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EXTENDED GENERAL VARIATIONAL INEQUALITY FOR RELAXED COCOERCIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we consider an extended general variational inequality with three nonlinear operators, more precisely, relaxed (α, r)-cocoercive mappings. Using the projection technique, we show that the extended general variational inequality is equivalent to the nonlinear projection equation. This alternative equivalent formulation is used to discuss the existence and convergence(or approximate solvability) of a solution of the extended general variational inequality under suitable conditions.

1. Introduction

In recent years, many theory of variational inequalities and its special forms have been extended and generalized to study a wide applications and problems arising from several fields such as applied mathematics, optimization, control theory, equilibrium problems and nonlinear programming problems, etc. Standard variational inequality problem was introduced by Stampacchia [18] in 1964.

In 2009, Noor [11] introduced and studied the existence of solution for extended general variational inequality with three strongly monotone mappings.

In this paper, we consider an extended general variational inequality with three nonlinear operators, more precisely, relaxed cocoercive mappings. We

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show that the extended general variational inequality includes general variational inequalities and several other classes of variational inequalities as special cases. One can show that the extended general variational inequalities provide us with a unified, simple and natural framework in which to study a wide class of problems which arise in various areas of pure and applied sciences. Using the projection technique, it is shown that the extended general variational inequalities are equivalent to the nonlinear projection equation. This alternative equivalent formulation is used to discuss the existence and convergence of a solution of the extended general variational inequalities under suitable conditions.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed subset in H.

For given nonlinear operators $T, g, h : H \to H$, we consider the problem of finding $u \in H, h(u) \in K$ such that

$$\langle Tu, g(v) - h(u)\rangle \geq 0, \quad \forall\, v \in H,\ g(v) \in K. \tag{2.1}$$

An inequality of type (2.1) is called the *extended general variational inequality involving three operators*.

We would like to emphasize that problem (2.1) is equivalent to that of finding $u \in H$, $h(u) \in K$ such that

$$\langle \rho Tu + h(u) - g(u), g(v) - h(u)\rangle \geq 0, \quad \forall\, v \in H,\ g(v) \in K, \tag{2.2}$$

where $\rho > 0$ is constant. The inequality of type (2.2) is called the *auxiliary extended general variational inequality associated with the problem* (2.1). This equivalent formulation is also useful from the applications point of view.

Now, we list some special cases of the extended general variational inequalities.

I. If $g = h$, then problem (2.1) and problem (2.2) are equivalent to that of finding $u \in H$, $g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall\, v \in H,\ g(v) \in K,$$

which is known as general variational inequality (see, [12]). It turned out that odd order and nonsymmetric obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via general variational inequalities, see [13, 15].

II. For $h = I$, the identity operator, then problem (2.1) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \quad \forall\, v \in H,\ g(v) \in K,$$

which is also called the general variational inequality, see [10].

III. For $g = h = I$, the identity operator, the extended general variational inequality problem (2.1) is equivalent to that of finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall\, v \in K,$$

which is known as the classical variational inequality, see [18].

IV. If $K^* = \{u \in H : \langle u, v\rangle \geq 0,\ \forall\, v \in K\}$ is a polar (dual) convex cone of a closed convex cone K in H, then problem (2.1) is equivalent to that of finding $u \in H$ such that

$$g(u) \in K,\ \ Tu \in K^*,\ \ \langle g(u), Tu\rangle = 0, \tag{2.3}$$

which is known as the general complementarity problem. If $g = I$, the identity operator, then problem (2.3) is called the generalized complementarity problem. For $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem (2.3) is called the quasi (implicit) complementarity problem, see [15, 22].

From the above discussion, it is clear that the extended general variational inequality (2.1) is most general and includes several known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences.

We also need the following concepts and results.

Lemma 2.1. *Let K be a closed and convex subset in a Hilbert space H. Then for a given $z \in H$, $u \in K$ satisfies*

$$\langle u - z, v - u\rangle \geq 0,\ \ \forall\, v \in K \tag{2.4}$$

if and only if

$$u = P_K(z),$$

where P_K is the projection of H onto the closed and convex set K in H.

