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CERTAIN EXTENDED SPECIAL FUNCTIONS AND FRACTIONAL INTEGRAL AND DERIVATIVE OPERATORS VIA AN EXTENDED BETA FUNCTION

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Abstract. Various extensions of the Euler's beta function have, recently, been presented and investigated. Here, choosing to use a fully extended beta function, we introduce an extended hypergeometric function, an extended confluent hypergeometric function, and an extension of the Appell function F_1. We, also, use the fully extended beta function to introduce an extended Riemann-Liouville type integral operator and investigate its associated formulas and generating relations. The results presented here, being very general, can be specialized to yield some known and new results.

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1. Introduction and preliminaries

The beta function $B(\alpha,\beta)$ is defined by

$$B(\alpha,\,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\min\{\Re(\alpha),\,\Re(\beta)\}>0)\\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),\end{cases} \tag{1.1}$$

where Γ is the familiar gamma function (see, e.g., [12, Section 1.1]). Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively.

The Gauss hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$ are defined by (see, e.g., [11, 13])

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\quad\left(|z|<1;\ a,\,b\in\mathbb{C},\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right) \tag{1.2}$$

and

$$\Phi(a;c;z)={}_1F_1(a;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}\quad\left(a\in\mathbb{C},\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right). \tag{1.3}$$

Here and in the following, $(\lambda)_\nu$ is the Pochhammer symbol defined (for $\lambda,\,\nu\in\mathbb{C}$) by (see [12, p. 2 and p. 5])

$$\begin{aligned}(\lambda)_\nu :&=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}\quad(\lambda,\,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-)\\ &=\begin{cases}1 & (\nu=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N}).\end{cases}\end{aligned} \tag{1.4}$$

The Appell's series or bivariate hypergeometric series F_1 is defined by (see, e.g., [13, p. 22])

$$F_1(a,b_1,b_2;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \tag{1.5}$$

$$\left(\max\{|x|,\,|y|\}<1;\ a,\,b_1,\,b_2\in\mathbb{C},\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

We recall integral representations for the above functions ${}_2F_1$, ${}_1F_1$, and F_1 (see, e.g, [12, Section 1.5]; see also [13, p. 276])

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt \tag{1.6}$$

$$(\Re(c)>\Re(b)>0,\ |\arg(1-z)|<\pi);$$

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt}\, dt \tag{1.7} $$
$$ (\Re(c) > \Re(a) > 0)\,; $$

$$ \begin{aligned} F_1\left(a, b_1, b_2; c; x, y\right) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \\ &\times \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}\, dt \end{aligned} \tag{1.8} $$
$$ (\Re(c) > \Re(a) > 0;\ |\arg(1-x)| < \pi,\ |\arg(1-y)| < \pi)\,. $$

Chaudhry et al. [2] introduced and investigated the following extended beta function

$$ B(\alpha, \beta; p) = B_p(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, \mathsf{e}^{-\frac{p}{t(1-t)}}\, dt \tag{1.9} $$
$$ (\min\{\Re(\alpha),\ \Re(\beta)\} > 0;\ \Re(p) > 0)\,. $$

Obviously $B(\alpha, \beta; 0) = B(\alpha, \beta)$.

Chaudhry et al. [3] used the extended beta function $B_p(\alpha, \beta)$ to extend the hypergeometric function ${}_2F_1$ and the confluent hypergeometric function Φ as follows:

$$ F_p(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!} \quad (p \geq 0) \tag{1.10} $$

and

$$ \Phi_p(b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \quad (p \geq 0)\,. \tag{1.11} $$

Clearly $F_0(a, b; c; z) = {}_2F_1(a, b; c; z)$ and $\Phi_0(b; c; z) = \Phi(b; c; z)$. They [3] presented the following integral representations

$$ \begin{aligned} F_p(a, b; c; z) &= \frac{1}{B(b, c-b)} \\ &\times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left(-\frac{p}{t(1-t)}\right) dt \end{aligned} \tag{1.12} $$
$$ (p \geq 0;\ \Re(c) > \Re(b) > 0;\ |\arg(1-z)| < \pi) $$

and

$$ \Phi_p(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left(zt - \frac{p}{t(1-t)}\right) dt \tag{1.13} $$
$$ (p \geq 0;\ \Re(c) > \Re(b) > 0)\,. $$

