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EXISTENCE OF SOLUTIONS FOR MULTI-VALUED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we establish the sufficient conditions for the existence of solutions for multi-valued equilibrium problems.

1. Introduction

The equilibrium problem is a unified model of several problem, for examples, optimization problems, variational inequality problems, complementarity problems and saddle point problems. In the literature, existence results for various type of equilibrium problems have been investigated intensively (see [1, 2, 3, 5, 10, 16, 21, 24, 25]).

Recently Kristly and Varga [17] considered the weak in the sense that the convexity and continuity assumptions must not hold on the whole domain, but just on a special type of dense subset of it that we call self segment dense (see [19]). This new concepts are related to but different from that of a segment

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dense set introduced by Luc [22] in the context of densely quasi-monotone respectively densely pseudo monotone operators.

Inspired and motivated by the recent research works [6, 7, 12, 13, 14, 15, 18, 20, 23, 26, 27, 28], we discuss the role of segment dense sets in the context of multi-valued equilibrium problems both with and without compactness assumptions and proved some existence theorems of multi-valued equilibrium problems.

2. Preliminaries

Throughout this paper, let X and Y be two real Hausdorff topological spaces. For a nonempty set $D \subseteq X$, we denotes by $int(D)$ its interior and by $cl(D)$ its closure. We say that $P \subseteq D$ is dense in D if $D \subseteq cl(P)$ and $P \subseteq X$ is closed regarding D if $cl(P) \cap D = P \cap D$. Let $T : X \to Y$ be a multi-valued mapping.

We denote by $D(T) = \{x \in X : T(x) \neq \emptyset\}$ its domain and by $R(T) = \bigcup_{x \in D(T)} T(x)$ its range. The graph T is the set $G(T) = \{(x, y) \in X \times Y : y \in T(x)\}$. T is said to be upper semicontinuous at $x \in D(T)$ if for every open set $N \subseteq Y$ containing $T(x)$, there exists a neighborhood $M \subseteq X$ of x such that $T(M) \subseteq N$. T is said to be lower semicontinuous at $x \in D(T)$ if for every open set $N \subseteq Y$ satisfying $T(x) \cap N \neq \emptyset$, there exists a neighborhood $M \subseteq X$ of x such that for every $y \in M \cap D(T)$ has $T(y) \cap N \neq \emptyset$. T is upper semi continuous (lower semicontinuous) on $D(T)$ if it is upper semicontinuous (lower semicontinuous) at every $x \in D(T)$. For $V \subseteq Y$, define the following sets

$$T^{-1}(v) = \{x \in X : T(x) \cap V \neq \emptyset\}$$

and

$$T^{+}(v) = \{x \in X : T(x) \subseteq V\},$$

called the inverse image of V and the cone of V, respectively.

Lemma 2.1. ([4]) *Let $T : X \to Y$ be a set-valued mapping. Then*

(i) *T is lower semicontinuous at $x \in D(T)$ if and only if for every net $\{x_\alpha\} \subseteq D(T)$ such that $x_\alpha \to x$ and for every $x^* \in T(x)$, there exists a net $x^*_\alpha \in T(x_\alpha)$ such that $x^*_\alpha \to x^*$.*

(ii) *T is upper semicontinuous at $x \in D(T)$ if and only if for every net $\{x_\alpha\} \subseteq D(T)$ such that $x_\alpha \to x$ and for every open set $V \subseteq Y$ such that $T(x) \subseteq V$ and $F(x_\alpha) \subseteq V$ for sufficiently large α.*

(iii) *T is lower semicontinuous if and only if for every closed set $V \subseteq Y$, $T^{+}(V)$ is closed in X.*

(iv) *T is upper semicontinuous if and only if for every closed set $V \subseteq Y$, $T^{-}(V)$ is closed in X.*

For a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, we denote by $domf$ its domain that is $domf = \{x \in X | f(x) \in \mathbb{R}\}$. We say that f is upper semi continuous at $x_0 \in domf$ if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that

$$f(x) \leq f(x_0) + \epsilon$$

for all $x \in U$. The function f is called upper semicontinuous if it is upper semi continuous at every point of its domain. Also we say that f is lower semicontinuity at $x_0 \in domf$ if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that

$$f(x) \geq f(x_0) - \epsilon$$

for all $x \in U$. The function f is called lower semicontinuous if it is lower semicontinuous at every point of its domain.

Lemma 2.2. *Let $f : X \to \overline{\mathbb{R}} = \mathbb{R} \bigcup \{+\infty\}$ be a function. Then*

(i) *f is uppersemi continuous at x_0 if and only if*

$$\limsup_{x_\alpha \to x_0} f(x_\alpha) \leq f(x_0)$$

where $\{x_\alpha\}$ is a net converging to x_0.