Remark 2.2. It is well known that the projection operator P_K is nonexpansive, that is,

$$\|P_K(u) - P_K(v)\| \leq \|u - v\|,\ \ \forall\, u, v \in H. \tag{2.5}$$

Definition 2.3. ([19]) Let H be a Hilbert space.

(1) A mapping $T : H \to H$ is called *α-strongly monotone*, if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y\rangle \geq \alpha\|x - y\|^2,$$

for a constant $r > 0$. This implies that

$$\|T(x) - T(y)\| \geq \alpha\|x - y\|,$$

that is, T is α-expansive and when $\alpha = 1$, it is expansive.

(2) A mapping $T : H \to H$ is called *β-Lipschitz continuous*, if there exists a constant $\beta \geq 0$ such that

$$\|T(x) - T(y)\| \leq \beta\|x - y\|, \quad \forall\, x, y \in H.$$

(3) A mapping $T : H \to H$ is called *μ-cocoercive*, if there exists a constant $\mu > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \mu\|T(x) - T(y)\|^2, \quad \forall\, x, y \in H.$$

Clearly, every μ-cocoercive mapping T is $\frac{1}{\mu}$-Lipschitz continuous.

(4) A mapping $T : H \to H$ is called *relaxed α-cocoercive*, if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\alpha)\|T(x) - T(y)\|^2, \quad \forall\, x, y \in H.$$

(5) A mapping $T : H \to H$ is called *relaxed (α, r)-cocoercive*, if there exists a constant $\alpha, r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\alpha)\|T(x) - T(y)\|^2 + r\|x - y\|^2,$$

for all $x, y \in H$. For $\alpha = 0$, T is r-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. It is easy to see that we have the following implication:

$$r - \text{strongly monotonicity} \quad \Rightarrow \quad \text{relaxed } \ (\alpha, r) - \text{cocoercivity}$$

3. Existence of a solution of the extended general variational inequality

In this section, we establish the equivalence between the extended general variational inequality (2.2) and the fixed point problem. This alternative equivalent formulation is used to study the existence of a solution of the extended general variational inequality under suitable conditions. We prove that the extended general variational inequality (2.2) is equivalent to the fixed point problem by invoking Lemma 2.1.

Lemma 3.1. ([11]) *The function $u \in H$, $h(u) \in K$ is a solution of the extended general variational inequality* (2.2) *if and only if $u \in H$, $h(u) \in K$ satisfies the relation*

$$h(u) = P_K(g(u) - \rho Tu), \tag{3.1}$$

where P_K is the projection operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the extended general variational inequality problem of (2.2) is equivalent to the fixed point problem. This alternative equivalence formulation is very useful from the numerical and theoretical points of view. Zhao and Sun [22] used the concept of the exceptional family to study the existence of a solution of the nonlinear projection equation (3.1). We rewrite the relation (3.1) in the following form:

$$F(u) = u - h(u) + P_K(g(u) - \rho Tu), \tag{3.2}$$

which is used to study the existence of a solution of the extended general variational inequality (2.2).

Now, we study those conditions under which the extended general variational inequality (2.2) has a solution and this is the main motivation for obtaining our next result.

Theorem 3.2. *Let the operators $T, g, h : H \to H$ be relaxed (α_T, r_T)-cocoercive, relaxed (α_g, r_g)-cocoercive, relaxed (α_h, r_h)-cocoercive and β_T-Lipschitz continuous, β_g-Lipschitz continuous, β_h-Lipschitz continuous, respectively. If*

$$\begin{aligned} &\left|\rho - \frac{r_T - \alpha_T\beta_T^2}{\beta_T^2}\right| < \frac{\sqrt{(r_T - \alpha_T\beta_T^2)^2 - \beta_T^2\delta(2-\delta)}}{\beta_T^2}, \\ &\beta_T^2\delta(2-\delta) < (r_T - \alpha_T\beta_T^2)^2 \le \beta_T^2, \quad \delta < 1, \end{aligned} \tag{3.3}$$

where

$$\delta = \sqrt{1 - 2r_h + (2\alpha_h + 1)\beta_h^2} + \sqrt{1 - 2r_g + (2\alpha_g + 1)\beta_g^2},$$

then there exists a unique solution $u \in H$, $h(u) \in K$ of the extended general variational inequality (2.2).

