Nonlinear Functional Analysis and Applications
Vol. 24, No. 1 (2019), pp. 15-28
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

SYSTEM OF HOMOGENEOUS DIFFERENTIAL EQUATION AT INFINITY

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Abstract. The main purpose of this paper is to study about the existence and uniqueness of a solution of a system of differential equation obtained from a new class of second order differential equation at infinity. To solve the problem, first we develop a system of differential equation and then obtain the general solution of the same with some conditions.

1. Introduction

System of differential equations are used to solve various types of problems arises in applied mathematics, physical sciences, finance and engineering branches. The researchers have developed many interesting results on this field of work. For our reference we recall the works of the researchers, such as Rubinstein [10], Kreyszig [5], Agarwal and O'Regan [1], Boyce and DiPrima [2], to name only few. For our need, we recall some known definitions and results.

[0]Received March 1, 2018. Revised October 24, 2018.
[0]2010 Mathematics Subject Classification: 34A05.
[0]Keywords: Differential equation of second order, system of differential equation.
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Let the system of differential equations be

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned} \tag{1.1}$$

The system of differentials can be written as a matrix form

$$Y' = AY, \tag{1.2}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^T$ and $Y' = \begin{pmatrix} y_1' & y_2' \end{pmatrix}^T$ and A^T denotes the transpose of a matrix A and the prime ($'$) denotes the first derivative with respect to t. The system of differential equations (1.1) has a unique tangent direction

$$\frac{dy_2}{dy_1} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$

of the trajectory passing through point $P = P(y_1, y_2)$. Since the tangent $\frac{dy_2}{dy_1}$ is undetermined at point $P_0 = P(0,0)$, so is a critical point for the system of differential equations (1.1). Let X_1 and X_2 be two linearly independent eigen vectors correspond to eigenvalues λ_1 and λ_2 of the coefficient matrix A respectively. According to the nature of the eigenvalues the general solution of the system of differentials can be written as

$$Y = \begin{cases} C_1X_1e^{\lambda_1 t} + C_2X_2e^{\lambda_2 t}, & \text{for } \lambda_1 \neq \lambda_2; \\ C_1X_1e^{\lambda_1 t} + C_2X_2e^{\lambda_2 t}, & \text{for } \lambda_1 = \lambda_2,; \\ C_1X_1e^{\lambda_1 t} + C_2X_2e^{\lambda_2 t}, & \text{for } \lambda_1 = \lambda_2,\ (A - \lambda I)\alpha = X_1 \\ & \text{and } X_2 = tX_1 + \alpha; \\ C_1X_1e^{\lambda_1 t} + C_2X_2e^{\lambda_2 t}, & \text{for } \lambda_1 = a + bi, \quad \lambda_2 = a - bi. \end{cases} \tag{1.3}$$

2. Second order differential equations at infinity

In 2015, Das [3] has studied a new class of second order differential equation at point of infinity. To solve the problem, Das [3] has generated the auxiliary equation via a pre-auxiliary equation and obtained the general solution of it. He has also expressed the higher order of the differential equation in a matrix form and expressed the problem with a change in variable. In this section, we consider a generalized homogeneous differential equation of second order at point of infinity. Consider the second order initial value problem having generalized homogeneous logarithmic differential equation of the form

$$e^{-2x}y'' + \phi(e^{-x}, e^{-2x})y' + by = 0 \tag{2.1}$$

with initial conditions

$$y(x_0) = z_0 \quad \text{and} \quad y'(x_0) = z_1$$

where b is real and the primes are the order of derivatives of the entire function y with respect to x and the function $\phi(e^{-x}, e^{-2x})$ is linear combination of e^{-x} and e^{-2x} given by

$$\phi(e^{-x}, e^{-2x}) = a_1 e^{-x} + a_2 e^{-2x}$$

with a_2 independent of a_1 where a_1 and a_2 are real. Now

$$\lim_{x\to 0} \phi(e^{-x}, e^{-2x}) = a_1 + a_2$$

exists for finite real values of a_1 and a_2. Considering $y = y_1$ and $y' = y_1' = y_2$, we have $y'' = y_2'$. Then (2.1) can be written as a system of differential equations given by

$$\begin{aligned} y_1' &= 0y_1 + y_2 \\ y_2' &= -be^{2x} y_1 + (-a_1 e^x - a_2) y_2 \end{aligned} \tag{2.2}$$

using matrix the system can be represented as

$$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -be^{2x} & -a_1 e^x - a_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A_x Y.$$

2.1. **System of homogeneous differential equation at infinity.** In this section, we solve (2.1) by reducing to a system of differential equations.

Theorem 2.1. *The problem* (2.1) *is solvable under the following cases:*

(i) $b = a_1 a_2$ *with* $a_1 \neq a_2$,
(ii) $b = a_1^2$ *with* $a_1 = a_2$,
(iii) $b \neq a_1 a_2$ *with* $(a_1 + a_2)^2 - 4b < 0$.

Proof. Since $\lim_{x\to 0} e^x = 1$ and $\lim_{x\to 0} e^{2x} = 1$, by Das [3], the pre-characteristic equation for the eigenvalue λ of A_x is

$$\lim_{x\to 0} \det(A_x - \lambda I) = 0 \Rightarrow \lambda^2 + (a_1 + a_2)\lambda + b = 0.$$

For the solution of the above equations, three cases are considered.