(ii) *f is lower semicontinuous at x_0 if and only if*

$$\liminf_{x_\alpha \to x_0} f(x_\alpha) \geq f(x_0)$$

where $\{x_\alpha\}$ is a net converging to x_0.

(iii) *f is upper semicontinuous on X if and only if the super level set $\{x \in X : f(x) \geq a\}$ is a closed set for every $a \in R$.*

(iv) *f is lower semicontinuous on X if and only if the super level set $\{x \in X : f(x) \leq a\}$ is a closed set for every $a \in R$.*

Lemma 2.3. ([11]) *If T is compact valued, then T is upper semicontinuous if and only if for every net $\{x_i\} \subseteq X$ such that $x_i \to x_0 \in X$ and for every $z_i \in T(x_i)$ there exists $z_0 \in T(x_0)$ and a subnet $\{z_{i_j}\}$ of $\{z_i\}$ such that $z_{i_j} \to z_0$.*

3. Multi-valued equilibrium problems

Let K be a nonempty subset of a real normed space X. Let $F : K \times K \to \mathbb{R}$ be a multi-valued mapping and let $T : K \to 2^K$ be a multi-valued mapping.

We consider a multi-valued equilibrium problem: for finding $x_0 \in K$ with $u_0 \in T(x_0)$ such that

$$F(u_0, y) \geq 0, \ \forall y \in K, \tag{3.1}$$

that is,

$$F(u_0, y) \subseteq [0, \infty) = \mathbb{R}_+.$$

Again, consider a multi-valued equilibrium problem: for finding $x_0 \in K$ with $u_0 \in T(x_0)$ such that

$$F(u_0, y) \cap \mathbb{R}_- \neq \emptyset, \ \forall y \in K. \tag{3.2}$$

Proposition 3.1. ([17]) *Let K be a a nonempty convex compact subset of a real normed space X. Let $F : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in K, x \to F(x, y)$ is lower semicontinuous on K,*
(ii) *for all $y \in K, y \to F(x, y)$ is convex on K,*
(iii) *for all $x \in K, F(x, x) \geq 0$.*

Then there exists $x_0 \in K$ such that

$$F(x_0, y) \geq 0, \ \forall y \in K.$$

Proposition 3.2. *Let K be a a nonempty convex compact subset of a real normed space X. Let $F : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in K, x \to F(x, y)$ is upper semicontinuous on K,*
(ii) *for all $y \in K, y \to F(x, y)$ is convex on K,*
(iii) *for all $x \in K, F(x, x) \cap \mathbb{R}_- \neq \emptyset$.*

Then there exists $x_0 \in K$ such that

$$F(x_0, y) \cap \mathbb{R}_- \neq \emptyset, \forall y \in K.$$

The convexity of multi-valued mapping $F : D \subset X \to \mathbb{R}$ where X is Hausdorff topological space, is understood in sense that for all $x_1, x_2 \cdots, x_n \in D$ and $\lambda_i \geq 0$, $(i = 1, 2, \cdots, n), \sum_{i=1}^n \lambda_i = 1$ such that $\sum_{i=1}^n \lambda_i x_i \in D$ and

$$\sum_{i=1}^n \lambda_i F(x_i) \subseteq F(\sum_{i=1}^n \lambda_i x_i). \tag{3.3}$$

To define the convexity in similar setting by

$$\sum_{i=1}^n \lambda_i F(x_i) \supseteq F(\sum_{i=1}^n \lambda_i x_i), \tag{3.4}$$

here we do not assume that D is convex, then classical equilibrium problems for $\phi : K \times K \to \mathbb{R}$ is to find $x_0 \in K$ such that

$$\phi(x_0, y) \geq 0, \ \forall y \in K.$$

Theorem 3.3. ([9]) *Let K be a nonempty convex compact subset of a Hausdorff topological space X and $\phi : K \times K \to \mathbb{R}$ be a mapping satisfying:*

(i) *for all $y \in K, x \to \phi(x, y)$ is upper semicontinuous on K,*
(ii) *for all $y \in K, y \to \phi(x, y)$ is quasi convex on K,*
(iii) *for all $x \in K, \phi(x, x) \geq 0$.*

Then there exists $x_0 \in K$ *such that*

$$\phi(x_0, y) \geq 0, \forall y \in K.$$

Definition 3.4. Let X be a Hausdorff topological vector space and $M \subseteq X$. $G : M \to X$ is called a KKM-mapping if for every finite number of elements $x_1, \cdots, x_n \in M$ we have

$$co\{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i).$$

Lemma 3.5. ([8]) *Let* X *be a Hausdorff topological vector space,* $M \subseteq X$ *and* $G : M \to X$ *a KKM-mapping. If* $G(x)$ *is closed for every* $x \in M$ *and there exists* $x_0 \in M$ *such that* $G(x_0)$ *is compact, then*