Proof. From Lemma 3.1, it follows that problems (3.2) and (2.2) are equivalent. Thus it is enough to show that the mapping $F(u)$, defined by (3.2), has a fixed point. For all $u \neq v \in H$, we have

$$\begin{aligned}
&\|F(u) - F(v)\| \\
&= \|u - v - (h(u) - h(v)) + P_K(g(u) - \rho Tu) - P_k(g(v) - \rho Tv)\| \\
&\leq \|u - v - (h(u) - h(v))\| + \|P_K(g(u) - \rho Tu) - P_k(g(v) - \rho Tv)\| \\
&\leq \|u - v - (h(u) - h(v))\| + \|u - v - (g(u) - g(v))\| \\
&\quad + \|u - v - \rho(Tu - Tv)\|,
\end{aligned} \tag{3.4}$$

where we have used the fact (2.5). Since the operator T is relaxed (α_T, r_T)-cocoercive and β_T-Lipschitz continuous, it follows that

$$\begin{aligned}
&\|u - v - \rho(Tu - Tv)\|^2 \\
&= \|u - v\|^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2\|Tu - Tv\|^2 \\
&\leq \|u - v\|^2 + 2\rho\alpha_T\|Tu - Tv\|^2 - 2\rho r_T\|u - v\|^2 + \rho^2\|Tu - Tv\|^2 \\
&= \Big(1 - 2\rho r_T + (2\rho\alpha_T + \rho^2)\beta_T^2\Big)\|u - v\|^2.
\end{aligned} \tag{3.5}$$

In a similar way, we have

$$\begin{aligned}
&\|u - v - (g(u) - g(v))\|^2 \\
&= \|u - v\|^2 - 2\langle u - v, g(u) - g(v)\rangle + \|g(u) - g(v)\|^2 \\
&\leq \|u - v\|^2 + 2\alpha_g\|g(u) - g(v)\|^2 - 2r_g\|u - v\|^2 + \|g(u) - g(v)\|^2 \\
&\leq \Big(1 - 2r_g + (2\alpha_g + 1)\beta_g^2\Big)\|u - v\|^2
\end{aligned} \tag{3.6}$$

and

$$\|u - v - (h(u) - h(v))\|^2 \leq \Big(1 - 2r_h + (2\alpha_h + 1)\beta_h^2\Big)\|u - v\|^2. \tag{3.7}$$

From (3.4)-(3.7), we get

$$\begin{aligned}
&\|F(u) - F(v)\| \\
&\leq \Big(\sqrt{1 - 2r_h + (2\alpha_h + 1)\beta_h^2} + \sqrt{1 - 2r_g + (2\alpha_g + 1)\beta_g^2} \\
&\quad + \sqrt{1 - 2\rho r_T + (2\rho\alpha_T + \rho^2)\beta_T^2}\Big)\|u - v\| \\
&= \Big(\delta + k(\rho)\Big)\|u - v\| \\
&= \theta\|u - v\|,
\end{aligned} \tag{3.8}$$

where

$$k(\rho) = \sqrt{1 - 2\rho r_T + (2\rho\alpha_T + \rho^2)\beta_T^2}$$

and

$$\theta = \delta + k(\rho).$$

From conditions (3.3), it follows that

$$\theta < 1.$$

Thus (3.8) implies that F is a contractive mapping and so there exists a unique point $u^* \in H$ such that $F(u^*) = u^*$. Therefore, from the mapping $F(u)$, defined by (3.2), has a fixed point which is the unique solution of (2.2). This completes the proof. □

Corollary 3.3. *Let the operators $T, g, h : H \to H$ be r_T-strongly monotone, r_g-strongly monotone, r_h-strongly monotone and β_T-Lipschitz continuous, β_g-Lipschitz continuous, β_h-Lipschitz continuous, respectively. If*

$$\left|\rho - \frac{r_T}{\beta_T^2}\right| < \frac{\sqrt{r_T^2 - \beta_T^2\delta(2-\delta)}}{\beta_T^2},$$
$$\beta_T^2\delta(2-\delta) < r_T^2 \le \beta_T^2, \quad \delta < 1,$$

where

$$\delta = \sqrt{1 - 2r_h + \beta_h^2} + \sqrt{1 - 2r_g + \beta_g^2},$$

then there exists a unique solution $u \in H$, $h(u) \in K$ of the extended general variational inequality (2.2).