Özarslan and Özergin [8] used the $B_p(\alpha,\beta)$ to extend the Appell's function F_1 and presented its integral representation

$$F_1(a,b_1,b_2;c;x,y;p)=\sum_{n=0}^{\infty}\frac{B_p(a+m+n,c-a)}{B(a,c-a)}(b_1)_m(b_2)_n\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.14}$$

and

$$\begin{aligned}F_1(a,b_1,b_2;c;x,y;p)&=\frac{1}{B(a,c-a)}\\ &\times\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2}\exp\Big(-\frac{p}{t(1-t)}\Big)\,dt\end{aligned} \tag{1.15}$$

$$\left(p\geq 0;\ \Re(c)>\Re(a)>0;\ |\arg(1-x)|<\pi,\ |\arg(1-y)|<\pi\right).$$

Obviously, the particular cases $p=0$ of (1.14) and (1.15) reduce, respectively, to (1.5) and (1.8).

Choi et al. [4] extended the beta function $B(\alpha,\beta)$

$$B(\alpha,\beta;p,q)=B_{p,q}(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,dt \tag{1.16}$$

$$\left(\min\{\Re(\alpha),\ \Re(\beta)\}>0;\ \min\{p,\ q\}>0\right).$$

Obviously, $B_{p,p}(\alpha,\beta)=B_p(\alpha,\beta)$ and $B_{0,0}(\alpha,\beta)=B(\alpha,\beta)$. They [4] used (1.16) to further extend the F_p and Φ_p and investigate

$$F_{p,q}(a,b;c;z)=\sum_{n=0}^{\infty}\frac{B_{p,q}(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!}\quad(p,\ q\geq 0) \tag{1.17}$$

and

$$\Phi_{p,q}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_{p,q}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\quad(p,\ q\geq 0), \tag{1.18}$$

with their integral representations

$$\begin{aligned}F_{p,q}(a,b;c;z)&=\frac{1}{B(b,c-b)}\\ &\times\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\exp\Big(-\frac{p}{t}-\frac{q}{(1-t)}\Big)\,dt\end{aligned} \tag{1.19}$$

$$\left(p,\ q\geq 0;\ \Re(c)>\Re(b)>0;\ |\arg(1-z)|<\pi\right)$$

and

$$\begin{aligned}\Phi_{p,q}(b;c;z)&=\frac{1}{B(b,c-b)}\\ &\times\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp\Big(zt-\frac{p}{t}-\frac{q}{(1-t)}\Big)\,dt\end{aligned} \tag{1.20}$$

$$\left(p,\ q\geq 0;\ \Re(c)>\Re(b)>0\right).$$

Baleanu et al. [1] used $B_{p,q}(\alpha,\beta)$ to further extend the Appell's function F_1

$$F_1(a,b_1,b_2;c;x,y;p,q)=\sum_{n=0}^{\infty}\frac{B_{p,q}(a+m+n,c-a)}{B(a,c-a)}(b_1)_m(b_2)_n\,\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.21}$$

$$(\max\{|x|,\ |y|\}<1;\ p,\ q\geq 0)$$

and presented its integral representation

$$\begin{aligned}F_1(a,b_1,b_2;c;x,y;p,q)&=\frac{1}{B(a,c-a)}\\ \times\int_0^1 t^{a-1}(1-t)^{c-a-1}&(1-xt)^{-b_1}(1-yt)^{-b_2}\,\exp\Big(-\frac{p}{t}-\frac{q}{1-t}\Big)dt,\end{aligned} \tag{1.22}$$

$$(p,\ q\geq 0;\ \Re(c)>\Re(a)>0;\ |\arg(1-x)|<\pi,\ |\arg(1-y)|<\pi)\,.$$

Clearly $F_1(a,b_1,b_2;c;x,y;p,p)=F_1(a,b_1,b_2;c;x,y;p)$.