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

Self segment-dense sets: Let X be a Hausdorff topological vector space. The open and *respectively* closed line segments in X with the end points x and y are defined by:

$$(x, y) = \{z \in X : z = x + t(y - x), t \in (0, 1)\},$$
$$[x, y] = \{z \in X : z = x + t(y - x), t \in [0, 1]\}.$$

Let $V \subseteq X$ be a convex set and U be a segment-dense subset in V. Then for each $x \in V$ there exists $y \in U$ such that x is a cluster point of the set $[x, y] \cap U$ (see Luc [22]).

Lemma 3.6. ([19]) *Let* X *be a Hausdorff topological vector space. Let* U *and* V *be two subsets of* X *with* $U \subseteq V$ *and assume that* V *is convex. Then* U *is self segment-dense in* V *if* U *is dense in* V *and for all* $x, y \in U$*, the set* $[x, y] \cap U$ *is dense in* $[x, y]$.

4. Self segment-dense sets and multi-valued equilibrium problems

Let X be a Hausdorff locally convex topological vector space. Then the origin has a local base of convex, balanced and absorbent sets and recall the set

$$coreD = \{u \in D |\ x \in X,\ \exists \delta > 0 \text{ such that } \forall \epsilon \in [0, \delta] : u + \epsilon x \in D\}$$

is called the algebraic interior (or core) of $D \subseteq X$. If D is convex with nonempty interior, then $int(D) = core(D)$ (see [29]).

Lemma 4.1. ([19]) *Let X be a Hausdorff locally convex topological vector space, $V \subseteq X$ a convex set and $U \subseteq V$ a self segment-dense set in V. Then for all finite subset $\{u_1 \cdots, u_n\} \subseteq U$, we have*

$$cl(co\{u_1 \cdots, u_n\} \cap U) = co\{u_1 \cdots, u_n\}.$$

Theorem 4.2. *Let X be a Hausdorff locally convex topological vector space, K a nonempty convex compact subset of X and D a self segment-dense subset of K. Let $T : K \to 2^K$ be a multi-valued mapping and $F : K \times K \to \mathbb{R}$ be a set-valued mapping satisfying:*

(i) *for all $y \in D, x \to F(x, y)$ is lower semicontinuous on K,*
(ii) *for all $x \in K, x \to F(x, y)$ is lower semicontinuous on $K \setminus D$,*
(iii) *for all $x \in D, y \to F(x, y)$ is convex on D,*
(iv) *for all $x \in D$, T is lower semicontinuous and $T(x)$ is compact,*
(v) *for all $x \in D, F(x, x) \geq 0$.*

Then there exists $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$F(u_0, y) \geq 0, \ \forall y \in K.$$

Proof. Consider a map $G : D \to K$ defined by

$$G(y) = \{x \in K | \ \exists \ u \in T(x) \text{ such that } F(u, y) \geq 0, \forall y \in K\}.$$

We prove that

$$\bigcap_{y \in D} G(y) \neq \emptyset,$$

or there exists $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$F(u_0, y) \geq 0, \ \forall y \in D.$$

First, we prov that $G(y)$ is closed for all $y \in D$. To this end, let for $y \in D$ and net $\{x_\alpha\} \subseteq G(y)$, $\lim x_\alpha = x \in K$. Then there exists $u_\alpha \in T(x_\alpha)$ such that

$$F(u_\alpha, y) \geq 0.$$

From Lemma 2.3, we obtain that $\{u_\alpha\}$ contains a subsequence $\{u_{\alpha_k}\}$ that converges to a $u^* \in T(x)$ in the norm topology of X. From the lower semicontinuity assumption (i), we have that for every $x_\alpha \in F(u_\alpha, y)$, there exists a net $\{x^*_\alpha\} \subseteq F(u^*_\alpha, y)$ such that $x^*_\alpha \to x^*$ and also $u^*_\alpha \to u^*$. Since $x^*_\alpha \geq 0$ for all α and $T(x^*)$ is compact, we have $x^* \geq 0$. Thus

$$F(u, y) \geq 0$$

which show that $x \in G(y)$ and the set $G(y) \subseteq K$ is closed. Since K is compact also $G(y)$ is compact for all $y \in D$. Hence G is a KKM-mapping satisfies the assumption of Ky Fan's Lemma, so we have

$$\bigcap_{y\in K} G(y) \neq \emptyset.$$

This means that there exists $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$F(u_0, y) \geq 0,\ y \in K.$$

In other worlds there exists $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$F(u_0, y) \geq 0,\ \forall y \in D.$$