Proof. For the mapping T be a relaxed (α, r)-cocoercive, if we take $\alpha = 0$, then relaxed (α, r)-cocoercive mapping being a r-strongly monotone mapping. So, we should take $\alpha = 0$ in Theorem 3.2, we can easily deduce the result of Corollary 3.3. □

Remark 3.4. Theorem 3.2 extends and improves the main result in Noor [11].

4. Convergence theorem of a solution of the extended general variational inequality

In this section, we study the approximation solvability of the extended general variational inequality problem (2.2) involving three operators for projection methods and its special iterative algorithm.

Most of all, Lemma 3.1 implies that (2.2) is equivalent to the fixed point problem (3.1). Using the fixed point formulation (3.2), we suggest and analyze iterative form:

$$u = (1 - a_n)u + a_n\{u - h(u) + P_K(g(u) - \rho Tu)\}, \tag{4.1}$$

where $0 \le a_n \le 1$ for all $n \ge 0$.

This alternative formulation is used to suggest the following algorithm for solving an extended general variational inequality (2.2) and its variant form.

Algorithm 4.1. For arbitrary chosen initial points $u_0 \in K$ compute the iterative sequence $\{u_n\}$ such that

$$u_{n+1} = (1 - a_n)u_n + a_n\{u_n - h(u_n) + P_K(g(u_n) - \rho Tu_n)\}, \tag{4.2}$$

where P_K is the projection of H onto K, $\rho > 0$ is constant and $\{a_n\}$ be sequence in $[0, 1]$.

Lemma 4.1. ([21]) *If $\{\lambda_n\}$ is a nonnegative sequence satisfying the following inequality:*

$$\lambda_{n+1} \le (1 - t_n)\lambda_n + \mu_n, \ \ \forall\, n \ge 0$$

with $0 \le t_n \le 1$, $\sum_{n=0}^{\infty} t_n = \infty$ and $\mu_n = o(t_n)$, then

$$\lim_{n\to\infty} \lambda_n = 0.$$

Theorem 4.2. *Let the operators $T, g, h : H \to H$ be relaxed (α_T, r_T)-cocoercive, relaxed (α_g, r_g)-cocoercive, relaxed (α_h, r_h)-cocoercive and β_T-Lipschitz continuous, β_g-Lipschitz continuous, β_h-Lipschitz continuous, respectively. Let $\{u_n\}$ be the iterative sequence generated by Algorithm 4.1. If $\{a_n\}$ is sequence in $[0, 1]$ satisfying the following conditions:*

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n = \infty, \\ &\left|\rho - \frac{r_T - \alpha_T\beta_T^2}{\beta_T^2}\right| < \frac{\sqrt{(r_T - \alpha_T\beta_T^2)^2 - \beta_T^2\delta(2-\delta)}}{\beta_T^2}, \\ &\beta_T^2\delta(2-\delta) < (r_T - \alpha_T\beta_T^2)^2 \le \beta_T^2, \quad \delta < 1, \end{aligned} \tag{4.3}$$

where

$$\delta = \sqrt{1 - 2r_h + (2\alpha_h + 1)\beta_h^2} + \sqrt{1 - 2r_g + (2\alpha_g + 1)\beta_g^2},$$

then the sequence $\{u_n\}$ converges strongly to u.

Proof. Since $u \in H$, $h(u) \in K$ is a solution of the extended general variable inequality (2.2), by Lemma 3.1, we know that

$$h(u) = P_K(g(u) - \rho Tu), \quad \rho > 0.$$

It follows from (4.1) and (4.2) that

$$\begin{aligned}
&\|u_{n+1} - u\| \\
&= \|(1 - a_n)u_n + a_n\{u_n - h(u_n) + P_K(g(u_n) - \rho T u_n)\} \\
&\quad - (1 - a_n)u - a_n\{u - h(u) + P_K(g(u) - \rho T u)\}\| \\
&\le (1 - a_n)\|u_n - u\| + a_n\|u_n - u - (h(u_n) - h(u))\| \\
&\quad + a_n\|u_n - u - (g(u_n) - g(u))\| + a_n\|u_n - u - \rho(T u_n - T u)\|.
\end{aligned} \tag{4.4}$$

