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STRONG CONVERGENCE THEOREMS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE-TYPE NONSELF-MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce and study a new type of multistep iterative sequence with errors for a finite family of asymptotically quasi-nonexpansive-type nonself-mappings in Banach spaces. The strong convergence of a multistep iterative scheme with errors to a common fixed point of a finite family of asymptotically quasi-nonexpansive-type nonself-mappings on a nonempty closed convex subset of a real Banach space is proved. Furthermore, a sufficient condition for convergence of the iteration process to a common fixed point of mappings under our setting is also established in a real uniformly convex Banach space. The results obtained in this paper extend and improve the several recent results in this area.

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1. Introduction

Fixed point iteration process for nonexpansive self-mappings including Mann and Ishikawa iteration processes have been studied extensively by various authors [8, 10, 14]. For nonexpansive nonself-mappings, some authors [9, 12, 18, 19, 23] have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach space. In 1972, Goebel and Kirk [7] introduced the class of asymptotically nonexpansive self-mappings, who proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point.

Let C be a nonempty closed convex subset of real normed linear space X. A self-mapping $T : C \to C$ is said to be nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y \in C$. A self-mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$. A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$. A self-mapping $T : C \to C$ is called asymptotically quasi-nonexpansive if $F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.3}$$

for all $x \in C$, for all $y \in F(T)$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly $L-$Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$ and asymptotically quasi-nonexpansive.

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al.[1]. It is known [11] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space X and T is a self-mapping of C which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

A self-mapping T is said to be asymptotically nonexpansive in the intermediate sense (see, e.g., [1]) if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \quad \leq \quad 0. \tag{1.4}$$

If $F(T) \neq \emptyset$ and (1.4) holds for all $x \in C$, $y \in F(T)$, then T is called asymptotically quasi-nonexpansive in the intermediate sense.

In 2006, Jing Quan et al.[15] studied necessary and sufficient conditions for the so called finite-step iterative sequences with mean errors for a finite family of asymptotically quasi-nonexpansive-type mappings in Banach spaces to converge to a common fixed point of members of the family. A mapping T is said to be the asymptotically quasi-nonexpansive-type if T is continuous and

$$\limsup_{n\to\infty} \{ \sup_{x\in C,\, p\in F(T)} (\|T^n x - p\|^2 - \|x-p\|^2) \} \quad \leq \quad 0. \tag{1.5}$$

Observe again that (1.5) implies

$$\limsup_{n\to\infty} \{ \sup_{x\in C,\, p\in F(T)} (\|T^n x - p\| - \|x-p\|)(\|T^n x - p\| + \|x-p\|) \} \quad \leq \quad 0$$

which implies

$$\limsup_{n\to\infty} \{ \sup_{x\in C,\, p\in F(T)} (\|T^n x - p\| - \|x-p\|) \} \quad \leq \quad 0, \tag{1.6}$$

so that asymptotically quasi-nonexpansive-type mappings studied by Jing Quan et al.[15] reduce to mappings which are asymptotically quasi-nonexpansive in the intermediate sense.

The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume et al. [6] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows:

Definition 1.1. [6] *Let C be a nonempty subset of a real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ as $n \to \infty$ such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \quad \leq \quad k_n\|x-y\| \tag{1.7}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \quad \leq \quad L\|x-y\| \tag{1.8}$$

for all $x, y \in C$ and $n \geq 1$.

By studying the following iteration process:

$$x_1 \in C, \quad x_{n+1} \;=\; P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \qquad (1.9)$$

Chidume et al. [6] proved weak and strong convergence theorems for asymptotically nonexpansive nonself-mappings in Banach spaces and extended the corresponding results of [2, 13, 16, 17].

In 2007, Tian, Chang and Huang [22] introduced the concept of asymptotically quasi-nonexpansive-type nonself-mappings and studied necessary and sufficient conditions for the so called N-step iterative sequences with errors for a finite family of asymptotically quasi-nonexpansive-type nonself-mappings in Banach spaces to converge to a common fixed point of members of the family.

Definition 1.2. [22] *Let C be a nonempty subset of a real Banach space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ as $n \to \infty$ such that*

$$\|T(PT)^{n-1}x - p\| \;\leq\; k_n\|x-p\| \qquad (1.10)$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$. T is said to be asymptotically nonexpansive-type nonself-mapping if

$$\limsup_{n\to\infty}\{\sup_{x,y\in C}(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x-y\|)\} \;\leq\; 0. \quad (1.11)$$

T is said to be asymptotically quasi-nonexpansive-type nonself-mapping if $F(T)$ $\neq \emptyset$ and

$$\limsup_{n\to\infty}\{\sup_{x\in C,\, p\in F(T)}(\|T(PT)^{n-1}x - p\| - \|x-p\|)\} \;\leq\; 0. \qquad (1.12)$$

If T is a self-mapping, then P becomes the identity mapping so that (1.7), (1.8) and (1.10) reduce to (1.1), (1.2) and (1.3), respectively. (1.11) reduces to (1.4). (1.12) reduces to (1.6).

It is easy to see that if $T : C \to X$ is an asymptotically nonexpansive nonself-mapping, then T is an asymptotically nonexpansive-type nonself-mapping. If $T : C \to X$ is an asymptotically quasi-nonexpansive nonself-mapping, then T is an asymptotically quasi-nonexpansive-type nonself-mapping. If $F(T) \neq \emptyset$ and $T : C \to X$ is an asymptotically nonexpansive-type nonself-mapping, then T is an asymptotically quasi-nonexpansive-type nonself-mapping.