At this point we make use of the assumption (ii) to extend the previous statement to the whole set K. Consider $y \in K\backslash D$, since D is dense in K there exists a net $\{y_\alpha\} \subseteq D$ such that $\lim y_\alpha = y$. From assumption (ii) and Definition 2.1, for every $y^* \in F(u_0, y)$ there exists a net $\{y^*_\alpha\} \subseteq F(u_0, y^\alpha)$ such that $y^*_\alpha = y^*$. But obviously $y^*_\alpha \geq 0$, hence $y^* \geq 0$ and finally

$$F(u_0, y) \geq 0,\ \forall y \in K, u_0 \in T(x_0).$$

□

Theorem 4.3. *Let X, K, D, T be as in Theorem 4.2. Let $F : K \times K \to \mathbb{R}$ be a set-valued mapping satisfying:*

(i) *for all $y \in D, x \to F(x, y)$ is upper semicontinuous on K,*
(ii) *for all $x \in K, y \to F(x, y)$ is upper semicontinuous on $K \setminus D$,*
(iii) *for all $x \in D, y \to F(x, y)$ is concave on D,*
(iv) *T is upper semicontinuous, concave and $T(x)$ is compact,*
(v) *for all $x \in D, F(x, x) \cap \mathbb{R}_+ \neq \emptyset$.*

Then there exists $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$F(u_0, y) \cap \mathbb{R}_+ \neq \emptyset,\ \forall y \in K.$$

Proof. Consider a map $G : D \to K$ defined by

$$G(y) = \{x \in K \mid \exists\ u \in T(x) : F(u, y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in K\}.$$

First, we prove that $G(y)$ is closed for all $y \in D$. For a fixed $y \in D$ we have $F(y) = f_y^{-1}(\mathbb{R}_+)$, where $f_y : K \to \mathbb{R}, f_y(x) = f(x, y)$. From (i) we have f_y is upper semicontinuous on K and $\mathbb{R}_+$ is closed. From Lemma 2.1, $f_y^{-1}(\mathbb{R}_+)$ is closed. Since T is upper semicontinuous and concave, $G(y) \subseteq K$ is closed for all $y \in D$ and by the compactness of K we get $G(y)$ is compact for every $y \in D$. From Theorem 4.2 we prove that

$$\bigcap_{y\in D} G(y) \neq \emptyset,$$

that is, there exists $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$f(u_0, y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in D.$$

Now, let for $y \in K \setminus D$ and assume that

$$F(u_0, y) \subset (-\infty, 0).$$

Since the set-valued function $F(x_0, \cdot)$ is upper semicontinuous at y, we obtain that there exists an open neighborhood U of y such that

$$F(u_0, U) \not\subseteq (-\infty, 0).$$

Since D is dense in K, there exists $z \in U$ such that $z \in D$, so we have

$$F(u_0, z) \cap \mathbb{R}_+ \neq \emptyset$$

which is a contradiction. Thus we obtain that

$$F(u_0, y) \cap \mathbb{R}_+ \neq \emptyset, \forall y \in K.$$

This is completes the proof. □

Let $f : K \subseteq X \to X$ be a mapping. Then we say that f is convex (resp. concave) on K, if for $x_1, \cdots, x_n \in K$ and $\lambda_i \geq 0, i \in \{1, 2, \cdots, n\}, \sum_{i=1}^n \lambda_i = 1$ such that $\sum_{i=1}^n \lambda_i x_i \in K$, we have

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq f(\sum_{i=1}^{n} \lambda_i x_i) \quad \Big(\text{resp.} \quad \sum_{i=1}^{n} \lambda_i f(x_i) \leq f(\sum_{i=1}^{n} \lambda_i x_i)\Big).$$

Theorem 4.4. *Let X, K, D, T be as in Theorem 4.2. Let $\phi : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D, x \to \phi(x, y)$ is upper semicontinuous on K,*
(ii) *for all $x \in K, y \to \phi(x, y)$ is upper semicontinuous on $K \setminus D$,*
(iii) *for all $x \in D$, the function $y \to \phi(x, y)$ is concave on D,*
(iv) *T is upper semicontinuous, concave and $T(x)$ is compact,*
(v) *for all $x \in D, \phi(x, x) \geq 0$.*

Then there exist $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$\phi(u_0, y) \geq 0, \ \forall y \in K.$$

Proof. From similar proof of Theorem 4.2, consider a map $G : D \to K$ defined by

$$G(y) = \{x \in K \mid \exists \ u \in T(x) : \phi(u, y) \geq 0\}.$$

Observe that for a fixed $y \in D$, the set $G(y)$ is the super level set

$$\{x \in K : f_y(x) \geq 0\}$$

of the function $f_y : K \to \mathbb{R}$ defined by $f_y(x) = \phi(u, y)$ for $u \in T(x)$. From the assumptions (i), (iv) and (v), we have that $G(y)$ is closed for all $y \in D$.

