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EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR VOLTERRA-FREDHOLM INTEGRO DIFFERENTIAL EQUATIONS

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Abstract. In this article, variational iteration technique is successfully applies to find the approximate solutions of nonlinear Volterra-Fredholm integro-differential equations. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. Moreover, we prove the existence and uniqueness results. Finally, the examples are included to demonstrate the validity and applicability of the proposed technique.

1. Introduction

In this chapter, we consider nonlinear Volterra-Fredholm integro-differential equation of the form:

$$\sum_{j=0}^{k} \xi_j(x) u^{(j)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t) G_1(u(t))dt \tag{1.1}$$

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$$+ \lambda_2 \int_a^b K_2(x,t) G_2(u(t)) dt,$$

with the initial conditions

$$u^{(r)}(a) = b_r, \quad r = 0, 1, 2, \cdots, (k-1), \tag{1.2}$$

where $u^{(j)}(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined, $K_i(x,t), i = 1, 2$ are the kernels of the equation, $f(x)$ and $\xi_j(x)$ are an analytic function, G_1 and G_2 are nonlinear functions of u and $a, b, \lambda_1, \lambda_2$, and b_r are real finite constants.

In recent years, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, we can remember the following works. Abbasbandy and Elyas [1] studied some applications on variational iteration method for solving system of nonlinear Volterra integro-differential equations, Alao *et al.* [2] used Adomian decomposition and variational iteration methods for solving integro-differential equations, Hamoud and Ghadle [8] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [10] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Behzadi *et al.* [3] solved some class of nonlinear Volterra-Fredholm integro-differential equations by homotopy analysis method. Moreover, several authors have applied the Adomian decomposition method and the variational iteration method to find the approximate solutions of various types of integro-differential equations [4, 5, 6, 7].

The main objective of the present chapter is to study the behavior of the solution that can be formally determined by semi-analytical approximated method as variational iteration method. Moreover, we proved the existence, uniqueness results and convergence of the solutions of the nonlinear Volterra-Fredholm integro-differential equations.

2. Variational Iteration Method (VIM)

This method is applied to solve a large class of linear and nonlinear problems with approximations converging rapidly to exact solutions. The main idea of this method is to construct a correction functional form using general Lagrange multipliers. These multipliers should be chosen such that its correction solution is superior to its initial approximation, called trial function. It is the best within the flexibility of trial functions. Accordingly, Lagrange multipliers can be identified by the variational theory [5, 6, 11]. A complete review of variational iteration method is available in [9]. The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing boundary/initial conditions. To illustrate, we consider the following general differential equation:

$$Lu(t) + Nu(t) = f(t),$$

where L is a linear operator, N is a nonlinear operator and $f(t)$ is inhomogeneous term. According to variational iteration method [2], the terms of a sequence u_n are constructed such that this sequence converges to the exact solution. The terms u_n are calculated by a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)(Lu_n(\tau) + N\tilde{y}(\tau) - f(\tau))d\tau. \qquad (2.1)$$

The successive approximation $u_n(t), n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected using any function that just satisfies at least the initial and boundary conditions, with λ determined, several approximations $u_n(t), n \geq 0$ follow immediately.

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

To obtain the approximation solution of IVB $(1.2) - (1.2)$, according to the VIM, the iteration formula (2.1) can be written as follows:

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + L^{-1}\Big[\lambda(x)\Big[\sum_{j=0}^{k} \xi_j(x) u_n^{(j)}(x) - f(x) \\ &\quad -\lambda_1 \int_a^x K_1(x,t) G_1(u_n(t))dt - \lambda_2 \int_a^b K_2(x,t) G_2(u_n(t))dt\Big]\Big], \end{aligned}$$

where L^{-1} is the multiple integration operator given as follows:

$$L^{-1}(\cdot) = \int_a^x \int_a^x \cdots \int_a^x (\cdot) dx dx \cdots dx \quad (k - times).$$

To find the optimal $\lambda(x)$, we proceed as follows:

$$\begin{aligned} \delta u_{n+1}(x) &= \delta u_n(x) + \delta L^{-1}\Big[\lambda(x)\Big[\sum_{j=0}^{k} \xi_j(x) u_n^{(j)}(x) - f(x) \\ &\quad -\lambda_1 \int_a^x K_1(x,t) G_1(u_n(t))dt \\ &\quad -\lambda_2 \int_a^b K_2(x,t) G_2(u_n(t))dt\Big]\Big] \\ &= \delta u_n(x) + \lambda(x)\delta u_n(x) - L^{-1}\Big[\delta u_n(x)\lambda'(x)\Big]. \end{aligned} \qquad (2.2)$$