Tian, Chang and Huang [22] considered the following iteration process:

Let X be a real Banach space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C. Let $T_1, T_2, \dots, T_N : C \to X$ be an asymptotically quasi-nonexpansive-type nonself-mapping.

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1 - a_{n1} - b_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}y_{n1} + b_{n1}u_{n1}), \\ y_{n1} = P((1 - a_{n2} - b_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}y_{n2} + b_{n2}u_{n2}), \\ \qquad\qquad \vdots \\ y_{nN-2} = P((1 - a_{nN-1} - b_{nN-1})x_n + a_{nN-1}T_{N-1}(PT_{N-1})^{n-1}y_{nN-1} \\ \qquad\qquad +b_{nN-1}u_{nN-1}), \\ y_{nN-1} = P((1 - a_{nN} - b_{nN})x_n + a_{nN}T_N(PT_N)^{n-1}x_n + b_{nN}u_{nN}), \end{cases}$$

where $\{a_{ni}\}_{n=1}^{\infty}$, $\{b_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \dots, N$ are real sequences in $[0, 1]$ satisfying the conditions $a_{ni} + b_{ni} \le 1$ for all $n \ge 1$, $i = 1, 2, \dots, N$, and $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \dots, N$ are bounded sequences in C.

Very recently, a new iterative scheme which is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space was defined and constructed by Thianwan [21]. It is given as follows:

$$\begin{aligned} y_n &= P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1 - \alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \quad n \ge 1, \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$. He gave some strong and weak convergence theorems of such iterations under some suitable conditions in a uniformly convex Banach space.

If $T_1 = T_2$ and $\beta_n = 0$ for all $n \ge 1$, then (1.13) reduces to (1.9).

Inspired and motivated by these facts, a new type of multistep iterative sequence is introduced and studied in this paper. This iterative sequence can be viewed as an extension for Ishikawa type iterative schemes of Thianwan [21].

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C and $T_1, T_2, \dots, T_N : C \to X$ given mappings. We define the iterative sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1-a_{n1}-b_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}x_n + b_{n1}u_{n1}), \text{ if } N=1, \ n \geq 1, \\ x_1 \in C, \\ x_{n+1} = P((1-a_{n1}-b_{n1})y_{n1} + a_{n1}T_1(PT_1)^{n-1}y_{n1} + b_{n1}u_{n1}), \\ y_{n1} = P((1-a_{n2}-b_{n2})y_{n2} + a_{n2}T_2(PT_2)^{n-1}y_{n2} + b_{n2}u_{n2}), \\ \qquad \vdots \\ y_{nN-2} = P((1-a_{nN-1}-b_{nN-1})y_{nN-1} + a_{nN-1}T_{N-1}(PT_{N-1})^{n-1}y_{nN-1} \\ \qquad + b_{nN-1}u_{nN-1}), \\ y_{nN-1} = P((1-a_{nN}-b_{nN})x_n + a_{nN}T_N(PT_N)^{n-1}x_n + b_{nN}u_{nN}), \text{ if } N \geq 2, \\ n \geq 1, \end{cases} \tag{1.14}$$

where $\{a_{ni}\}_{n=1}^{\infty}$, $\{b_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ are appropriate sequences in $[0, 1]$ and $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ are bounded sequences in C.

The iterative scheme (1.14) is called the projection type multistep iterative scheme with errors for a finite family of asymptotically quasi-nonexpansive-type nonself-mappings.

If $N = 2$ and $b_{ni} = 0$ for all $n \geq 1$, $i = 1, 2, 3, 4, \ldots, N$, then (1.14) reduces to (1.13).

The purpose of this paper is to construct an iteration scheme for approximating common fixed points of finite family of asymptotically quasi-nonexpansive-type nonself-mappings and give necessary and sufficient conditions for the convergence of the scheme to common fixed points of the mappings in arbitrary real Banach spaces. Furthermore, in the case that X is a real uniformly convex Banach space, a sufficient condition for convergence of the iteration process to a common fixed point of mappings under our setting is established.

Now, we recall some well known concepts and results.

A subset C of X is said to be retract if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to C$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, the range of P.

For studying the strong convergence of fixed points of a finite family of nonexpansive mappings, Chidume and Shahzad [4] introduced a condition (B) which is more weaker than T is demicompact.

A finite family $\{T_i : i = 1, 2, \ldots, N\}$ of N mappings from C to X with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy

(1) condition (B) [4] if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ and all $x\in C$ such that $\max\limits_{1\leqslant i\leqslant N}\{\|x-T_ix\|\}\geqslant f(d(x,F))$;

(2) condition $(\overline{C})$ [3] if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ and all $x\in C$ such that $\|x-T_ix\|\geqslant f(d(x,F))$ for at least one T_i, $i=1,2,\dots,N$.

Note that condition (B) and condition $(\overline{C})$ are equivalent (see [3]).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.3. *[[20]] Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1}\leq a_n+t_n \text{ for all } n\geq 1.$$

If $\sum_{n=1}^{\infty}t_n<\infty$, then $\lim_{n\to\infty}a_n$ exists.

