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FIXED POINTS AND HYERS-ULAM-RASSIAS STABILITY OF THE QUADRATIC AND JENSEN FUNCTIONAL EQUATIONS

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Abstract. In this paper, we apply a fixed point theorem to the proof of Hyers-Ulam-Rassias stability property for the quadratic functional equation

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = f(x)+f(y), \quad x,y\in E_1$$

and for the Jensen functional equation

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = f(x), \quad x,y\in E_1$$

from a normed space E_1 into a quasi Banach space E_2, where K is a finite cyclic transformation group of E_1.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [40] concerning the stability of group homomorphisms: Given a group

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F, a metric group H with a metric $d(.,.)$ and an $\epsilon > 0$, find $\delta > 0$ such that, if $f : F \to H$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in F$, then there exists a homomorphisms $g : F \to H$ such that $d(f(x), g(x)) \leq \epsilon$ for all $x \in F$.
If the answer is affirmative, we would say that the equation of homomorphism $f(xy) = f(x)f(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which arts as a perturbation of the equation. Thus, the stability question of functional equations is that "how do the solutions of the inequality differ from those of the given functional equation"?
Hyers [11] gave a first partial affirmative answer to the question of Ulam for Banach spaces.

Let F and H be Banach spaces. Assume that $f : F \to H$ satisfies

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon$$

for all $x, y \in F$ and some $\epsilon \geq 0$. Then, there exists a unique additive mapping $T : F \to H$ such that

$$\| f(x) - T(x) \| \leq \epsilon$$

for all $x \in F$.
Th. M. Rassias [27] provided a generalization of Hyers theorem which allows the Cauchy difference to be unbounded.

Theorem 1.1. *(Th. M. Rassias) Let $f : F \to H$ be a mapping from a normed vector space F into a Banach space H subject to the inequality*

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon(\| x \|^p) + (\| y \|^p)$$

for all $x, y \in F$, where ϵ and p are constants such that $\epsilon > 0$ and $p < 1$. Then, the limit

$$L(x) = \lim_{n \to +\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in F$ and $L : F \to H$ is the unique additive mapping which satisfies

$$\| f(x) - L(x) \| \leq \frac{2\epsilon}{2 - 2^p} \| x \|^p$$

for all $x \in F$. Also, if for each $x \in F$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

This result provided a remarkable generalization of Theorem proved by Hyers. What is more important here is that Rassias Theorem simulated several mathematicians working in functional equations to investigate this kind of stability for many important functional equations. Taking this fact in consideration, the terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. Beginning around the year 1980, several results for the Hyers-Ulam-Rassias stability of very many functional equations have been proved by several researchers. For more detailed, we can refer to [5],...[39]

Let E_1 be a real vector space and E_2 be a real Banach space. Let K be a finite cyclic subgroup of $Aut(E_1)$ (the group of automorphisms of G), $|K|$ denotes the order of K. Writing the action of $k \in K$ on $x \in G$ as $k \cdot x$, we will say that a function $f : E_1 \to E_2$ is a solution of the quadratic functional equation, if

$$\frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) = f(x) + f(y), \quad x, y \in E_1 \tag{1.1}$$

and that f is a solution of the Jensen functional equation, if

$$\frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) = f(x), \quad x, y \in E_1 \tag{1.2}$$

The above functional equations appeared in several works by H. Stetkær (see, [37]-[39]).

Recently, Belaid et al have proved the Hyers-Ulam-Rassias stability of the quadratic functional (1.1) and and the Jensen functional equation (1.2) (see [1], [3] and [4]).

In [2] L. Cădariu and V. Radu applied the fixed point method to the investigation of the Cauchy additive functional equation.

In this paper, we will apply the fixed point method as in [2] to prove the Hyers-Ulam-Rassias stability of the functional equations (1.1) and (1.2). In this case the range of relevant functions is extended to any complete β-normed space.

In 1996, G. Isac and Th. M. Rassias [16] were the first to provide applications of stability theory of functional equation for the proof of new fixed point theorems with applications.

First we shall recall two fundamental results in fixed point theory. The reader is referred to the book of D. H. Hyers, G. Isac and Th. M. Rassias [13] for an extensive account of fixed point theory with several applications.

