was used by Shen [17].

Theorem 2.4. *Let $b > a$, $\min\{\Re(\mu), \Re(\nu)\} > -1$, and $p, q \in \mathbb{N}_0$. Then the following integral formula holds true.*

$$\begin{aligned}
&\int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,\{\ln(x-a)\}^p\,\{\ln(b-x\}^q\,dx \\
&\quad = (b-a)^{\mu+\nu+1}\sum_{\ell=0}^{q}\sum_{k=0}^{p}\binom{p}{k}\binom{q}{\ell}\{\ln(b-a)\}^{p+q-k-\ell} \\
&\qquad \times \frac{\partial^{\ell}}{\partial\nu^{\ell}}\Big\{\frac{\partial^{k}}{\partial\mu^{k}}B(\mu+1,\nu+1)\Big\} \quad (p,\,q\in\mathbb{N}_0)\,.
\end{aligned} \tag{2.10}$$

Here the last partial derivative can be obtained from the recurrence formulas in Lemma 2.3.

Proof. In view of Remark 2.2, taking partial derivatives of (2.1) of order p with respect to the variable μ under the integral sign and, then, taking partial derivatives of the resulting identity of order q with respect to the variable ν, we find (2.10). In each step, we used Leibniz's higer-order derivative formula of product of two functions. □

Some special cases of the integral formula in (2.10) are demonstrated in the following corollary.

Corollary 2.5. *Let $b > a$ and $\min\{\Re(\mu), \Re(\nu)\} > -1$. Then*

$$\begin{aligned}
&\int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,\ln(x-a)\,dx \\
&\quad = B(\mu+1,\nu+1)\{\psi(\mu+1)-\psi(\mu+\nu+2)\};
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,\ln(b-x)\,dx \\
&\quad = B(\mu+1,\nu+1)\{\psi(\nu+1)-\psi(\mu+\nu+2)\};
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,\{\ln(x-a)\}^2\,dx \\
&\quad = B(\mu+1,\nu+1)\Big[\{\psi(\mu+1)-\psi(\mu+\nu+2)\}^2 \\
&\qquad\qquad + \psi'(\mu+1)-\psi'(\mu+\nu+2)\Big];
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&\int_a^b (x-a)^{\mu}\,(b-x)^{\nu}\,\{\ln(b-x)\}^2\,dx \\
&\quad = B(\mu+1,\nu+1)\Big[\{\psi(\nu+1)-\psi(\mu+\nu+2)\}^2 \\
&\qquad\qquad + \psi'(\nu+1)-\psi'(\mu+\nu+2)\Big];
\end{aligned} \tag{2.14}$$

$$\int_a^b (x-a)^\mu\,(b-x)^\nu\,\ln(x-a)\,\ln(b-x)\,dx$$
$$= B(\mu+1,\nu+1)\Big[\{\psi(\mu+1)-\psi(\mu+\nu+2)\}\{\psi(\nu+1)-\psi(\mu+\nu+2)\} - \psi'(\mu+\nu+2)\Big]. \tag{2.15}$$

Further, setting $\mu,\,\nu\in\mathbb{N}$ in the identities in Corollary 2.5, in view of (1.5) and (1.9), we have corresponding integral formulas as in the following corollary.

Corollary 2.6. *Let $b>a$ and $\mu,\,\nu\in\mathbb{N}$. Then*

$$\int_a^b (x-a)^\mu\,(b-x)^\nu\,\ln(x-a)\,dx = -\frac{\mu!\,\nu!}{(\mu+\nu+1)!}\,H_{\nu+1}(\mu); \tag{2.16}$$

$$\int_a^b (x-a)^\mu\,(b-x)^\nu\,\ln(b-x)\,dx = -\frac{\mu!\,\nu!}{(\mu+\nu+1)!}\,H_{\mu+1}(\nu); \tag{2.17}$$