Lemma 1.4. *[[5]] Let X be a uniformly convex Banach space and $B_r=\{x\in X:\|x\|\leq r\}$, $r>0$. Then there exists a continuous, strictly increasing, and convex function $g:[0,\infty)\to[0,\infty), g(0)=0$ such that*

$$\|\lambda x+\beta y+\gamma z\|^2\leq\lambda\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\lambda\beta g(\|x-y\|),$$

for all $x,y,z\in B_r$, and all $\lambda,\beta,\gamma\in[0,1]$ with $\lambda+\beta+\gamma=1$.

2. Main Results

2.1. Necessary and sufficient conditions for convergence.

Theorem 2.1. *Let X be a real Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1,T_2,\dots,T_N:C\to X$ be N asymptotically quasi-nonexpansive-type nonself-mappings such that $F=\cap_{i=1}^{N}F(T_i)\neq\emptyset$ and closed (here $F(T_i)$ denotes the set of fixed points of T_i). Let $\{a_{ni}\}_{n=1}^{\infty}$, $i=1,2,\dots,N$ and $\{b_{ni}\}_{n=1}^{\infty}$, $i=1,2,\dots,N$ be real sequences in $[0,1]$ such that $a_{ni}+b_{ni}\leq 1$ for all $n\geq 1$, $i=1,2,\dots,N$, and $\sum_{n=1}^{\infty}a_{ni}<\infty$, $\sum_{n=1}^{\infty}b_{ni}<\infty$, $i=1,2,\dots,N$, and let $\{u_{ni}\}_{n=1}^{\infty}$, $i=1,2,\dots,N$ be bounded sequences in C. From an arbitrary $x_1\in C$, define the sequence $\{x_n\}$ by (1.14). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1,T_2,\dots,T_N$ if and only if*

$$\liminf_{n\to\infty}d(x_n,F)=0,$$

where $d(x_n,F)=\inf_{y\in F}\|x_n-y\|$, $n\geq 1$.

Proof. The necessity of the conditions is obvious. Thus, we will only prove the sufficiency. Let $p \in F$, by boundedness of the sequences $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$, so we can put

$$M = \sup_{n \geq 1,\ i=1,2,\ldots,N} \|u_{ni} - p\|.$$

Since $T_1, T_2, \ldots, T_N : C \to X$ are asymptotically quasi-nonexpansive-type nonself-mappings, for any given $\varepsilon > 0$, there exists a natural number n_0 such that for all $n \geq n_0$ and $x^* \in F$

$$\sup_{x \in C,\ i=1,2,\ldots,N} \{\|T_i(PT_i)^{n-1}x - x^*\| - \|x - x^*\|\} < \varepsilon.$$

Since $\{x_n\}$ and $\{y_{ni}\} \subset C$, $i = 1, 2, \ldots, N-1$, for any $n \geq n_0$ and $x^* \in F$, we have

$$\begin{cases} \|T_1(PT_1)^{n-1}y_{n1} - x^*\| - \|y_{n1} - x^*\| < \varepsilon, \\ \|T_2(PT_2)^{n-1}y_{n2} - x^*\| - \|y_{n2} - x^*\| < \varepsilon, \\ \qquad\vdots \\ \|T_{N-1}(PT_1)^{n-1}y_{nN-1} - x^*\| - \|y_{nN-1} - x^*\| < \varepsilon, \\ \|T_N(PT_N)^{n-1}x_n - x^*\| - \|x_n - x^*\| < \varepsilon. \end{cases} \tag{2.1}$$

For each $n \geq n_0$, using (1.14) and (2.1), we have

$$\begin{aligned} \|y_{n1} - p\| &= \|P((1 - a_{n2} - b_{n2})y_{n2} + a_{n2}T_2(PT_2)^{n-1}y_{n2} + b_{n2}u_{n2}) - P(p)\| \\ &\leq \|(1 - a_{n2} - b_{n2})y_{n2} + a_{n2}T_2(PT_2)^{n-1}y_{n2} + b_{n2}u_{n2} - p\| \\ &\leq (1 - a_{n2} - b_{n2})\|y_{n2} - p\| + a_{n2}\|T_2(PT_2)^{n-1}y_{n2} - p\| \\ &\quad + b_{n2}\|u_{n2} - p\| \\ &= (1 - b_{n2})\|y_{n2} - p\| + a_{n2}(\|T_2(PT_2)^{n-1}y_{n2} - p\| - \|y_{n2} - p\|) \\ &\quad + b_{n2}\|u_{n2} - p\| \\ &\leq (1 - b_{n2})\|y_{n2} - p\| + a_{n2}\varepsilon + b_{n2}M \\ &\leq \|y_{n2} - p\| + a_{n2}\varepsilon + b_{n2}M. \end{aligned}$$

Continuing, we get that

$$\|y_{ni} - p\| \leq \|y_{ni+1} - p\| + a_{ni+1}\varepsilon + b_{ni+1}M,\ i = 1, 2, \ldots, N-2. \tag{2.2}$$