Further from assumption (iii), (v) and Lemma 4.1, we obtain that G is a KKM-mapping. Then from Ky Fan's Lemma,

$$\bigcap_{y\in D} G(y) \neq \emptyset.$$

Hence there exist $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$\phi(u_0, y) \geq 0, \forall y \in D.$$

Finally, let $y \in K \setminus D$. Then from the density of D in K, there exists a net $\{y_\alpha\} \subseteq D$ such that $\lim y_a = y$. Since $\phi(u_0, y)$ is the upper semicontinuous on $K \setminus D$ from the assumptions (ii) and (iv), this assure that

$$0 \leq \limsup_{y_\alpha \to y} \phi(u_0, y_\alpha) \leq \phi(u_0, y).$$

Thus

$$\phi(u_0, y) \geq 0, \ \forall y \in K.$$

This completes the proof. □

5. Densely defined multi-valued equilibrium problems without compactness

In this section, we will prove the multi-valued equilibrium problems without compactness.

Theorem 5.1. *Let X be a Hausdorff locally convex topological vector space, K a nonempty convex compact subset of X and D a self segment-dense subset of K. Let $T : K \to 2^K$ be a multi-valued mapping and $F : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D, x \to F(x, y)$ is lower semicontinuous on K,*
(ii) *for all $x \in K, y \to F(x, y)$ is lower semicontinuous on $K \setminus D$,*
(iii) *for all $x \in D, y \to F(x, y)$ is convex on D,*
(iv) *T is lower semi continuous and convex,*
(v) *for all $x \in D, F(x, x) \geq 0$,*
(vi) *there exists a compact set $K_0 \subseteq X$ such that for $y_0 \in D \cap K_0$,*

$$F(u, y_0) \cap (-\infty, 0) \neq \emptyset, \forall x \in K \setminus K_0, u \in T(x).$$

Then there exist $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$F(u_0, y) \geq 0, \ \forall y \in K.$$

Proof. Consider a map $G : D \to K$ defined by

$$G(y) = \{x \in K \mid \exists\ u \in T(x) : F(u, y) \geq 0\}.$$

From the proof of Theorem 4.4, $G(y)$ is closed for all $y \in D$. We show that $G(y_0)$ is compact and the rest of the proof is similar to the proof of Theorem 4.4. It is enough to show that $G(y_0) \subseteq K_0$. Assume the contrary that there exists $z \in G(y_0)$ such that $z \notin K_0$, then

$$F(z, y_0) \geq 0,$$

which is contradicts to (vi). □

Theorem 5.2. *Let X, K, D, T be as in Theorem 5.1. Let $F : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D, x \to F(x, y)$ is upper semicontinuous on K,*
(ii) *for all $x \in K, y \to F(x, y)$ is upper semicontinuous on $K \setminus D$,*
(iii) *T is upper semicontinuous, concave and compact,*
(iv) *for all $x \in D, y \to F(x, y)$ is concave on D,*
(v) *for all $x \in D, F(x, x) \cap \mathbb{R}_+ \neq \emptyset$,*
(vi) *there exists a compact set $K_0 \subseteq X$ such that for $y_0 \in D \cap K_0$,*

$$F(x, y_0) \cap \mathbb{R}_+ \neq \emptyset, \forall x \in K \setminus K_0.$$

Then there exist $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$F(u_0, y) \cap \mathbb{R}_+ \neq \emptyset, \ \forall y \in K.$$

Theorem 5.3. *Let X, K, D, T be as in Theorem 5.1. Let $\phi : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D, x \to \phi(x, y)$ is upper semicontinuous on K,*
(ii) *for all $x \in D, y \to \phi(x, y)$ is upper semicontinuous on $K \setminus D$,*
(iii) *T is upper semicontinuous, convex and compact,*
(iv) *for all $x \in D, y \to \phi(x, y)$ is convex on D,*
(v) *for all $x \in D, \phi(x, x) \geq 0$,*
(vi) *there exists a compact set $K_0 \subseteq X$ such that for $y_0 \in D \cap K_0$,*

$$\phi(u, y_0) < 0, \ \ \forall x \in K \setminus K_0, \ u \in T(x).$$

Then there exists $x_0 \in K$ such that for $u_0 \in T(x_0)$,

$$\phi(u_0, y) \geq 0, \ \forall y \in K.$$

The condition (vi) in Theorem 5.1, Theorem 5.2 and Theorem 5.3 seem to be not so easy to verify. However it is well known that the closed ball $\overline{B}_r = \{x \in X : \|x\| \leq r\}, r > 0$ is weakly compact in a reflexive Banach space X. Therefore, for the reflexive Banach space X with the weak topology, condition (vi) of the previous theorem becomes:

(vi') there exists $r > 0$ and $y_0 \in D, \|y_0\| \leq r$ such that for all $x \in K, u \in T(x), \|x\| > r, F(u, y_0) \cap (-\infty, 0) \neq \emptyset$ holds,
(vi") there exists $r > 0$ and $y_0 \in D, \|y_0\| \leq r$ such that for all $x \in K, u \in T(x), \|x\| > r, F(u, y_0) \cap \mathbb{R}_+ = \emptyset$ holds,

(vi"') there exists $r > 0$ and $y_0 \in D, \|y_0\| \leq r$ such that for all $x \in K,\ u \in T(x),\ \|x\| > r,\ \phi(u, y_0)) < 0$ holds.

Furthermore, condition (vi) in the hypothesis of Theorem 5.1, 5.2 and 5.3 can be weaken by assuming that there exists $r > 0$ such that for all $x \in K, u \in T(x),\ \|x\| > r$ there exists $y_0 \in K$ with $\|y_0\| < \|x\|$ and the appropriate conditions:

(i) $F(u, y_0) \cap (-\infty, 0) \neq \emptyset$,
(ii) $F(u, y_0) \cap \mathbb{R}_+ = \emptyset$,
(iii) $F(u, y_0) < 0$ holds.

More precisely we have the following results.

Theorem 5.4. *Let X be a reflexive Banach space, $K \subseteq X$ a nonempty convex closed subset and $D \subseteq K$ a self segment dense set in the weak topology of X. Assume that $T : K \to 2^K$ is a multi-valued mapping. Let $F : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D, x \to F(x, y)$ is weak lower semi continuous on K,*
(ii) *for all $x \in K, y \to F(x, y)$ is weak lower semi continuous on $K \setminus D$,*
(iii) *T is weak lower semicontinuous, convex and compact,*
(iv) *for all $x \in K, y \to F(x, y)$ is convex on K,*
(v) *for all $x \in K, F(x, x) \geq 0$ and $\{0\} \subseteq F(x, x)$,*
(vi) *there exists $r > 0$ such that for all $x \in K, u \in T(x), \|x\| > 0$ there exists $y_0 \in K$ with $\|y_0\| < \|x\|$ such that*

$$F(u, y_0) \cap (-\infty, 0) \neq \emptyset.$$

Then there exists $x_0 \in K, u_0 \in T(x_0)$ such that

$$F(u_0, y) \geq 0,\ \forall y \in K.$$

Proof. Let $r > 0$ such that (v) holds and let $r_1 > r_2$. Let $K_0 = K \cap \overline{B}_{r_1}$. Since K is convex and closed it is also weakly closed, $\overline{B}_{r_1}$ is weakly compact, hence K_0 is convex and weakly compact. From Theorem 4.2 there exists $x_0 \in K_0, u_0 \in T(x_0)$ such that

$$F(u_0, y) \geq 0,\ \forall y \in K_0.$$

Next we prove that there exists $z_0 \in K_0, \|z_0\| < r_1$ such that

$$\{0\} \subseteq F(u_0, z_0), \forall u_0 \in T(x_0).$$

If $\|x_0\| < r_1$ then let $z_0 = x_0$ and the conclusion follows by (iv). If $\|x_0\| = r_1 > r$ then by (vi) we have that there exists $z_0 \in K$ with $\|z_0\| < \|x_0\|$ such that

$$F(u_0, z_0) \cap (-\infty, 0] \neq \emptyset.$$

On the other hand since $z_0 \in K_0$ we have

$$F(u_0, z_0) \geq 0, \forall u_0 \in T(x_0),$$

hence

$$\{0\} \subseteq F(u_0, z_0), \forall u_0 \in T(x_0).$$

Let $y \in K$. Then there exists $\lambda \in [0,1]$ such that $\lambda z_0 + (1-\lambda)y \in K_0$. Therefore

$$F(u_0, \lambda z_0 + (1-\lambda)y) \geq 0.$$

From (iv) we have

$$\lambda F(u_0, z_0) + (1-\lambda)F(u_0, y) \subseteq F(u_0, \lambda z_0 + (1-\lambda)y) \subseteq [0, \infty).$$