From Eq. (2.2), the stationary conditions can be obtained as follows:

$$\lambda'(x) = 0, \text{ and } 1 + \lambda(x)|_{x=t} = 0.$$

As a result, the Lagrange multipliers can be identified as $\lambda(x) = -1$ and by substituting in Eq. (2.2), the following iteration formula is obtained:

$$\begin{aligned} u_0(x) &= L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r, \\ u_{n+1}(x) &= u_n(x) - L^{-1}\Big[\sum_{j=0}^{k} \xi_j(x) u_n^{(j)}(x) - f(x) \\ &\quad -\lambda_1 \int_a^x K_1(x,t) G_1(u_n(t)) dt \\ &\quad -\lambda_2 \int_a^b K_2(x,t) G_2(u_n(t)) dt\Big], n \geq 0. \end{aligned} \tag{2.3}$$

The term $\sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r$ is obtained from the initial conditions, $\xi_k(x) \neq 0$. Relation (2.3) will enable us to determine the components $u_n(x)$ recursively for $n \geq 0$. Consequently, the approximation solution may be obtained by using

$$u(t) = \lim_{n \to \infty} u_n(t).$$

3. Main results

In this section, we shall give an existence and uniqueness results of Eq. (1.2), with the initial condition (1.2) and prove it.

We can be written Eq. (1.2) in the form of:

$$\begin{aligned} & u(x) \\ &= L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r + \lambda_1 L^{-1}\Big[\int_a^x \frac{1}{\xi_k(x)} K_1(x,t) G_1(u_n(t)) dt\Big] \\ &\quad + \lambda_2 L^{-1}\Big[\int_a^b \frac{1}{\xi_k(x)} K_2(x,t) G_2(u_n(t)) dt\Big] - L^{-1}\Big[\sum_{j=0}^{k-1} \frac{\xi_j(x)}{\xi_k(x)} u^{(j)}(x)\Big], \end{aligned}$$

such that,

$$L^{-1}\Big[\int_a^x \frac{1}{\xi_k(x)} K_1(x,t) G_1(u_n(t)) dt\Big] = \int_a^x \frac{(x-t)^k}{k!\xi_k(x)} K_1(x,t) G_1(u_n(t)) dt,$$

$$\sum_{j=0}^{k-1} L^{-1}\Big[\frac{\xi_j(x)}{\xi_k(x)}\Big] u^{(j)}(x) = \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1} \xi_j(t)}{k-1!\xi_k(t)} u^{(j)}(t) dt.$$

We set,

$$\Psi(x) = L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r.$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exist two constants α, β and $\gamma_j > 0, j = 0, 1, \cdots, k$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|G_1(u_1)) - G_1(u_2))| \leq \alpha\, |u_1 - u_2|\,,$$

$$|G_2(u_1) - G_2(u_2)| \leq \beta\, |u_1 - u_2|$$

and

$$\left|D^j(u_1) - D^j(u_2)\right| \leq \gamma_j\, |u_1 - u_2|\,,$$

we suppose that the nonlinear terms $G_1(u(x))), G_2(u(x)))$ and $D^j(u) = (\frac{d^j}{dx^j})u(x) = \sum_{i=0}^{\infty} \gamma_{i_j}$, ($D^j$ is a derivative operator), $j = 0, 1, \cdots, k$, are Lipschitz continuous.

(H2) We suppose that for all $a \leq t \leq x \leq b$, and $j = 0, 1, \cdots, k$:

$$\left|\frac{\lambda_1 (x-t)^k K_1(x,t)}{k!\xi_k(x)}\right| \leq \ \theta_1, \qquad \left|\frac{\lambda_1 (x-t)^k K_1(x,t)}{k!}\right| \leq \theta_2,$$

$$\left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!\xi_k(t)}\right| \leq \theta_3, \qquad \left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!}\right| \leq \theta_4,$$

$$\left|\lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\right| \leq \ \theta_5, \qquad \left|\lambda_2 L^{-1}\Big[K_2(x,t)\Big]\right| \leq \theta_6,$$

(H3) There exist three functions $\theta_3^*, \theta_4^*,$ and $\gamma^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that:

$$\theta_3^* = \max|\theta_3|\,,\ \theta_4^* = \max|\theta_4|\,,\ \text{and}\ \gamma^* = \max|\gamma_j|\,.$$

(H4) $\Psi(x)$ is bounded function for all x in $J = [a, b]$.