Mubeen et al. [7] presented a further extension of the extended beta function $B_{p,q}(\alpha,\beta)$

$$\begin{aligned}B^{\lambda,\rho}(\alpha,\beta;p,q)&=B^{\lambda,\rho}_{p,q}(\alpha,\beta)\\ &=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{p}{t}\Big]\,{}_1F_1\Big[\lambda;\rho;-\frac{q}{1-t}\Big]dt\end{aligned} \tag{1.23}$$

$$\big(\min\{\Re(p),\ \Re(q)\}>0;\ \min\{\Re(\alpha),\ \Re(\beta)\}>0;\ \lambda\in\mathbb{C},\ \rho\in\mathbb{C}\setminus\mathbb{Z}_0^-\big)\,.$$

Obviously $B^{\rho,\rho}_{p,q}(\alpha,\beta)=B_{p,q}(\alpha,\beta)$.

Here, we introduce further extensions of the (p,q)-extended functions $F_{p,q}(a,b;c;z)$ in (1.17) and $\Phi_{p,q}(b;c;z)$ in (1.18) as follows:

$$F^{\lambda,\rho}_{p,q}(a,b;c;z):=\sum_{n=0}^{\infty}\frac{B^{\lambda,\rho}_{p,q}(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!}\quad(p,\ q\geq 0;\ |z|<1) \tag{1.24}$$

and

$$\Phi^{\lambda,\rho}_{p,q}(b;c;z):=\sum_{n=0}^{\infty}\frac{B^{\lambda,\rho}_{p,q}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\quad(p,\ q\geq 0), \tag{1.25}$$

together with the following extended Appell function:

$$F_1\Big(a,b_1,b_2;c;x,y;p,q;\lambda,\rho):=\sum_{m,n=0}^{\infty}(b_1)_m(b_2)_n\frac{B^{\lambda,\rho}_{p,q}(a+m+n,c-a)}{B(a,c-a)}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.26}$$

$$(p,\ q\geq 0;\ \max\{|x|,\ |y|\}<1)\,.$$

Clearly, $F^{\rho;\rho}_{p,q}(a,b;c;z)=F_{p,q}(a,b;c;z)$, $\Phi^{\rho;\rho}_{p,q}(b;c;z)=\Phi_{p,q}(b;c;z)$, and $F_1\Big(a,b_1,b_2;c;x,y;p,q;\rho,\rho)=F_1\Big(a,b_1,b_2;c;x,y;p,q)$.

Then we aim to present integral representations for the extended functions (1.24), (1.25) and (1.26). Also we introduce a fractional differential operator involving the extended function in (1.24) and investigate some of its properties.

2. Integral representations

Here we present certain integral representations for the functions in (1.24), (1.25) and (1.26).

Theorem 2.1. *Each of the following integral representations holds.*

$$\begin{aligned} F_{p,q}^{\lambda,\rho}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 & t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \\ & \times {}_1F_1\left[\lambda;\rho;-\frac{p}{t}\right] {}_1F_1\left[\lambda;\rho;-\frac{q}{1-t}\right] dt \end{aligned} \tag{2.1}$$

$$\big(\min\{\Re(p),\, \Re(q)\} > 0;\ \Re(c) > \Re(b) > 0;$$
$$\lambda \in \mathbb{C},\, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ |\arg(1-z)| < \pi\big);$$

$$\begin{aligned} \Phi_{p,q}^{\lambda,\rho}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 & t^{b-1}(1-t)^{c-b-1}\, \mathrm{e}^{zt} \\ & \times {}_1F_1\left[\lambda;\rho;-\frac{p}{t}\right] {}_1F_1\left[\lambda;\rho;-\frac{q}{1-t}\right] dt \end{aligned} \tag{2.2}$$

$$\big(\min\{\Re(p),\, \Re(q)\} > 0;\ \Re(c) > \Re(b) > 0;\ \lambda \in \mathbb{C},\, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-\big);$$