Since

$$\{0\} \subseteq F(u_0, z_0), \forall u_0 \in T(x_0),$$

we have

$$F(u_0, y) \subseteq [0, \infty).$$

$\square$

If we replace (vi) with a condition that assures the existence of a solution under the original assumption (iv) and (v). In fact we show that if for all $x \in K, y \to F(x,y)$ is convex on D *respectively*, for all $x \in D, u \in T(x),\ F(u,x) \geq 0$ instead of (iv) *respectively*, (v) in the previous theorem then we can replace by (vi), there exists $r > 0$ such that for all $x \in K, u \in T(x), \|x\| < r$ there exists $y_0 \in D$ with $\|y_0\| < r$ such that

$$\{0\} \subseteq F(u, y_0) \geq 0, \forall u \in T(x).$$

Theorem 5.5. *Let X be a reflexive Banach space, $K \subseteq X$ a nonempty convex closed subset and $D \subseteq K$ a self segment dense set in the weak topology of X. Assume that $T : K \to 2^K$ is a multi-valued mapping. Let $F : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D, x \to F(x,y)$ is weak lower semi continuous on K,*
(ii) *for all $x \in K, y \to F(x,y)$ is weak lower semi continuous on $K \setminus D$,*
(iii) *T is weak lower semi continuous, convex and compact,*
(iv) *for all $x \in K, y \to F(x,y)$ is convex on D,*
(v) *for all $x \in D, F(x,x) \geq 0$,*
(vi) *there exists $r > 0$ such that for all $x \in K, u \in T(x),\ \|x\| < 0$ there exists $y_0 \in K$ with $\|y_0\| < r$ such that*

$$\{0\} \subseteq F(u, y_0).$$

Then there exists $x_0 \in K, u_0 \in T(x_0)$ such that

$$F(u_0, y) \geq 0,\ \forall y \in K.$$

Proof. Let $r > 0$ such that (vi) holds and consider weakly compact set $K_0 = K \cap \overline{B}_r$. From Theorem 4.2 there exists $x_0 \in K_0, u_0 \in T(x_0)$ such that

$$F(u_0, y) \geq 0, \ \forall y \in K_0.$$

From (vi) there exists $z_0 \in D, \|z_0\| < r$ such that

$$\{0\} \subseteq F(u_0, z_0).$$

Let $z \in D \setminus K_0$, by virtue of self segment denseness of D in K there exists $\lambda \in (0, 1)$ such that

$$\lambda z_0 + (1 - \lambda) z_0 \in K_0 \cap D.$$

From (iv)

$$F(u_0, \lambda z_0 + (1 - \lambda) z) \supseteq \lambda F(u_0, z_0) + (1 - \lambda) F(u_0, z_0).$$

But

$$F(u_0, \lambda z_0 + (1 - \lambda) z) \geq 0$$

and

$$\{0\} \subseteq F(u, z_0) \geq 0,$$

which lead to

$$F(u_0, z) \geq 0.$$

Hence

$$F(u_0, z) \geq 0, \ \forall z \in D.$$

Let $y \in K \setminus D$, since D is dense in K, there exists a net $y_\alpha \in D$ such that $\lim y_\alpha = y$ where the limit is taken in the weak topology of X. From (ii) $F(x_0, \cdot)$ is weakly lower semi continuous at y. From Definition 2.1 for every $y^* \in F(u_0, y)$ there exists a net

$$y^*_\alpha \in F(u_0, y^*_\alpha) \geq 0$$

such that $y^*_\alpha = y^*$. But $y^*_\alpha \geq 0$. Hence $y^* \geq 0$ and

$$F(u_0, y) \geq 0, \ \forall y \in K.$$

□

Theorem 5.6. *Let X be a reflexive Banach spaces, $K \subseteq X$ a nonempty convex closed subset and $D \subseteq K$ a self segment dense set in the weak topology of X. Assume that $T : K \to 2^K$ is a multi-valued mapping. Let $F : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D, x \to F(x, y)$ is weak upper semi continuous on K,*
(ii) *for all $x \in K, y \to F(x, y)$ is weak upper semi continuous on $K \setminus D$,*
(iii) *T is weak upper semi continuous,*
(iv) *for all $x \in K, y \to F(x, y)$ is concave on D,*
(v) *for all $x \in D, F(x, x) \cap \mathbb{R}_+ \neq \emptyset$,*

(vi) *there exists* $r > 0$ *such that for all* $x \in K, u \in T(x), \|x\| < r$ *there exists* $y_0 \in D$ *with* $\|y_0\| < r$ *such that*

$$F(u, y_0) \leq o.$$

Then there exists $x_0 \in K, u_0 \in T(x_0)$ *such that*

$$F(u_0, y) \cap \mathbb{R}_+ \neq \emptyset, \ \forall y \in K.$$

Proof. Let $r > 0$ such that (vi) holds and consider the weakly compact set $K_0 = K \cap \overline{B}_r$. From Theorem 4.4, there exists $x_0 \in K_0, u_0 \in T(x_0)$ such that