By using (1.14) and (2.1), we have

$$
\begin{aligned}
&\|y_{nN-1} - p\| \\
&= \|P((1 - a_{nN} - b_{nN})x_n + a_{nN}T_N(PT_N)^{n-1}x_n + b_{nN}u_{nN}) - P(p)\| \\
&\leq \|(1 - a_{nN} - b_{nN})x_n + a_{nN}T_N(PT_N)^{n-1}x_n + b_{nN}u_{nN} - p\| \\
&\leq (1 - a_{nN} - b_{nN}\|x_n - p\| + a_{nN}\|T_N(PT_N)^{n-1}x_n - p\| \\
&\quad + b_{nN}\|u_{nN} - p\| \\
&= (1 - b_{nN})\|\|x_n - p\| + a_{nN}(\|T_N(PT_N)^{n-1}x_n - p\| - \|x_n - p\|) \\
&\quad + b_{nN}\|u_{nN} - p\| \\
&\leq (1 - b_{nN})\|x_n - p\| + a_{nN}\varepsilon + b_{nN}M \\
&\leq \|x_n - p\| + a_{nN}\varepsilon + b_{nN}M. \qquad (2.3)
\end{aligned}
$$

From (2.2) and (2.3), we have

$$
\begin{aligned}
\|y_{nN-2} - p\| &\leq \|y_{nN-1} - p\| + a_{nN-1}\varepsilon + b_{nN-1}M \\
&\leq \|x_n - p\| + a_{nN}\varepsilon + b_{nN}M + a_{nN-1}\varepsilon + b_{nN-1}M \\
&= \|x_n - p\| + (a_{nN} + a_{nN-1})\varepsilon + (b_{nN} + b_{nN-1})M.
\end{aligned}
$$

By induction, we can show that for any $i = 1, 2, \ldots, N-1$,

$$
\|y_{nN-i} - p\| \leq \|x_n - p\| + (\sum_{j=0}^{i-1} a_{nN-j})\varepsilon + (\sum_{j=0}^{i-1} b_{nN-j})M. \qquad (2.4)
$$

By taking $i = N - 1$ in (2.4), we have

$$
\|y_{n1} - p\| \leq \|x_n - p\| + (\sum_{j=0}^{N-2} a_{nN-j})\varepsilon + (\sum_{j=0}^{N-2} b_{nN-j})M. \qquad (2.5)
$$

Hence for any $n \geq n_0$, it follows from (1.14), (2.1) and (2.5) that

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|P((1-a_{n1}-b_{n1})y_{n1}+a_{n1}T_1(PT_1)^{n-1}y_{n1}+b_{n1}u_{n1})-P(p)\| \\
&\le \|(1-a_{n1}-b_{n1})y_{n1}+a_{n1}T_1(PT_1)^{n-1}y_{n1}+b_{n1}u_{n1}-p\| \\
&\le (1-a_{n1}-b_{n1})\|y_{n1}-p\|+a_{n1}\|T_1(PT_1)^{n-1}y_{n1}-p\| \\
&\quad + b_{n1}\|u_{n1}-p\| \\
&= (1-b_{n1})\|y_{n1}-p\|+a_{n1}(\|T_1(PT_1)^{n-1}y_{n1}-p\|-\|y_{n1}-p\|) \\
&\quad + b_{n1}\|u_{n1}-p\| \\
&\le (1-b_{n1})\|y_{n1}-p\|+a_{n1}\varepsilon+b_{n1}M \\
&\le \|y_{n1}-p\|+a_{n1}\varepsilon+b_{n1}M \\
&\le \|x_n-p\|+(\sum_{j=0}^{N-2}a_{nN-j})\varepsilon+(\sum_{j=0}^{N-2}b_{nN-j})M+a_{n1}\varepsilon+b_{n1}M \\
&= \|x_n-p\|+(a_{nN}+a_{nN-1}+a_{nN-2}+\ldots+a_{n2})\varepsilon \\
&\quad +(b_{nN}+b_{nN-1}+b_{nN-2}+\ldots+b_{n2})M+a_{n1}\varepsilon+b_{n1}M \\
&= \|x_n-p\|+(\sum_{j=1}^{N}a_{nj})\varepsilon+(\sum_{j=1}^{N}b_{nj})M \\
&= \|x_n-p\|+A_n, \qquad (2.6)
\end{aligned}
$$

where $A_n=(\sum_{j=1}^{N}a_{nj})\varepsilon+(\sum_{j=1}^{N}b_{nj})M, n\ge 1$. Since $\sum_{n=1}^{\infty}a_{ni}<\infty$, $\sum_{n=1}^{\infty}b_{ni}<\infty$, $i=1,2,\ldots,N$, it follows that $\sum_{n=1}^{\infty}A_n<\infty$. We obtained by Lemma 1.3 that $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence $\{x_n\}$ is bounded.

By the arbitrariness of $p\in F$, taking the inf on both sides in the inequality (2.6), we have

$$\inf_{p\in F}\|x_{n+1}-p\|\le \inf_{p\in F}\|x_n-p\|+A_n$$

for all $n\ge n_0$, and so

$$d(x_{n+1},F)\le d(x_n,F)+A_n$$

for all $n\ge n_0$.