$$\begin{aligned} F_1\Big(a,b_1,b_2;c;x,y;p,q;\lambda,\rho\Big) &= \frac{1}{B(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \\ \times (1-tx)^{-b_1}(1-ty)^{-b_2} & {}_1F_1\left[\lambda;\rho;-\frac{p}{t}\right] {}_1F_1\left[\lambda;\rho;-\frac{q}{1-t}\right] dt \end{aligned} \tag{2.3}$$

$$\big(\min\{\Re(p),\, \Re(q)\} > 0;\ \Re(c) > \Re(a) > 0;$$
$$\lambda \in \mathbb{C},\, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ |\arg(1-x)| < \pi,\, |\arg(1-y)| < \pi\big).$$

Proof. Using the definition (1.23) in (1.24) and interchanging the order of integral and summation, which is verified under the assumptions given in this theorem, we have

$$\begin{aligned} F_{p,q}^{\lambda,\rho}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 & t^{b-1}(1-t)^{c-b-1} \\ \times {}_1F_1\left[\lambda;\rho;-\frac{p}{t}\right] {}_1F_1\left[\lambda;\rho;-\frac{q}{1-t}\right] & \left\{\sum_{n=0}^{\infty} \frac{(a)_n (tz)^n}{n!}\right\} dt. \end{aligned} \tag{2.4}$$

Recalling the following generalized binomial theorem

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n = (1-z)^{-\alpha} \quad (|z|<1;\ \alpha \in \mathbb{C}) \tag{2.5}$$

in the summation in (2.4), we obtain the desired result (2.1).

A similar argument as in the proof of (2.1) will establish the results in (2.2) and (2.3). We omit the details. □

3. Fractional integral and derivative operators

In this section, we define further extension of extended Riemann-Liouville fractional derivative.

The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ of a function f is defined by (see, e.g., [6])

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)\,(x-t)^{\alpha-1}\,dt \quad (x>0), \tag{3.1}$$

where the right side exits. The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ of a function $f(x)$ is defined by

$$\begin{aligned} D_x^{\alpha} f(x) =& \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x f(t)\,(x-t)^{n-\alpha-1}\,dt \\ =& \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) \quad (n = [\Re(\alpha)]+1,\ x>0)\,. \end{aligned} \tag{3.2}$$

Özarslan and Özergin [8] extended the Riemann-Liouville integral and derivative of order α as follows:

$$I_x^{\alpha,p} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)\,(x-t)^{\alpha-1} \exp\Big(-\frac{px^2}{t(x-t)}\Big)\,dt \tag{3.3}$$

$$(\Re(\alpha)>0,\ \Re(p)>0,\ x>0)$$

and

$$\begin{aligned} D_x^{\alpha,p} f(x) =& \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x f(t)\,(x-t)^{n-\alpha-1} \exp\Big(-\frac{px^2}{t(x-t)}\Big)\,dt \\ =& \frac{d^n}{dx^n} I_x^{n-\alpha,p} f(x) \quad (\Re(\alpha) \geq 0,\ n=[\Re(\alpha)]+1,\ \Re(p)>0,\ x>0)\,. \end{aligned} \tag{3.4}$$

Baleanu et al. [1] extended the fractional integral and derivative (3.3) and (3.4) as follows:

$$I_x^{\alpha}\{f(x);p,q\} = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)\,(x-t)^{\alpha-1} \exp\Big(-\frac{px}{t} - \frac{qx}{x-t}\Big)\,dt \tag{3.5}$$

$$(\Re(\alpha) > 0,\, \min\{\Re(p),\, \Re(q)\} > 0,\, x > 0)$$

and

$$\begin{aligned} D_x^{\alpha}\{f(x);p,q\} =& \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x f(t)\,(x-t)^{n-\alpha-1}\exp\left(-\frac{px}{t}-\frac{qx}{x-t}\right)dt \\ =& \frac{d^n}{dx^n}\, I_x^{n-\alpha}\{f(x);p,q\} \\ & (\Re(\alpha) \geq 0,\, n = [\Re(\alpha)]+1,\, \min\{\Re(p),\, \Re(q)\} > 0,\, x > 0)\,. \end{aligned} \tag{3.6}$$