$$F(u_0, y) \cap \mathbb{R}_+ \neq \emptyset, \ \forall y \in K_0$$

From (vi) there exists $z_0 \in D, \|z_0\| < r$ such that

$$F(u, z_0) \leq 0, \forall u \in T(x).$$

Let $z \in D \setminus K_0$, by virtue of self segment denseness of D in K there exists $\lambda \in (0, 1)$ such that

$$\lambda z_0 + (1 - \lambda)z \in K_0 \cap D.$$

From (iv)

$$F(u_0, \lambda z_0 + (1 - \lambda)z) \subseteq \lambda F(u_0, z_0) + (1 - \lambda)F(u_0, z).$$

which lead to

$$F(u_0, z) \cap \mathbb{R}_+ \neq \emptyset.$$

Hence

$$F(u_0, z) \cap \mathbb{R}_+ \neq \emptyset, \ \forall z \in D, u_0 \in T(x_0).$$

Let $y \in K \setminus D$ and from (ii), $F(x_0, \cdot)$ is weakly upper semi continuous at y, hence for any open set $V \subseteq X$ there exists an open neighborhood U of y such that for any $u \in U$, we have

$$F(u_0, u) \subseteq V.$$

Since D is dense in K we have $D \cap U \neq \emptyset$. Assume that

$$F(u_0, y) \subseteq (-\infty, 0), \ \forall y \in D, u_0 \in T(x_0).$$

We take $V = (-\infty, 0)$ and let $z \in U \cap D$. Then

$$F(u_0, z) \subseteq (-\infty, 0)$$

which is contradiction the fact that

$$F(u_0, z) \cap \mathbb{R}_+ \neq \emptyset, \forall u_0 \in T(x_0).$$

□

Theorem 5.7. *Let X be a reflexive Banach space, $K \subseteq X$ a nonempty convex closed subset and $D \subseteq K$ a self segment dense set in the weak topology of X. Assume that $T : K \to 2^K$ is a multi-valued mapping. Let $\phi : K \times K \to \mathbb{R}$ be a multi-valued mapping satisfying:*

(i) *for all $y \in D,\ x \to \phi(x, y)$ is weak upper semi continuous on K,*
(ii) *for all $x \in K, y \to \phi(x, y)$ is weak upper semi continuous on $K \setminus D$,*
(iii) *T is weak upper semi continuous on K, convex and $T(x)$ is compact,*
(iv) *for all $x \in K,\ y \to \phi(x, y)$ is convex on D,*
(v) *for all $x \in D, \phi(x, x) \geq 0$,*
(vi) *there exists $r > 0$ such that for all $x \in K, u \in T(x), \|x\| \leq r$ there exists $y_0 \in D$ with $\|y_0\| < r$ such that*

$$\phi(u, y_0) = 0.$$

Then there exists $x_0 \in K, u_0 \in T(x_0)$ such that

$$\phi(u_0, y) \geq 0,\ \forall y \in K.$$

Proof. Let $r > 0$ such that (vi) holds and consider the weakly compact set $K_0 = K \cap \overline{B}_r$. From Theorem 4.4, there exists $x_0 \in K_0, u_0 \in T(x_0)$ such that

$$\phi(u_0, y) \geq 0,\ \forall y \in K_0.$$

From (vi) there exists $z_0 \in D \cap K_0, \|z_0\| < r$ such that

$$0 = \phi(u_0, z_0).$$

Consider $z \in D \setminus K_0$, since D is self segment denseness in K there exists $\lambda \in (0, 1)$ such that

$$\lambda z_0 + (1 - \lambda)z \in K_0 \cap D.$$

From (iv)

$$\phi(u_0, \lambda z_0 + (1 - \lambda)z) \leq \lambda\phi(u_0, z_0) + (1 - \lambda)\phi(u_0, z),$$

or equivalently

$$(1 - \lambda)\phi(u_0, z_0) \geq \phi(u_0, \lambda z_0 + (1 - \lambda)z) \geq 0.$$

This show that

$$\phi(u_0, z) \geq 0,\ \forall z \in D, u_0 \in T(x_0).$$

Finally if $y \in K \setminus D$. By the denseness of D in K there exists a net $y_\alpha \subseteq D$ such that $\lim y_\alpha = y$ where the limit is taken in the weak topology of X. At this point the assumption (ii)

$$\phi(u_0, y) \geq 0,\ \forall y \in D, u_0 \in T(x_0).$$

From the upper semi continuity of $\phi(u_0, y)$ in $K \setminus D$ we have

$$0 \leq \lim_{y_\alpha \to y} \sup \phi(u_0, y^\alpha) \leq \phi(u_0, y).$$

Thus

$$\phi(u_0, y) \geq 0,\ \forall y \in K, u_0 \in T(x_0).$$

□

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