$$\int_a^b (x-a)^\mu\,(b-x)^\nu\,\{\ln(x-a)\}^2\,dx = \frac{\mu!\,\nu!}{(\mu+\nu+1)!}\Big[H_{\nu+1}^2(\mu)+H_{\nu+1}^{(2)}(\mu)\Big]; \tag{2.18}$$

$$\int_a^b (x-a)^\mu\,(b-x)^\nu\,\{\ln(b-x)\}^2\,dx = \frac{\mu!\,\nu!}{(\mu+\nu+1)!}\Big[H_{\mu+1}^2(\nu)+H_{\mu+1}^{(2)}(\nu)\Big]; \tag{2.19}$$

$$\int_a^b (x-a)^\mu\,(b-x)^\nu\,\ln(x-a)\,\ln(b-x)\,dx = \frac{\mu!\,\nu!}{(\mu+\nu+1)!}\Big[H_{\nu+1}(\mu)\,H_{\mu+1}(\nu)-\zeta(2,\mu+\nu+2)\Big]. \tag{2.20}$$

3. Integral formulas deducible from certain formulas expressed in terms of log gamma function

We begin by recalling a known integral formula (see, e.g., [14, p. 547, Entry 4.267-31])

$$\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{x^s\,dx}{(1-x)\,\ln x}$$
$$= \ln\left\{\frac{\Gamma(p+1+s)\,\Gamma(q+1+s)\,\Gamma(r+1+s)\,\Gamma(p+q+r+1+s)}{4\,\Gamma(p+q+1+s)\,\Gamma(p+r+1+s)\,\Gamma(q+r+1+s)\,\Gamma(1+s)}\right\} \tag{3.1}$$
$$(\min\{p,\,q,\,r\}>0,\ s>-1)\,.$$

We present some integral formulas, which are deducible from (3.1), asserted by the following theorem.

Theorem 3.1. *Each of the following integral formulas holds.*

$$\begin{aligned}
&\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{x^s\,dx}{1-x}\\
&= \psi(p+1+s)+\psi(q+1+s)+\psi(r+1+s)+\psi(p+q+r+1+s)\\
&\quad -\psi(p+q+1+s)-\psi(p+r+1+s)-\psi(q+r+1+s)-\psi(1+s)
\end{aligned} \tag{3.2}$$

$$(\min\{p,\,q,\,r\}>0,\; s>-1).$$

$$\begin{aligned}
&\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{(\ln x)^k\,x^s\,dx}{1-x}\\
&= \psi^{(k)}(p+1+s)+\psi^{(k)}(q+1+s)+\psi^{(k)}(r+1+s)\\
&\quad +\psi^{(k)}(p+q+r+1+s)-\psi^{(k)}(p+q+1+s)-\psi^{(k)}(p+r+1+s)\\
&\quad -\psi^{(k)}(q+r+1+s)-\psi^{(k)}(1+s)
\end{aligned} \tag{3.3}$$

$$(\min\{p,\,q,\,r\}>0,\; s>-1;\; k\in\mathbb{N}).$$

$$\begin{aligned}
&\int_0^1 (1-x^q)\,(1-x^r)\,\frac{(\ln x)^{k+\ell}\,x^{s+p}}{1-x}\,dx\\
&\quad = \psi^{(k+\ell)}(p+q+1+s)+\psi^{(k+\ell)}(p+r+1+s)\\
&\qquad -\psi^{(k+\ell)}(p+1+s)-\psi^{(k+\ell)}(p+q+r+1+s)
\end{aligned} \tag{3.4}$$

$$(\min\{p,\,q,\,r\}>0,\; s>-1;\; k,\,\ell\in\mathbb{N}).$$

$$\begin{aligned}
&\int_0^1 (1-x^r)\,\frac{(\ln x)^{k+\ell+m}\,x^{s+p+q}}{1-x}\,dx\\
&\quad = \psi^{(k+\ell+m)}(p+q+r+1+s)-\psi^{(k+\ell+m)}(p+q+1+s)
\end{aligned} \tag{3.5}$$