Here we introduce further extensions of the extended fractional integral and derivative (3.5) and (3.6) defined by

$$\begin{aligned} I_x^{\alpha}\{f(x);p,q;\lambda,\rho\} =& \frac{1}{\Gamma(\alpha)}\int_0^x f(t)\,(x-t)^{\alpha-1} \\ & \times {}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]\,{}_1F_1\left[\lambda;\rho;-\frac{qx}{x-t}\right]dt \end{aligned} \tag{3.7}$$

$$\left(\Re(\alpha) > 0,\, \min\{\Re(p),\, \Re(q)\} > 0,\, x > 0,\, \lambda \in \mathbb{C},\, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-\right)$$

and

$$\begin{aligned} D_x^{\alpha}\{f(x);p,q;\lambda,\rho\} =& \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x f(t)\,(x-t)^{n-\alpha-1} \\ & \times {}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]\,{}_1F_1\left[\lambda;\rho;-\frac{qx}{x-t}\right]dt \\ =& \frac{d^n}{dx^n}\, I_x^{n-\alpha}\{f(x);p,q;\lambda,\rho\} \end{aligned} \tag{3.8}$$

$$\left(\Re(\alpha) \geq 0,\, n = [\Re(\alpha)]+1,\, \min\{\Re(p),\, \Re(q)\} > 0,\, x > 0,\, \lambda \in \mathbb{C},\, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Obviously, the extended fractional integral and derivative (3.7) and (3.8) when $\lambda = \rho$ reduce, respectively, to (3.5) and (3.6).

We present some formulas involving the operators (3.7) and (3.8).

Theorem 3.1. *The following formula holds true.*

$$I_x^{\alpha}\{x^{\eta};p,q;\lambda,\rho\} = \frac{B_{p,q}^{\lambda,\rho}(\eta+1,\alpha)}{\Gamma(\alpha)}\, x^{\alpha+\eta}. \tag{3.9}$$

$$\left(\Re(\alpha) > 0,\, \Re(\eta) > -1,\, \min\{\Re(p),\, \Re(q)\} > 0,\, x \in \mathbb{R}^+,\, \lambda \in \mathbb{C},\, \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-\right)$$

Proof. We set $f(t) = t^{\eta}$ in (3.7). Then, letting $t = xu$ in the resulting identity and using (1.23), we obtain the desired result. □

Theorem 3.2. *Let* $f(t) = \sum_{n=0}^{\infty} a_n\, t^n$ $(|t| < \delta)$ *for some* $\delta \in \mathbb{R}^+$*. Then, for each* $0 < x < \delta$*, the following formula holds true.*

$$\begin{aligned} I_x^{\alpha}\{f(x); p, q; \lambda, \rho\} &= \sum_{n=0}^{\infty} a_n\, I_x^{\alpha}\{x^n; p, q; \lambda, \rho\} \\ &= \frac{x^{\alpha}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} a_n\, B_{p,q}^{\lambda,\rho}(n+1, \alpha)\, x^n \end{aligned} \tag{3.10}$$

$$\left(\Re(\alpha) > 0,\ \min\{\Re(p),\ \Re(q)\} > 0,\ \lambda \in \mathbb{C},\ \rho \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Proof. We set the power series of $f(t)$ in $I_x^{\alpha}\{f(x); p, q; \lambda, \rho\}$. Then we find that the integrand converges uniformly under the given conditions. This verifies term-by-term integration. Finally, using (3.9), we get the desired result. □