Theorem 3.1. *Assume that* (H1)–(H4) *hold. If*

$$0 < \psi = (\alpha\theta_1 + \beta\theta_5 + k\gamma^*\theta_3^*)(b-a) < 1, \tag{3.1}$$

then there exists a unique solution $u(x) \in C(J)$ *to IVB* (1.2) − (1.2).

Proof. Let u_1 and u_2 be two different solutions of IVB (1.2) – (1.2). Then

$$\begin{aligned}
\Big|u_1-u_2\Big| &= \Big|\int_a^x \frac{\lambda_1(x-t)^k K_1(x,t)}{\xi_k(x)k!}[G_1(u_1)-G_1(u_2))]dt \\
&\quad + \int_a^b \lambda_1 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big][G_2(u_1)-G_2(u_2))]dt \\
&\quad - \sum_{j=0}^{k-1}\int_a^x \frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}[D^j(u_1)-D^j(u_2))]dt\Big| \\
&\le \int_a^x \Big|\frac{\lambda_1(x-t)^k K_1(x,t)}{\xi_k(x)k!}\Big|\Big|G_1(u_1)-G_1(u_2))\Big|dt \\
&\quad + \int_a^b \Big|\lambda_1 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big|\Big|G_2(u_1)-G_2(u_2))\Big|dt \\
&\quad - \sum_{j=0}^{k-1}\int_a^x \Big|\frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}\Big|\Big|D^j(u_1)-D^j(u_2))\Big|dt \\
&\le (\alpha\theta_1+\beta\theta_5+k\gamma^*\theta_3^*)(b-a)|u_1-u_2|,
\end{aligned}$$

it implies that $(1-\psi)|u_1-u_2| \le 0$. Since $0 < \psi < 1$, so $|u_1-u_2| = 0$. Therefore, $u_1 = u_2$ and the proof is completed. □

Theorem 3.2. *If problem* (1.2)–(1.2) *has a unique solution, then the solution* $u_n(x)$ *obtained from the recursive relation* (2.3) *using VIM converges when* $0 < \phi = (\alpha\theta_5+\beta\theta_6+k\gamma^*\theta_4^*)(b-a) < 1$.

Proof. We have from equation (2.3):

$$\begin{aligned}
u_{n+1}(x)-u(x) &= u_n(x)-u(x)-\Big(L^{-1}\Big[\sum_{j=0}^{k}\xi_j(x)[u_n^{(j)}(x)-u^{(j)}(x)]\Big] \\
&\quad -L^{-1}\Big[\lambda_1\int_a^x K_1(x,t)[G_1(u_n(t))-G_1(u(t))]dt \\
&\quad -L^{-1}\Big[\lambda_2\int_a^b K_2(x,t)[G_2(u_n(t))-G_2(u(t))dt]\Big]\Big).
\end{aligned}$$

If we set, $\xi_k(x) = 1$, and $W_{n+1}(x) = u_{n+1}(x)-u(x)$, $W_n(x) = u_n(x)-u(x)$ since $W_n(a) = 0$, then

$$\begin{aligned} W_{n+1}(x) &= W_n(x) + \int_a^x \frac{\lambda_1 K_1(x,t)(x-t)^k}{k!}[G_1(u_n(t)) - G_1(u(t))]dt \\ &+ \int_a^b \lambda_2 L^{-1}\Big[K_2(x,t)[G_2(u_n(t)) - G_2(u(t))dt]\Big] \\ &- \sum_{j=0}^{k-1}\int_a^x \frac{\lambda_1 \xi_j(t)(x-t)^{k-1}}{(k-1)!}[D^j(u_n(t)) - D^j(u(t))]dt \\ &-(W_n(x) - W_n(a)). \end{aligned}$$