Theorem 1.2. *(Banach's contraction principal) Let (X, d) be a complete metric space, and consider a mapping $J : X \to X$, which is strictly contractive, that is*

$$d(Jx, Jy) \leq Ld(x, y), \forall x, y \in X,$$

for some (Lipshitz constant) $L < 1$. Then,
(1) the mapping J has one, and only one, fixed point $x^* = J(x^*)$,
(2) the fixed point x^* is globally attractive, that is,

$$\lim_{n \to +\infty} J^n x = x^*$$

for any starting point $x \in X$.
(3) One has the following estimation inequalities:

$$d(J^n x, x^*) \leq L^n d(x, x^*)$$

$$d(J^n x, x^*) \leq \frac{1}{1-L} d(J^n x, J^{n+1} x)$$

$$d(x, x^*) \leq \frac{1}{1-L} d(x, Jx)$$

for all nonnegative integers n and all $x \in X$.

Let X be a set. A function $d : X \times X \to [0, +\infty]$ is called a *generalized metric* on X if d satisfies the following:
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(2) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.3. *(The alternative of fixed point)* [7] *Suppose we are given complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$, white the Lipshitz constant $L < 1$. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < +\infty$, $\forall n \geq n_0$;
(2) the sequence $J^n x$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < +\infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

This paper is organized as followings. In section 2, using the fixed point method, we prove the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). In section 3, using the fixed point method, we prove the Hyers-Ulam-Rassias stability of the Jensen functional equation (1.2).

Throughout this paper, we fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Suppose E is a vector space over $\mathbb{K}$. A function $\|.\|_\beta$: $E \longrightarrow [0, \infty)$ is called a β-norm if and only if it satisfies
(1) $\|x\|_\beta = 0$, if and only if $x = 0$;
(2) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in E$;
(3) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$, for all $x, y \in E$.

2. Hyers-Ulam-Rassias stability of the quadratic functional equation

In this section we prove the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1)

Theorem 2.1. *Let E_1 be a vector space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$, where β is a fixed real number with $0 < \beta \leq 1$. Let K be a finite cyclic subgroup of the group of automorphisms of the abelian group $(E_1, +)$. Let $f\colon E_1 \longrightarrow E_2$ be a mapping for which there exists a function $\varphi : E_1 \times E_1 \to \mathbb{R}^+$ and a constant L, $0 < L < 1$, such that*

$$\| \mid \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - f(x) - f(y) \|_\beta \leq \varphi(x, y), \tag{2.1}$$

$$\sum_{k \in K} \varphi(x + k \cdot x, y + k \cdot y) \leq (2|K|)^\beta L \varphi(x, y) \tag{2.2}$$

for all $x, y \in E_1$. Then, there exists a unique solution $q\colon E_1 \longrightarrow E_2$ of equation (1.1) such that

$$\|f(x) - q(x)\|_\beta \leq \frac{1}{2^\beta} \frac{1}{1 - L} \varphi(x, x) \tag{2.3}$$

for all $x \in E_1$.

Proof. Consider the set

$$X := \{g : \ E_1 \longrightarrow E_2\}$$

and introduce the *generalized metric* on X as follows:

$$d(g, h) = \inf\{C \in [0, \infty] : \ \|g(x) - h(x)\|_\beta \leq C\varphi(x, x), \ \forall x \in E_1\}.$$

It easy to show that (X, d) is complete.
Now, we consider the linear mapping $J : X \to X$ such that

$$(Jf)(x) = \frac{1}{2|K|} \sum_{k \in K} f(x + k \cdot x)$$

for all $x \in E_1$.
From [1], we can verified that

$$(J^n f)(x) = \frac{1}{(2|K|)^n} \sum_{k_1, \ldots, k_n \in K} f \left(x + \sum_{i_j < i_{j+1}, k_{ij} \in \{k_1, \ldots, k_n\}} (k_{i_1} \ldots k_{i_p}) \cdot x \right)$$

for all integer n.

Next, we are going to prove that J is a strictly contractive on X with the *Lipschitz constant* L. Indeed, for given g and h in X and $C \geq 0$ an arbitrary constant with $d(g, h) \leq C$, that is,

$$\|g(x) - h(x)\|_\beta \leq C\varphi(x, x) \tag{2.4}$$

for all $x \in E_1$. Thus from (2.1), (2.2) and (2.4) we get

$$\begin{aligned}\|(Jg)(x)-(Jh)(x)\|_\beta &= \|\frac{1}{2|K|}\sum_{k\in K} g(x+k\cdot x)-\frac{1}{2|K|}\sum_{k\in K} h(x+k\cdot x)\|_\beta\\ &= \frac{1}{(2|K|)^\beta}\|\sum_{k\in K} g(x+k\cdot x)-h(x+k\cdot x)\|_\beta\\ &\le \frac{1}{(2|K|)^\beta}\sum_{k\in K}\|g(x+k\cdot x)-h(x+k\cdot x))\|_\beta\\ &\le \frac{1}{(2|K|)^\beta}C\sum_{k\in K}\varphi(x+k\cdot x,x+k\cdot x)\\ &\le CL\varphi(x,x)\end{aligned}$$

for all $x \in E_1$, that is, $d(Jg, Jh) \le LC$. This means that $d(Jg, Jh) \le Ld(g, h)$ for any $g, h \in X$.