Theorem 3.3. *The following formula holds true.*

$$I_x^{\mu-\eta}\{x^{\eta-1}(1-x)^{-\beta}; p, q; \lambda, \rho\} = \frac{\Gamma(\eta)}{\Gamma(\mu)} x^{\mu-1}\, F_{p,q}^{\lambda,\rho}\Big(\beta; \eta; \mu; x\Big) \tag{3.11}$$

$$\left(x \in \mathbb{R}^+,\ \Re(\mu) > \Re(\eta) > 0,\ \min\{\Re(p),\ \Re(q)\} > 0,\ \lambda \in \mathbb{C},\ \rho \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Proof. Using (3.7) and letting $t = xu$, we have

$$\begin{aligned} I_x^{\mu-\eta}\{x^{\eta-1}(1-x)^{-\beta}; p, q; \lambda, \rho\} &= \frac{x^{\mu-1}}{\Gamma(\eta-\mu)} \int_0^1 u^{\eta-1}(1-ux)^{-\beta}(1-u)^{\mu-\eta-1} \\ &\quad \times {}_1F_1\Big[\lambda; \rho; -\frac{p}{u}\Big]\, {}_1F_1\Big[\lambda; \rho; -\frac{q}{1-u}\Big]\, du, \end{aligned}$$

which, in view of (2.1), leads to the right side of (3.11). □

Theorem 3.4. *Let* $x \in \mathbb{R}^+$*,* $|ax| < 1$ *and* $|bx| < 1$*. Then*

$$\begin{aligned} & I_x^{\mu-\eta}\{x^{\eta-1}(1-ax)^{-\alpha}(1-bx)^{-\beta}; p, q; \lambda, \rho\} \\ &\quad = \frac{\Gamma(\eta)}{\Gamma(\mu)} x^{\mu-1}\, F_1\Big(\eta, \alpha, \beta; \mu; ax, bx; p, q; \lambda, \rho\Big) \end{aligned} \tag{3.12}$$

$$\left(\Re(\mu) > \Re(\eta) > 0,\ \min\{\Re(p),\ \Re(q)\} > 0,\ \lambda \in \mathbb{C},\ \rho \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Proof. Using (2.5), we have

$$(1-ax)^{-\alpha}(1-bx)^{-\beta} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (\alpha)_m (\beta)_n \frac{(ax)^m}{m!}\, \frac{(bx)^n}{n!}. \tag{3.13}$$

Applying (3.10) to (3.13), we get

$$\begin{aligned} & I_x^{\mu-\eta}\{x^{\eta-1}(1-ax)^{-\alpha}(1-bx)^{-\beta};p,q;\lambda,\rho\} \\ & = \frac{x^{\mu-1}}{\Gamma(\mu-\eta)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\beta)_n\, B_{p,q}^{\lambda,\rho}(\eta+m+n,\mu-\eta)\, \frac{(ax)^m}{m!}\frac{(bx)^n}{n!}, \end{aligned}$$

which, in terms of (1.26), leads to the right side of (3.12). □

Setting $b = 0$ in the result in Theorem 3.4 with the help of (2.1) and (2.3), we obtain a useful identity, which is asserted by the following corollary.

Corollary 3.5. *Let $x \in \mathbb{R}^+$ and $|ax| < 1$. Then*

$$I_x^{\mu-\eta}\{x^{\eta-1}(1-ax)^{-\alpha};p,q;\lambda,\rho\} = \frac{\Gamma(\eta)}{\Gamma(\mu)} x^{\mu-1} F_{p,q}^{\lambda,\rho}\Big(\alpha;\eta;\mu;ax\Big) \tag{3.14}$$

$$\big(\Re(\mu) > \Re(\eta) > 0,\ \min\{\Re(p),\ \Re(q)\} > 0,\ \lambda \in \mathbb{C},\ \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-\big).$$

Recall the extended gamma function (see [9, Eq. (3)])

$$\Gamma_\nu^{(\alpha,\beta)}(x) := \int_0^\infty t^{x-1}\, {}_1F_1\Big[\alpha;\beta;-t-\frac{\nu}{t}\Big]\, dt \tag{3.15}$$

$$(\min\{\Re(\alpha),\ \Re(\beta)\} > 0,\ \Re(\nu) > 0,\ \Re(x) > 0).$$

Also let $\Gamma^{(\alpha,\beta)}(x) := \Gamma_0^{(\alpha,\beta)}(x)$.