Therefore,

$$\begin{aligned} \Big|W_{n+1}(x)\Big| &\leq \int_a^x \Big|\frac{\lambda_1 K_1(x,t)(x-t)^k}{k!}\Big|\Big|W_n\Big|\alpha dt \\ &+ \int_a^b \Big|\lambda_2 L^{-1}\Big[\Big|K_2(x,t)\Big|\Big|W_n\Big|\beta dt\Big] \\ &+ \sum_{j=0}^{k-1}\int_a^x \Big|\frac{\lambda_1 \xi_j(t)(x-t)^{k-1}}{(k-1)!}\Big|\max|\gamma_j|\Big|W_n\Big|dt \\ &\leq \Big|W_n\Big|\Big[\int_a^x \alpha\theta_5 dt + \int_a^b \beta\theta_6 dt + \sum_{j=0}^{k-1}\int_a^x \theta_4^* \max|\gamma_j|\Big] \\ &\leq |W_n|(\alpha\theta_5 + \beta\theta_6 + k\gamma^*\theta_4^*)(b-a) \\ &= |W_n|\phi. \end{aligned}$$

Hence,

$$\|W_{n+1}\| = \max_{\forall x\in J}|W_{n+1}(x)| \leq \phi \max_{\forall x\in J}|W_n(x)| = \phi\|W_n\|.$$

Since $0 < \phi < 1$, we have $\|W_n\| \longrightarrow 0$. So, the series converges and the proof is complete. □

4. Illustrative Examples

In this section, we present the semi-analytical techniques based on ADM and VIM to solve Volterra-Fredholm integro-differential equations. To show the efficiency of the present methods for our problem in comparison with the exact solution we report absolute error.

Example 4.1. Consider the following Volterra-Fredholm integro-differential equation:

$$u'(x) + xu(x) = 2x + x^3 - \frac{x^5}{5} - \frac{0.9^7}{7}x + \int_0^x u^2(t)dt + \int_0^{0.9} xu^3(t)dt,$$

with the initial condition $u(0) = 0, u'(0) = 0$, and the the exact solution is $u(x) = x^2$.

Table 1. Numerical Results of the Example 1.

x	Exact	ADM	VIM	EADM	EVIM
0.1	0.010000	0.010397	0.010024	0.000397	0.000024
0.2	0.040000	0.043354	0.040394	0.003354	0.000394
0.3	0.090000	0.097463	0.091969	0.007463	0.001969
0.4	0.160000	0.148954	0.151274	0.011046	0.008726
0.5	0.250000	0.240548	0.243752	0.009452	0.006248
0.6	0.360000	0.348973	0.350874	0.011027	0.009126
0.7	0.490000	0.473681	0.483681	0.016319	0.006319
0.8	0.640000	0.627596	0.630257	0.012404	0.009743
0.9	0.810000	0.764797	0.801487	0.045203	0.008531

Example 4.2. Consider the following Volterra-Fredholm integro-differential equation:

$$u''(x) + u'(x) - u(x) = e^{x-1} - e^x - 1 + \int_0^x u(t)dt + \int_0^1 e^{t+x}u^2(t)dt,$$

with the initial condition

$$u(0) = 1, u'(0) = -1,$$

and the the exact solution is $u(x) = e^{-x}$.

Table 2. Numerical Results of the Example 2.

x	Exact	ADM	VIM	EADM	EVIM
0.1	0.904837418	0.896160501	0.904350694	0.008676917	0.000486724
0.2	0.818730753	0.783594511	0.817029618	0.035136242	0.001701135
0.3	0.740818221	0.660685557	0.737405770	0.080132664	0.003412451
0.4	0.670320046	0.525762821	0.664765171	0.144557225	0.005554875
0.5	0.606530659	0.377106516	0.598327465	0.229424144	0.008203194
0.6	0.548811636	0.212950800	0.537264396	0.335860836	0.011547240
0.7	0.496585303	0.031483915	0.480719900	0.465101388	0.015865403
0.8	0.449328964	-0.169154690	0.427831430	0.618483654	0.021497534
0.9	0.406569659	-0.390880529	0.377751090	0.797450188	0.028818569

5. Conclusion

The variational iteration method has been successfully applied to find the approximate solution of Volterra-Fredholm integro-differential equation. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear Volterra-Fredholm integro-differential equations. Moreover, we proved the existence and uniqueness of the solution. From the examples considered, VIM has an advantage over ADM due to non-requirement of Adomian polynomial and hence converges faster to the exact solution for some nonlinear problems. Also, it was observed that these methods were compared favorably with the exact solution.

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