Since $\sum_{n=1}^{\infty}A_n<\infty$ we obtained by Lemma 1.3 that $\lim_{n\to\infty}d(x_n,F)$ exists. By assumption, $\liminf_{n\to\infty}d(x_n,F)=0$ we have

$$\lim_{n\to\infty}d(x_n,F)=0. \qquad (2.7)$$

Next, we prove that the sequence $\{x_n\}$ defined by (1.14) is a Cauchy sequence in C. For all integer $m \geq 1$, any $n \geq n_0$, and any $p \in F$, by using (2.6), we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + A_{n+m-1} \\ &\leq \|x_{n+m-2} - p\| + (A_{n+m-1} + A_{n+m-2}) \\ &\quad \vdots \\ &\leq \|x_n - p\| + \sum_{k=n}^{n+m-1} A_k. \end{aligned} \tag{2.8}$$

For all integer $m \geq 1$, any $n \geq n_0$, by using (2.8), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &= \|x_{n+m} - p + p - x_n\| \\ &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq \|x_n - p\| + \sum_{k=n}^{n+m-1} A_k + \|x_n - p\| \\ &= 2\|x_n - p\| + \sum_{k=n}^{n+m-1} A_k. \end{aligned} \tag{2.9}$$

Using (2.9), by the arbitrariness of $p \in F$, we have

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F) + \sum_{k=n}^{\infty} A_k \tag{2.10}$$

for all $n \geq n_0$.

Now, since $\sum_{n=1}^{\infty} A_n < \infty$ and $\lim_{n\to\infty} d(x_n, F) = 0$, given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that for all $n \geq n_1$, $d(x_n, F) < \frac{\varepsilon}{4}$ and $\sum_{k=n}^{\infty} A_k < \frac{\varepsilon}{2}$. So for all integers $n \geq n_1$, $m \geq 1$, we obtain from (2.10) that $\|x_{n+m} - x_n\| < \varepsilon$. Hence, $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset of X, and so it is complete. Let $\lim_{n\to\infty} x_n = x^*$. Then $x^* \in C$. It remains to show that $x^* \in F$. Suppose for contradiction that $x^* \in F^c$ (where F^c denotes the complement of F). Since F is closed set, $d(x^*, F) > 0$. But, for all $p \in F$, we have $\|x^* - p\| \leq \|x^* - x_n\| + \|x_n - p\|$ which implies

$$d(x^*, F) \leq \|x^* - x_n\| + d(x_n, F). \tag{2.11}$$

Using (2.7) and letting $n \to \infty$ in (2.11) we obtain $d(x^*, F) \leq 0$ which contradicts $d(x^*, F) > 0$. Thus, $x^* \in F$. This completes the proof. $\square$

Since every asymptotically quasi-nonexpansive nonself-mapping is asymptotically quasi-nonexpansive-type nonself-mapping, so the following result is directly obtained by Theorem 2.1.

Theorem 2.2. *Let X be a real Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be N asymptotically quasi-nonexpansive nonself-mappings such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and closed. Let $\{a_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ and $\{b_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ be real sequences in $[0, 1]$ such that $a_{ni} + b_{ni} \leq 1$ for all $n \geq 1$, $i = 1, 2, \ldots, N$, and $\sum_{n=1}^\infty a_{ni} < \infty$, $\sum_{n=1}^\infty b_{ni} < \infty$, $i = 1, 2, \ldots, N$, and let $\{u_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ be bounded sequences in C. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.14). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_N$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Remark 2.1. If $T_1, T_2, \ldots, T_N : C \to X$ in theorem 2.1 are asymptotically nonexpansive nonself-mappings, then taking $n = 1$ in the inequality (1.10) (Definition 1.2), we see that $T_1, T_2, \ldots, T_N : C \to X$ are continuous asymptotically nonexpansive nonself-mappings. It is easy to prove that the common fixed point set F of $T_1, T_2, \ldots, T_N$ is closed.

In view of Remark 2.1, the following result can be obtained by Theorem 2.1.

Theorem 2.3. *Let X be a real Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be N asymptotically nonexpansive nonself-mappings such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{a_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ and $\{b_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ be real sequences in $[0, 1]$ such that $a_{ni} + b_{ni} \leq 1$ for all $n \geq 1$, $i = 1, 2, \ldots, N$, and $\sum_{n=1}^\infty a_{ni} < \infty$, $\sum_{n=1}^\infty b_{ni} < \infty$, $i = 1, 2, \ldots, N$, and let $\{u_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ be bounded sequences in C. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.14). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_N$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

2.2. Convergence theorem in real uniformly convex Banach spaces.

Theorem 2.4. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be N uniformly L-Lipschitzian and asymptotically quasi-nonexpansive-type nonself-mappings such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and closed. Let $\{a_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ and $\{b_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ be real sequences in $[0, 1]$ such that $a_{ni}, a_{ni} + b_{ni}$ are in $[s, 1 - s]$ for some $s \in (0, 1)$ and for all $n \geq 1$, $i = 1, 2, \ldots, N$, and $\sum_{n=1}^\infty a_{ni} < \infty$, $\sum_{n=1}^\infty b_{ni} < \infty$, $i = 1, 2, \ldots, N$, and let $\{u_{ni}\}_{n=1}^\infty$, $i = 1, 2, \ldots, N$ be bounded* *sequences in C. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.14). Then* $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0,\ i = 1, 2, \dots, N.$

Proof. We have $\{x_n\}$ is bounded (see proof Theorem 2.1). So, there exists $r > 0$ such that $\{x_n\} \subset B_r$, $\forall n \geq 1$; where B_r is the closed ball of X with center zero and radius r. Let $p \in F$, by boundedness of the sequences $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \dots, N$, so we can put

$$M = \sup_{n\geq 1,\ i=1,2,\dots,N} \|u_{ni} - p\|.$$

Since $T_1, T_2, \dots, T_N$ are asymptotically quasi-nonexpansive-type nonself-mappings. Applying (1.5), we have for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$ and $x^* \in F$

$$\sup_{x\in C,\ i=1,2,\dots,N} \{\|T_i(PT_i)^{n-1}x - x^*\|^2 - \|x - x^*\|^2\} < \varepsilon.$$