Theorem 3.6. *The following Mellin transform formula holds.*

$$\mathfrak{M}\{I_x^\alpha\{x^\eta;p,q;\lambda,\rho\};p\to r,\ q\to s\} = \frac{x^{\alpha+\eta}}{\Gamma(\alpha)}\, B(r+\eta+1,s+\alpha)\,\Gamma^{(\lambda,\rho)}(r)\,\Gamma^{(\lambda,\rho)}(s) \tag{3.16}$$

$$\begin{aligned} & \big(\Re(\alpha) > 0,\ x > 0,\ \min\{\Re(p),\ \Re(q)\} > 0, \\ & \min\{\Re(\lambda),\ \Re(\rho)\} > 0,\ \min\{\Re(r),\ \Re(s)\} > 0\big). \end{aligned}$$

Proof. Let $\mathcal{L}_1$ be the left side of (3.16). Taking the Mellin transform on (3.7), we obtain

$$\begin{aligned} \mathcal{L}_1 = \frac{1}{\Gamma(\alpha)} \int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\Big\{\int_0^x t^\eta(x-t)^{\alpha-1} \\ \times\, {}_1F_1\Big[\lambda;\rho;-\frac{px}{t}\Big]\, {}_1F_1\Big[\lambda;\rho;-\frac{qx}{(x-t)}\Big]dt\Big\}\, dp\, dq. \end{aligned} \tag{3.17}$$

Setting $t = ux$ in (3.17) and interchanging the order of integrals, which is verified under the conditions here, we get

$$\mathcal{L}_1 = \frac{x^{\alpha+\eta}}{\Gamma(\alpha)} \int_0^1 u^{\eta}\,(1-u)^{\alpha-1} \Big(\int_0^{\infty} p^{r-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{p}{u}\Big]dp\Big) \times \Big(\int_0^{\infty} q^{s-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{q}{1-u}\Big]dq\Big)\,du. \tag{3.18}$$

Letting $p = ua$ and $q = (1-u)b$ in the integrals in the first and second parentheses of (3.18), respectively, and using (3.15), we have the desired result. □

Theorem 3.7. *The following Mellin transform formula holds.*

$$\mathfrak{M}\left\{I_x^{\alpha}\{(1-x)^{\eta};p,q;\lambda,\rho\};p\to r,\,q\to s\right\} = \frac{x^{\alpha}\,r!}{\Gamma(\alpha)}\,\Gamma(r+s+\alpha+1) \times \Gamma(s+\alpha)\,\Gamma^{(\lambda,\rho)}(r)\,\Gamma^{(\lambda,\rho)}(s)\,{}_2F_1\Big(-\eta, r+1; r+s+\alpha+1; x\Big) \tag{3.19}$$

$$\Big(\Re(\alpha)>0,\,\Re(\eta)>-1,\,x>0,\,\min\{\Re(p),\,\Re(q)\}>0,\quad \min\{\Re(\lambda),\,\Re(\rho)\}>0,\,\min\{\Re(r),\,\Re(s)\}>0\Big).$$

Proof. Let $\mathcal{L}_2$ be the left side of (3.19). Expanding $(1-x)^{\eta} = \sum_{n=0}^{\infty}(-\eta)_n\,\frac{x^n}{n!}$ and taking term-by-term integration, we have

$$\mathcal{L}_2 = \sum_{n=0}^{\infty} \frac{(-\eta)_n}{n!}\,\mathfrak{M}\left\{I_x^{\alpha}\{x^n;p,q;\lambda,\rho\};p\to r,\,q\to s\right\}. \tag{3.20}$$

Applying the result in Theorem 3.6 to the right-sided Mellin transforms and simplifying, we arrive at the right side of (3.19). □

4. Generating relations

Here we establish two generating relations for the extended (p,q)-functions (1.24).