Since T is relaxed (α_T, r_T)-cocoercive and β_T-Lipschitz continuous, from (3.5), we have

$$\|u_n - u - \rho(T u_n - T u)\|^2 \le \Big(1 - 2\rho r_T + (2\rho\alpha_T + \rho^2)\beta_T^2\Big)\|u_n - u\|^2. \tag{4.5}$$

Similarly, from (3.6) and (3.7), we get

$$\|u_n - u - (g(u_n) - g(u))\|^2 \le \Big(1 - 2r_g + (2\alpha_g + 1)\beta_g^2\Big)\|u_n - u\|^2 \tag{4.6}$$

and

$$\|u_n - u - (h(u_n) - h(u))\|^2 \le \Big(1 - 2r_h + (2\alpha_h + 1)\beta_h^2\Big)\|u_n - u\|^2. \tag{4.7}$$

From (4.4)-(4.7), we get

$$\begin{aligned}
\|u_{n+1} - u\| &\le (1 - a_n)\|u_n - u\| \\
&\quad + a_n\sqrt{1 - 2r_h + (2\alpha_h + 1)\beta_h^2}\,\|u_n - u\| \\
&\quad + a_n\sqrt{1 - 2r_g + (2\alpha_g + 1)\beta_g^2}\,\|u_n - u\| \\
&\quad + a_n\sqrt{1 - 2\rho r_T + (2\rho\alpha_T + \rho^2)\beta_T^2}\,\|u_n - u\| \\
&= (1 - a_n(1 - \theta))\|u_n - u\|,
\end{aligned}$$

where

$$k(\rho) = \sqrt{1 - 2\rho r_T + (2\rho\alpha_T + \rho^2)\beta_T^2}$$

and

$$\theta = \delta + k(\rho).$$

From conditions (4.3), it follows that

$$\theta < 1.$$

Taking

$$\lambda_n = \|u_n - u\|, \quad t_n = a_n(1 - \theta) \quad \text{and} \quad \mu_n = 0,$$

we know that all conditions in Lemma 4.1 are satisfied. Therefore, $\|u_n - u\| \to 0$ as $n \to \infty$, *i.e.*,

$$u_n \to u \quad \text{as} \quad n \to \infty.$$

This completes the proof of Theorem 4.2. □

Corollary 4.3. *Let the operators* $T, g, h : H \to H$ *be* r_T*-strongly monotone,* r_g*-strongly monotone,* r_h*-strongly monotone and* β_T*-Lipschitz continuous,* β_g*-Lipschitz continuous,* β_h*-Lipschitz continuous, respectively. Let* $\{u_n\}$ *be the iterative sequence generated by Algorithm* 4.1. *If* $\{a_n\}$ *is sequence in* $[0, 1]$ *satisfying the following conditions:*

$$\sum_{n=0}^{\infty} a_n = \infty,$$

$$\left|\rho - \frac{r_T}{\beta_T^2}\right| < \frac{\sqrt{(r_T)^2 - \beta_T^2 \delta(2-\delta)}}{\beta_T^2},$$

$$\beta_T^2 \delta(2-\delta) < r_T^2 \leq \beta_T^2, \quad \delta < 1,$$

where

$$\delta = \sqrt{1 - 2r_h + \beta_h^2} + \sqrt{1 - 2r_g + \beta_g^2},$$

then the sequence $\{u_n\}$ *converges strongly to* u.

Proof. For the mapping T be a relaxed (α, r)-cocoercive, if we take $\alpha = 0$, then relaxed (α, r)-cocoercive mapping being a r-strongly monotone mapping. So, if we take $\alpha = 0$, then we can easily obtain the result of Corollary 4.3 from Theorem 4.2 immediately. □

5. Conclusion

The extended general variational inequalities include various classes of variational inequalities and optimization problems as special cases, its results proved in this paper continue to hold for these problems. It is expect that this class will inspire and motivate further research in this area (see, [1]-[9], [14], [16, 17], [20]).

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