Now, by letting $y = x$ in (2.1), we get

$$\|(Jf)(x)-f(x)\|_\beta = \frac{1}{2^\beta}\|\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot x)-2f(x)\|_\beta \le \frac{1}{2^\beta}\varphi(x,x)$$

for all $x \in E_1$ and it follows that

$$d(Jf,f) \le \frac{1}{2^\beta} < \infty \tag{2.5}$$

From the fixed point alternative we deduce the existence of a fixed point of J which is a function $q : E_1 \to E_2$ such that $\lim_{n\longrightarrow\infty} d(J^n f, q) = 0$. Since $d(J^n f, q) \to 0$ as $n \to \infty$, there exists a sequence $\{C_n\}$ such that $\lim_{n\longrightarrow\infty} C_n = 0$ and $d(J^n f, q) \le C_n$ for every $n \in \mathbb{N}$. Hence, from the definition of d, we get

$$\|(J^n f)(x)-q(x)\|_\beta \le C_n\varphi(x,x) \tag{2.6}$$

for all $x \in E_1$. Consequently, we obtain

$$\lim_{n\to\infty}\|(J^n f)(x)-q(x)\|_\beta = 0, \tag{2.7}$$

for each $x \in E_1$.

Now, we will prove that q is a solution of the quadratic functional equation (1.1). First, we use induction on n to prove the following inequality

$$\|\frac{1}{|K|}\sum_{k\in K} J^n f(x+k\cdot y)-J^n f(x)-J^n f(y)\|_\beta \le L^n\varphi(x,y) \tag{2.8}$$

For $n = 1$, by using the definition of J, the commutativity of K and inequalities (2.1), (2.2) we get

$$
\begin{aligned}
&\|\tfrac{1}{|K|}\textstyle\sum_{k\in K} Jf(x+k\cdot y)-Jf(x)-Jf(y)\|_\beta\\
&=\|\tfrac{1}{|K|}\textstyle\sum_{k\in K}\tfrac{1}{2|K|}\sum_{k_1\in K} f(x+k\cdot y+k_1\cdot x+k_1k\cdot y)\\
&\quad-\tfrac{1}{2|K|}\textstyle\sum_{k_1\in K} f(x+k_1\cdot x)\\
&\quad-\tfrac{1}{2|K|}\textstyle\sum_{k_1\in K} f(y+k_1\cdot y)\|_\beta\\
&\le \tfrac{1}{(2|K|)^\beta}\textstyle\sum_{k_1\in K}\|\tfrac{1}{|K|}\sum_{k\in K} f(x+k_1\cdot x+k\cdot(y+k_1\cdot y))\\
&\quad-f(x+k_1\cdot x)-f(y+k_1\cdot y)\|_\beta\\
&\le \tfrac{1}{(2|K|)^\beta}\textstyle\sum_{k_1\in K}\varphi(x+k_1\cdot x, y+k_1\cdot y)\\
&\le \tfrac{1}{(2|K|)^\beta}(2|K|)^\beta L\varphi(x,y)=L\varphi(x,y),
\end{aligned}
$$

which proves that the assertion (2.8) is true for $n=1$. Now, we assume that (2.8) is true for some for n. By using the definition of J, the commutativity of K, the inequalities (2.8), (2.2), we obtain