Theorem 4.1. *The following generating relation holds.*

$$(1-t)^{-\alpha}\,F_{p,q}^{\lambda,\rho}\Big(\alpha,\beta;\gamma;\frac{x}{1-t}\Big) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\,F_{p,q}^{\lambda,\rho}\Big(\alpha+n,\beta;\gamma;x\Big)t^n \tag{4.1}$$

$$\Big(|t|<|1-x|,\,\Re(\gamma)>\Re(\beta)>0,\,\min\{\Re(p),\,\Re(q)\}>0,\,\lambda\in\mathbb{C},\,\rho\in\mathbb{C}\setminus\mathbb{Z}_0^-\Big).$$

Proof. Consider the following function $f_\alpha(t,x) = (1-x-t)^{-\alpha}$ in two ways. Firstly we use (2.5) to find

$$\begin{aligned} f_\alpha(t,x) &= (1-x)^{-\alpha}\left(1-\frac{t}{1-x}\right)^{-\alpha} \\ &= (1-x)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\,\frac{t^n}{(1-x)^n} \quad (|t|<|1-x|). \end{aligned} \tag{4.2}$$

Multiplying both sides of (4.2) by $x^{\beta-1}$ and taking the operator $I_x^{\gamma-\beta}(\cdot;p,q;\lambda,\rho)$ on both sides of the resulting identity, term-by-term on its right side, and using (3.11), we get

$$I_x^{\gamma-\beta}\Big(x^{\beta-1}f_\alpha(t,x);p,q;\lambda,\rho\Big) = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\,x^{\gamma-1}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\,F_{p,q}^{\lambda,\rho}\Big(\alpha+n,\beta;\gamma;x\Big)t^n. \tag{4.3}$$

Secondly we consider

$$f_\alpha(t,x) = (1-t)^{-\alpha}\left(1-\frac{x}{1-t}\right)^{-\alpha}. \tag{4.4}$$

Multiplying both sides of (4.4) by $x^{\beta-1}$ and taking the operator $I_x^{\gamma-\beta}(\cdot;p,q;\lambda,\rho)$ on both sides of the resulting identity, and using (3.14), we obtain

$$I_x^{\gamma-\beta}\Big(x^{\beta-1}f_\alpha(t,x);p,q;\lambda,\rho\Big) = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\,x^{\gamma-1}\,(1-t)^{-\alpha}\,F_{p,q}^{\lambda,\rho}\Big(\alpha,\beta;\gamma;\frac{x}{1-t}\Big). \tag{4.5}$$

Equating (4.3) and (4.5), we get the desired result (4.1). □

Theorem 4.2. *The following generating relation holds.*

$$(1-t)^{-\beta}F_1\Big(\alpha,\delta,\beta;\gamma;x,\frac{xt}{t-1};p,q;\lambda,\rho\Big) = \sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\,F_{p,q}^{\lambda,\rho}\Big(\delta-n;\beta;\gamma;x\Big)\,t^n \tag{4.6}$$

$$\Big(|xt|<|t-1|,\ \Re(\gamma)>\Re(\alpha)>0,\ \min\{\Re(p),\ \Re(q)\}>0,\ \lambda\in\mathbb{C},\ \rho\in\mathbb{C}\setminus\mathbb{Z}_0^-\Big).$$

Proof. Consider the function $g_\beta(x,t) = (1-t+xt)^{-\beta}$. Similarly as in the proof of Theorem 4.1, we consider $g_\beta(x,t)$ in two ways to find

$$(1-t)^{-\beta}\left(1-\frac{xt}{t-1}\right)^{-\beta} = \sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}\,(1-x)^n\,t^n. \tag{4.7}$$

Multiplying both sides of (4.7) by $x^{\alpha-1}(1-x)^{-\delta}$ and taking the operator $I_x^{\gamma-\alpha}(\cdot;p,q;\lambda,\rho)$ on both sides of the resulting identity, and using (3.12) and

(3.11), respectively, on the left side and right side of the final resulting identity, we obtain the desired result (4.6). □

5. Concluding remarks

The results presented here are presumably new and potentially useful. They, being very general, can be specialized to yield some known results (see, e.g., [1, 5]) as well as new ones.

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