$$(\min\{p,\,q,\,r\}>0,\; s>-1;\; k,\,\ell,\,m\in\mathbb{N}).$$

$$\begin{aligned}
&\int_0^1 \frac{(\ln x)^{k+\ell+m+n}\,x^{s+p+q+r}}{1-x}\,dx\\
&\quad = -\psi^{(k+\ell+m+n)}(p+q+r+1+s)
\end{aligned} \tag{3.6}$$

$$(\min\{p,\,q,\,r\}>0,\; s>-1;\; k,\,\ell,\,m,\,n\in\mathbb{N}).$$

Proof. In view of Remark 2.2, we can differentiate the integral (3.1) with respect to the parameters p, q, r and s under the integral sign.

Differentiating both sides of (3.1) with respect to s, we obtain (3.2).

Differentiating both sides of (3.2), k times, with respect to s, we get (3.3).

Differentiating both sides of (3.3), ℓ times, with respect to p, we obtain (3.4).

Differentiating both sides of (3.4), m times, with respect to q, we get (3.5).

Differentiating both sides of (3.5), n times, with respect to r, we obtain (3.6). □

Applying (1.5), (1.9) and (1.10) in the results of Theorem 3.1, we get a set of integral formulas asserted in the following corollary.

Corollary 3.2. *Each of the following integral formulas holds.*

$$\begin{aligned} &\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{x^s\,dx}{1-x} \\ &= H_r(s) + H_r(p+q+s) - H_r(p+s) - H_r(q+s) \\ &\qquad (\min\{p,\,q\} > 0,\; s > -1,\; r \in \mathbb{N}). \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{(\ln x)^k\,x^s\,dx}{1-x} \\ &= (-1)^k k!\Big\{H_r^{(k+1)}(p+q+s) + H_r^{(k+1)}(s) \\ &\qquad - H_r^{(k+1)}(p+s) - H_r^{(k+1)}(q+s)\Big\} \\ &\qquad (\min\{p,\,q\} > 0,\; s > -1;\; k,\, r \in \mathbb{N}). \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\int_0^1 (1-x^q)\,(1-x^r)\,\frac{(\ln x)^k\,x^s}{1-x}\,dx \\ &= (-1)^k\,k!\Big\{H_r^{(k+1)}(s) - H_r^{(k+1)}(q+s)\Big\} \\ &\qquad (q > 0,\; s > -1;\; k,\, r \in \mathbb{N}). \end{aligned} \tag{3.9}$$

$$\int_0^1 (1-x^r)\,\frac{(\ln x)^k\,x^s}{1-x}\,dx = (-1)^k\,k!\,H_r^{(k+1)}(s) \qquad (s > -1;\; k,\, r \in \mathbb{N}). \tag{3.10}$$

$$\int_0^1 \frac{(\ln x)^k\,x^{s+r}}{1-x}\,dx = (-1)^k\,k!\,\zeta(k+1, s+r+1) \qquad (s > -1;\; k,\, r \in \mathbb{N}). \tag{3.11}$$

Concluding remarks

One can see a number of integral formulas which are expressed in terms of gamma function and its related functions (see, e.g., [2, 14]). Applying the same technique used in this paper to those formulas, we can derive a large number of integral formulas. For example, recall a known integral formula (see, e.g., [14, Entry 3.554-4, p. 388])

$$\int_0^\infty e^{-2\beta x}\left(\frac{1}{x}-\coth x\right)dx=\psi(\beta)-\ln\beta+\frac{1}{2\beta}\quad(\Re(\beta)>0). \tag{3.12}$$

Differentiating both sides of (3.12) with respect to β, k times, and using (1.5), we get

$$\begin{aligned}&\int_0^\infty x^k\,e^{-2\beta x}\left(\frac{1}{x}-\coth x\right)dx\\&\quad=\frac{(k-1)!}{2^k}\left\{\frac{1}{\beta^k}+\frac{k}{2\,\beta^{k+1}}-k\,\zeta(k+1,\beta)\right\}\quad(\Re(\beta)>0).\end{aligned} \tag{3.13}$$

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