Since $\{x_n\}$ and $\{y_{ni}\} \subset C$, $i = 1, 2, \dots, N-1$, for any $n \geq n_0$ and $x^* \in F$, we have

$$\begin{cases} \|T_1(PT_1)^{n-1}y_{n1} - x^*\|^2 - \|y_{n1} - x^*\|^2 < \varepsilon, \\ \|T_2(PT_2)^{n-1}y_{n2} - x^*\|^2 - \|y_{n2} - x^*\|^2 < \varepsilon, \\ \vdots \\ \|T_{N-1}(PT_1)^{n-1}y_{nN-1} - x^*\|^2 - \|y_{nN-1} - x^*\|^2 < \varepsilon, \\ \|T_N(PT_N)^{n-1}x_n - x^*\|^2 - \|x_n - x^*\|^2 < \varepsilon. \end{cases} \tag{2.12}$$

Now, for $N = 1$, we get from (1.14) that

$$x_1 \in C, \quad x_{n+1} = P((1 - a_{n1} - b_{n1})x_n + a_{n1}T_1(PT_1)^{n-1}x_n + b_{n1}u_{n1}). \tag{2.13}$$

For each $n \geq n_0$, using (2.12), (2.13) and Lemma 1.4, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|(1 - a_{n1} - b_{n1})(x_n - p) + a_{n1}(T_1(PT_1)^{n-1}x_n - p) \\ &\quad + b_{n1}(u_{n1} - p)\|^2 \\ &\leq (1 - a_{n1} - b_{n1})\|x_n - p\|^2 + a_{n1}\|T_1(PT_1)^{n-1}x_n - p\|^2 \\ &\quad + b_{n1}\|u_{n1} - p\|^2 - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\ &= (1 - b_{n1})\|x_n - p\|^2 + a_{n1}(\|T_1(PT_1)^{n-1}x_n - p\|^2 - \|x_n - p\|^2) \\ &\quad + b_{n1}\|u_{n1} - p\|^2 - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\ &\leq \|x_n - p\|^2 + a_{n1}\varepsilon^2 + b_{n1}M^2 \\ &\quad - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}x_n - x_n\|). \end{aligned} \tag{2.14}$$

Using (2.14), we have

$$
\begin{aligned}
& s^2 \sum_{n=n_0}^{m} g(\|T_1(PT_1)^{n-1}x_n - x_n\|) \\
& \leq \sum_{n=n_0}^{m} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\
& \quad + \varepsilon^2 \sum_{n=n_0}^{m} a_{n1} + M^2 \sum_{n=n_0}^{m} b_{n1} \\
& = \|x_{n_0} - p\|^2 + \varepsilon^2 \sum_{n=n_0}^{m} a_{n1} + M^2 \sum_{n=n_0}^{m} b_{n1}. \qquad (2.15)
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} a_{ni} < \infty$, $\sum_{n=1}^{\infty} b_{ni} < \infty$, $i = 1, 2, \ldots, N$, it follows that

$$\varepsilon^2 \sum_{n=n_0}^{m} a_{n1} < \infty$$

and

$$M^2 \sum_{n=n_0}^{m} b_{n1} < \infty.$$

By letting $m \to \infty$ in (2.15) we get

$$\sum_{n=n_0}^{\infty} g(\|T_1(PT_1)^{n-1}x_n - x_n\|) < \infty,$$

and therefore $\lim_{n\to\infty} g(\|T_1(PT_1)^{n-1}x_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0. \qquad (2.16)$$

For $N = 2$, (1.14) becomes

$$
\begin{cases}
x_1 \in C, \\
x_{n+1} = P((1 - a_{n1} - b_{n1})y_{n1} + a_{n1}T_1(PT_1)^{n-1}y_{n1} + b_{n1}u_{n1}), \\
y_{n1} = P((1 - a_{n2} - b_{n2})x_n + a_{n2}T_2(PT_2)^{n-1}x_n + b_{n2}u_{n2}).
\end{cases} \qquad (2.17)
$$

For each $n \geq n_0$, using (2.12), (2.17) and Lemma 1.4, we have

$$
\begin{aligned}
&\|y_{n1} - p\|^2 \\
&\le \|(1 - a_{n2} - b_{n2})(x_n - p) + a_{n2}(T_2(PT_2)^{n-1}x_n - p) \\
&\quad + b_{n2}(u_{n2} - p)\|^2 \\
&\le (1 - a_{n2} - b_{n2})\|x_n - p\|^2 + a_{n2}\|T_2(PT_2)^{n-1}x_n - p\|^2 \\
&\quad + b_{n2}\|u_{n2} - p\|^2 - a_{n2}(1 - a_{n2} - b_{n2})g(\|T_2(PT_2)^{n-1}x_n - x_n\|) \\
&= (1 - b_{n2})\|x_n - p\|^2 + a_{n2}(\|T_2(PT_2)^{n-1}x_n - p\|^2 - \|x_n - p\|^2) \\
&\quad + b_{n2}\|u_{n2} - p\|^2 - a_{n2}(1 - a_{n2} - b_{n2})g(\|T_2(PT_2)^{n-1}x_n - x_n\|) \\
&\le \|x_n - p\|^2 + a_{n2}\varepsilon^2 + b_{n2}M^2 \\
&\quad - a_{n2}(1 - a_{n2} - b_{n2})g(\|T_2(PT_2)^{n-1}x_n - x_n\|),
\end{aligned}
$$