$$
\begin{aligned}
&\|\tfrac{1}{|K|}\textstyle\sum_{k\in K} J^{n+1}f(x+k\cdot y)-J^{n+1}f(x)-J^{n+1}f(y)\|_\beta\\
&=\|\tfrac{1}{|K|}\textstyle\sum_{k\in K}\tfrac{1}{2|K|}\sum_{k'\in K} J^nf(x+k\cdot y+k'\cdot x+k'k\cdot y)\\
&\quad-\tfrac{1}{2|K|}\textstyle\sum_{k'\in K} J^nf(x+k'\cdot x)\\
&\quad-\tfrac{1}{2|K|}\textstyle\sum_{k'\in K} J^nf(y+k'\cdot y)\|_\beta\\
&\le \tfrac{1}{(2|K|)^\beta}\textstyle\sum_{k'\in K}\|\tfrac{1}{|K|}\sum_{k\in K} J^nf(x+k'\cdot x+k\cdot(y+k'\cdot y)\\
&\quad-J^nf(x+k'\cdot x)-J^nf(y+k'\cdot y)\|_\beta\\
&\le \tfrac{1}{(2|K|)^\beta}\textstyle\sum_{k'\in K} L^n\varphi(x+k'\cdot x, y+k'\cdot y)\\
&\le L^{n+1}\varphi(x,y),
\end{aligned}
$$

which implies the validity of the inequality (2.8) for $n+1$.
By letting $n\to\infty$, in (2.8), we get the desired result that

$$\frac{1}{|K|}\sum_{k\in K} q(x+k\cdot y)-q(x)-q(y)=0, \tag{2.9}$$

for all $x,y\in E_1$. From Theorem 1.3 and inequality (2.5), we deduce that

$$d(f,q)\le\frac{1}{1-L}d(Jf,f)\le\frac{1}{2^\beta}\frac{1}{(1-L)}, \tag{2.10}$$

which proves the inequality (2.3). Now, assume that $q_1: E_1\to E_2$ is another solution of (1.1) satisfying (2.3) so q_1 is a fixed point of J. From the definition of d and the inequality (2.3), the assertion (2.10) is also true with q_1 in place of q. By using Theorem 1.3 (3), we get the uniqueness of q. This ends the proof of Theorem 2.1. □

The following corollaries follows from Theorem 2.1.
With the new weak condition (2.11), we obtain

Corollary 2.2. [19] *Let E_1 be a vector space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$, where β is a fixed real number with $0 < \beta \leq 1$. Let $K = \{I, \sigma\}$, where σ is an involution of the abelian group $(E_1, +)$. Let f: $E_1 \longrightarrow E_2$ be a mapping for which there exists a function $\varphi : E_1 \times E_1 \to \mathbb{R}^+$ and a constant L, $0 < L < 1$, such that*

$$\varphi(2x, 2y) + \varphi(x + \sigma(x), y + \sigma(y)) \leq 4^\beta L\varphi(x, y), \tag{2.11}$$

for all $x, y \in E_1$. Assume that $f : E_1 \to E_2$ satisfies the inequality

$$\|\frac{1}{2}[f(x + y) + f(x + \sigma(y))] - f(x) - f(y)\|_\beta \leq \varphi(x, y) \tag{2.12}$$

for all $x, y \in E_1$. Then, there exists a unique solution q: $E_1 \longrightarrow E_2$ of equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \quad x, y \in E_1 \tag{2.13}$$

such that

$$\|f(x) - q(x)\|_\beta \leq \frac{1}{2^\beta}\frac{1}{(1 - L)}\varphi(x, x) \tag{2.14}$$

for all $x \in E_1$.

Corollary 2.3. *Let E_1 be a vector space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$, let K be a finite cyclic subgroup of the group of automorphisms of the abelian group $(E_1, +)$ and choose a constant p with $p < \beta + (\beta - 1)\frac{\log(|K|)}{\log(2)}$. Let f: $E_1 \longrightarrow E_2$ be a mapping such that*

$$\|\frac{1}{|K|}\sum_{k \in K} f(x + k \cdot y) - f(x) - f(y)\|_\beta \leq \theta(\|x\|^p + \|y\|^p), \tag{2.15}$$

and $\|x + k \cdot x\|^p \leq 2^p\|x\|^p$ for all $k \in K$ and $x \in E_1$. Then, there exists a unique solution q: $E_1 \longrightarrow E_2$ of equation (1.1) such that

$$\|f(x) - q(x)\|_\beta \leq \frac{2\theta|K|^\beta}{2^\beta|K|^\beta - 2^p|K|}\|x\|^p \tag{2.16}$$

for all $x \in E_1$.

3. Hyers-Ulam-Rassias stability of Jensen functional equation

In this section, we prove the Hyers-Ulam-Rassias stability of the functional equation (1.2).