and so

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \|(1 - a_{n1} - b_{n1})(y_{n1} - p) + a_{n1}(T_1(PT_1)^{n-1}y_{n1} - p) \\
&\quad + b_{n1}(u_{n1} - p)\|^2 \\
&\le (1 - a_{n1} - b_{n1})\|y_{n1} - p\|^2 + a_{n1}\|T_1(PT_1)^{n-1}y_{n1} - p\|^2 \\
&\quad + b_{n1}\|u_{n1} - p\|^2 - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\|) \\
&= (1 - b_{n1})\|y_{n1} - p\|^2 + a_{n1}(\|T_1(PT_1)^{n-1}y_{n1} - p\|^2 - \|y_{n1} - p\|^2) \\
&\quad + b_{n1}\|u_{n1} - p\|^2 - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\|) \\
&\le \|y_{n1} - p\|^2 + a_{n1}\varepsilon^2 + b_{n1}M^2 \\
&\quad - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\|) \\
&\le \|x_n - p\|^2 + a_{n2}\varepsilon^2 + b_{n2}M^2 + a_{n1}\varepsilon^2 + b_{n1}M^2 \\
&\quad - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\|) \\
&\quad - a_{n2}(1 - a_{n2} - b_{n2})g(\|T_2(PT_2)^{n-1}x_n - x_n\|) \\
&= \|x_n - p\|^2 + (a_{n1} + a_{n2})\varepsilon^2 + (b_{n1} + b_{n2})M^2 \\
&\quad - a_{n1}(1 - a_{n1} - b_{n1})g(\|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\|) \\
&\quad - a_{n2}(1 - a_{n2} - b_{n2})g(\|T_2(PT_2)^{n-1}x_n - x_n\|). \qquad (2.18)
\end{aligned}
$$

From (2.18), we obtain the following two important inequalities

$$\begin{aligned} & a_{n2}(1-a_{n2}-b_{n2})g(\|T_2(PT_2)^{n-1}x_n-x_n\|) \\ & \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\ & \quad +(a_{n1}+a_{n2})\varepsilon^2+(b_{n1}+b_{n2})M^2 \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} & a_{n1}(1-a_{n1}-b_{n1})g(\|T_1(PT_1)^{n-1}y_{n1}-y_{n1}\|) \\ & \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\ & \quad +(a_{n1}+a_{n2})\varepsilon^2+(b_{n1}+b_{n2})M^2. \end{aligned} \tag{2.20}$$

Using (2.19), we have

$$\begin{aligned} & s^2\sum_{n=n_0}^{m} g(\|T_2(PT_2)^{n-1}x_n-x_n\|) \\ & \leq \sum_{n=n_0}^{m}(\|x_n-p\|^2-\|x_{n+1}-p\|^2) \\ & \quad +\varepsilon^2\sum_{n=n_0}^{m}(a_{n1}+a_{n2})+M^2\sum_{n=n_0}^{m}(b_{n1}+b_{n2}) \\ & = \|x_{n_0}-p\|^2+\varepsilon^2\sum_{n=n_0}^{m}(a_{n1}+a_{n2}) \\ & \quad +M^2\sum_{n=n_0}^{m}(b_{n1}+b_{n2}). \end{aligned} \tag{2.21}$$

Since $\sum_{n=1}^{\infty} a_{ni}<\infty,\ \sum_{n=1}^{\infty} b_{ni}<\infty,\ i=1,2,\ldots,N$, it follows that

$$\varepsilon^2\sum_{n=n_0}^{m}(a_{n1}+a_{n2})<\infty$$

and

$$M^2\sum_{n=n_0}^{m}(b_{n1}+b_{n2})<\infty.$$

By letting $m\to\infty$ in (2.21) we get

$$\sum_{n=n_0}^{\infty} g(\|T_2(PT_2)^{n-1}x_n-x_n\|)<\infty,$$

and therefore $\lim_{n\to\infty} g(\|T_2(PT_2)^{n-1}x_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_{n\to\infty} \|T_2(PT_2)^{n-1}x_n - x_n\| = 0. \tag{2.22}$$

By using a similar method, together with (2.20), we can prove that

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\| = 0. \tag{2.23}$$

Since $\sum_{n=1}^{\infty} a_{ni} < \infty$, $\sum_{n=1}^{\infty} b_{ni} < \infty$, $i = 1, 2, \ldots, N$, and $\{x_n\}$, $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ are all bounded. Using (2.17) and (2.22), we have

$$\begin{aligned}\|y_{n1} - x_n\| &\le a_{n2}\|T_2(PT_2)^{n-1}x_n - x_n\| + b_{n2}\|u_{n2} - x_n\| \\ &\to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{2.24}$$

Since $\sum_{n=1}^{\infty} a_{ni} < \infty$, $\sum_{n=1}^{\infty} b_{ni} < \infty$, $i = 1, 2, \ldots, N$, and $\{x_n\}$, $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ are all bounded. Using (2.17), (2.23) and (2.24), we have

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\le (1 - a_{n1} - b_{n1})\|y_{n1} - x_n\| + a_{n1}\|T_1(PT_1)^{n-1}y_{n1} - x_n\| \\ &\quad + b_{n1}\|u_{n1} - x_n\| \\ &= (1 - a_{n1} - b_{n1})\|y_{n1} - x_n\| + a_{n1}\|T_1(PT_1)^{n-1}y_{n1} - y_{n1} + y_{n1} - x_n\| \\ &\quad + b_{n1}\|u_{n1} - x_n\| \\ &\le (1 - a_{n1} - b_{n1})\|y_{n1} - x_n\| + a_{n1}\|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\| \\ &\quad + a_{n1}\|y_{n1} - x_n\| + b_{n1}\|u_{n1} - x_n\| \\ &\le \|y_{n1} - x_n\| + a_{n1}\|T_1(PT_1)^{n-1}y_{n1} - y_{n1}\| + b_{n1}\|u_{n1} - x_n\| \\ &\quad \to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{2.25}$$

Since $T_1, T_2, \ldots, T_N$ are uniformly L-Lipschitzian. Using (2.16) and (2.25), we have

$$\begin{aligned}
&\|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| \\
&= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-1}x_n \\
&\quad + T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{n-1}x_n\| \\
&\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\le \|x_{n+1} - x_n\| + L\|x_{n+1} - x_n\| \\
&\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| \to 0 \quad (\text{as } n \to \infty).
\end{aligned} \tag{2.26}$$

In addition,

$$\begin{aligned}
&\|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\| \\
&= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-2}x_n \\
&\quad + T_1(PT_1)^{n-2}x_n - T_1(PT_1)^{n-2}x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-2}x_n - x_n\| \\
&\quad + \|T_1(PT_1)^{n-2}x_{n+1} - T_1(PT_1)^{n-2}x_n\| \\
&\le \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-2}x_n - x_n\| \\
&\quad + L\|x_{n+1} - x_n\|,
\end{aligned}$$

where $L > 0$. It follows from (2.25) and (2.26) that

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\| = 0. \tag{2.27}$$

We denote $(PT_1)^{1-1}$ to be the identity maps from C onto itself. Thus by the inequality (2.26) and (2.27), we have

$$\begin{aligned}
&\|x_{n+1} - T_1x_{n+1}\| \\
&= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1} + T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&\le \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + \|T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\quad + \|T_1(PT_1)^{1-1}(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{1-1}x_{n+1}\|
\end{aligned}$$

$$
\begin{aligned}
&\le \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|(PT_1)^{n-1}x_{n+1} - x_{n+1}\| \\
&= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\quad + L\|(PT_1)(PT_1)^{n-2}x_{n+1} - P(x_{n+1})\| \\
&\le \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|T_1(PT_1)^{n-2}x_{n+1} - x_{n+1}\| \\
&\quad \to 0 \quad (\text{as } n \to \infty),
\end{aligned}
$$

which implies that $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$. Similarly, we may show that $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$, $i = 1, 2, \ldots, N$. The proof is completed. □

In the next result, we prove the strong convergence of the scheme (1.14) under condition $(\overline{C})$ which is weaker than the compactness of the domain of the mappings.

Theorem 2.5. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be N uniformly L-Lipschitzian and asymptotically quasi-nonexpansive-type nonself-mappings such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and closed. Let $\{a_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ and $\{b_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ be real sequences in $[0, 1]$ such that $a_{ni}, a_{ni} + b_{ni}$ are in $[s, 1 - s]$ for some $s \in (0, 1)$ and for all $n \geq 1$, $i = 1, 2, \ldots, N$, and $\sum_{n=1}^{\infty} a_{ni} < \infty$, $\sum_{n=1}^{\infty} b_{ni} < \infty$, $i = 1, 2, \ldots, N$, and let $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ be bounded sequences in C. Suppose that there exists one of $T_1, T_2, \ldots, T_N$ satisfying condition $(\overline{C})$. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.14). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_N$.*

Proof. We may assume that T_1 satisfies condition $(\overline{C})$ without loss of generality. By Theorem 2.4, we have $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$, $i = 1, 2, \ldots, N$. It follows from condition $(\overline{C})$ that

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty} \|x_n - T_1x_n\| = 0.$$

This implies that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$. Hence, $\liminf_{n\to\infty} d(x_n, F) = 0$. Now apply Theorem 2.1. This completes the proof. □

In view of Remark 2.1, the following result can be obtained by Theorem 2.5.

Theorem 2.6. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be N asymptotically nonexpansive nonself-mappings such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{a_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ and $\{b_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ be real sequences in $[0, 1]$ such that $a_{ni}, a_{ni} + b_{ni}$*

are in $[s, 1-s]$ *for some* $s \in (0,1)$ *and for all* $n \geq 1$, $i = 1, 2, \ldots, N$, *and* $\sum_{n=1}^{\infty} a_{ni} < \infty$, $\sum_{n=1}^{\infty} b_{ni} < \infty$, $i = 1, 2, \ldots, N$, *and let* $\{u_{ni}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ *be bounded sequences in* C. *Suppose that there exists one of* $T_1, T_2, \ldots, T_N$ *satisfying condition* $(\overline{C})$. *From an arbitrary* $x_1 \in C$, *define the sequence* $\{x_n\}$ *by (1.14). Then* $\{x_n\}$ *converges strongly to a common fixed point of* $T_1, T_2, \ldots, T_N$.

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