Theorem 3.1. *Let E_1 be a vector space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$, where β is a fixed real number with $0 < \beta \leq 1$. Let K be a finite cyclic subgroup of the group of automorphisms of the abelian group*

$(E_1,+)$. Let $f\colon E_1 \longrightarrow E_2$ be a mapping for which there exists a function $\varphi : E_1 \times E_1 \to \mathbb{R}^+$ and a constant L, $0 < L < 1$, such that

$$\|\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) - f(x)\|_\beta \le \varphi(x,y), \tag{3.1}$$

$$\sum_{k\in K} \varphi(x-k\cdot x, y-k\cdot y) \le |K|^\beta L\varphi(x,y) \tag{3.2}$$

for all $x, y \in E_1$. Then, there exists a unique solution $j\colon E_1 \longrightarrow E_2$ of equation (1.2) such that

$$\|f(x) - j(x)\|_\beta \le \frac{1}{1-L}\varphi(x,x) \tag{3.3}$$

for all $x \in E_1$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$(Jf)(x) = \frac{1}{|K|}\sum_{k\in K} f(x-k\cdot x) \tag{3.4}$$

for all $x \in E_1$. Given $g, h \in X$ and $C \in [0,\infty]$ such that $d(g,h) \le C$, then we get

$$\begin{aligned}
\|(Jg)(x) - (Jh)(x)\|_\beta &= \|\frac{1}{|K|}\sum_{k\in K} g(x-k\cdot x) - \frac{1}{|K|}\sum_{k\in K} h(x-k\cdot x)\|_\beta \\
&= \frac{1}{|K|^\beta}\|\sum_{k\in K}[g(x-k\cdot x) - h(x-k\cdot x)]\|_\beta \\
&\le \frac{1}{|K|^\beta}\sum_{k\in K}\|g(x-k\cdot x) - h(x-k\cdot x)\|_\beta \\
&\le CL\varphi(x,x)
\end{aligned}$$

for all $x \in E_1$, which implies that J is a strictly contractive operator, that is $d(Jg, Jh) \le Ld(g,h)$.
Letting $y = -x$ in (3.1), we get

$$d(Jf, f) \le 1 \tag{3.5}$$

By Theorem 1.3, there exits a mapping $j : E_1 \to E_2$ such that the following hold.
(a) j is a fixed point of J, that is

$$\frac{1}{|K|}\sum_{k\in K} j(x-k\cdot x) = j(x),$$

for all $x \in G$. The mapping j is a unique fixed point of J in the set

$$Y = \{g \in X : \ d(f,g) < \infty\}$$

(b) $\lim_{n\longrightarrow\infty} d(J^n f, j) = 0$, that is

$$j(x) = \lim_{n\to\infty} \frac{1}{|K|^n} \sum_{k_1,\dots,k_n\in K} f\left(x + \sum_{i_j<i_{j+1},k_{ij}\in\{k_1,\dots,k_n\}} [(-k_{i_1})\cdots(-k_{i_p})]\cdot x\right)$$

(c) $d(f,j) \leq \frac{1}{1-L} d(f, Jf)$, so we have the inequality (3.3).

Now, by applying same computations used in the proof of Theorem 3.1, we will show by induction that

$$\|\frac{1}{|K|}\sum_{k\in K} J^n f(x+k\cdot y) - J^n f(x)\|_\beta \leq L^n\varphi(x,y) \tag{3.6}$$

for all $x, y \in E_1$.

Finally, By letting $n \to \infty$ in the formula (3.6), we get that j is a solution of equation (1.2). The uniqueness of j can be derived by using same argument as in the proof of Theorem 2.1. This completes the proof of our theorem. □

Corollary 3.2. *Let E_1 be a vector space over $\mathbb{K}$ and let E_2 be a complete β-normed space over $\mathbb{K}$, let K be a finite cyclic subgroup of the group of automorphisms of the abelian group $(E_1,+)$ with $|K| \geq 2$ and choose a constant p with $p < \frac{\beta\log(|K|)-\log(|K|-1)}{\log(2)}$. Let $f\colon E_1 \longrightarrow E_2$ be a mapping such that*

$$\|\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) - f(x)\|_\beta \leq \theta(\|x\|^p + \|y\|^p), \tag{3.7}$$

and $\|x + k\cdot x\|^p \leq 2^p\|x\|^p$ for all $k \in K$ and $x \in E_1$. Then, there exists a unique solution $j : E_1 \longrightarrow E_2$ of equation (1.2) such that

$$\|f(x) - j(x)\|_\beta \leq \frac{2\theta|K|^\beta\|x\|^p}{2^p + |K|^\beta - 2^p|K|} \tag{3.8}$$

for all $x \in E_1$.

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