(i) $b = a_1 a_2$ with $a_1 \neq a_2$:
For $b = a_1 a_2$, with $a_1 \neq a_2$, the eigenvalues of A are $\lambda = -a_1$, $\lambda = -a_2$ and their corresponding eigenvectors are

$$X_1 = \begin{pmatrix} 1 & -a_1 \end{pmatrix}^T \text{ and } X_2 = \begin{pmatrix} 1 & -a_2 \end{pmatrix}^T$$

respectively. The general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \exp(-a_1 e^x) \begin{pmatrix} 1 \\ -a_1 \end{pmatrix} + C_2 \exp(-a_2 e^x) \begin{pmatrix} 1 \\ -a_2 \end{pmatrix},$$

that is,

$$\begin{aligned} y_1 &= C_1 \exp(-a_1 e^x) + C_2 \exp(-a_2 e^x) \\ y_2 &= -C_1 a_1 \exp(-a_1 e^x) - C_2 a_2 \exp(-a_2 e^x). \end{aligned}$$

(ii) $b = a_1^2$ with $a_1 = a_2$:
For $b = a_1 a_2$ with $a_1 = a_2$, the eigenvalue $\lambda = -a_1$ is a double root and the corresponding eigenvectors are $X_1 = \begin{pmatrix} 1 & -a_1 \end{pmatrix}^T$ and $X_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T + e^x \begin{pmatrix} 1 & -a_1 \end{pmatrix}^T$. The general solution is

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= C_1 \exp(-a_1 e^x) \begin{pmatrix} 1 \\ -a_1 \end{pmatrix} \\ &\quad + C_2 \exp(-a_1 e^x) \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^x \begin{pmatrix} 1 \\ -a_1 \end{pmatrix} \right], \end{aligned}$$

that is,

$$\begin{aligned} y_1 &= [C_1 + C_2 e^x] \exp(-a_1 e^x) \\ y_2 &= -[C_1 a_1 - C_2 (1 - a_1 e^x)] \exp(-a_1 e^x). \end{aligned}$$

(iii) $b \neq a_1 a_2$ with $(a_1 + a_2)^2 - 4b < 0$:
For $b \neq a_1 a_2$, the eigenvalues obtained are $b = b_1 \pm i b_2$ where b_1 and b_2 are real numbers and corresponding eigenvectors are $X_1 = \begin{pmatrix} 1 & -b_1 - ib_2 \end{pmatrix}^T$ for $\lambda = -(b_1 + ib_2)$ and $X_2 = \begin{pmatrix} 1 & -b_1 + ib_2 \end{pmatrix}^T$ for $\lambda = -(b_1 - ib_2)$. The general solution is

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= C_1 \exp([-b_1 - ib_2] e^x) \begin{pmatrix} 1 \\ -b_1 - ib_2 \end{pmatrix} \\ &\quad + C_2 \exp([-b_1 + ib_2] e^x) \begin{pmatrix} 1 \\ -b_1 + ib_2 \end{pmatrix}, \end{aligned}$$

implying

$$\begin{aligned} y_1 &= C_1 \exp([-b_1 - ib_2] e^x) + C_2 \exp([-b_1 + ib_2] e^x) \\ y_2 &= C_1 [-b_1 - ib_2] \exp([-b_1 - ib_2] e^x) \\ &\quad + C_2 [-b_1 + ib_2] \exp([-b_1 + ib_2] e^x), \end{aligned}$$

that is,

$$y_1 = A \exp(-b_1 e^x) \cos(b_2 e^x),$$

and

$$y_2 = \exp(-b_1 e^x)[A_1 \cos(b_2 e^x) + B_1 \sin(b_2 e^x)].$$

This completes the proof of the theorem. □

3. Existence and uniqueness theorem

To make this work self contained, we recall the existence and uniqueness theorem as follows:

Theorem 3.1. [5] *Let $f_1(y_1, y_2)$, $f_2(y_1, y_2)$ be continuous functions having continuous partial derivatives $\frac{\partial f_1}{\partial y_1}$, $\frac{\partial f_2}{\partial y_2}$ in some domain* R *of xy_1y_2-space containing the point (x, k_1, k_2). Then the system has a solution on some interval $x_0 - \alpha < x < x_0 + \alpha$ satisfying the initial condition, and the solution is unique for $\alpha > 0$.*

Theorem 3.2. *The system of the differential equations* 2.2 *is stable and attractive if the eigenvalues of the coefficient matrix which are the solutions of the pre-characteristic equation $\lim_{x\to 0} \det(A_x - \lambda I) = 0$, where*

$$A = \lim_{x\to 0} A_x = \begin{pmatrix} 0 & 1 \\ -b & -a_1 - a_2 \end{pmatrix}$$

are satisfying the conditions of critical point $P_0 \in B_\delta = B(P_0, \delta)$ the neighborhood of radius $\delta > 0$ with center P_0 and every trajectory that has a point in B_δ approaches P_0 as $x \to 0$. The critical point P_0 of the system of differential equation is

(a) *stable and attractive if $a_1 + a_2 > 0$ and $b > 0$,*
(b) *stable if $a_1 + a_2 \geq 0$ and $b > 0$,*
(c) *unstable if $a_1 + a_2 < 0$ or $b < 0$.*

Proof. The pre-characteristic equation of the matrix

$$A = \lim_{x\to 0} A_x = \begin{pmatrix} 0 & 1 \\ -b & -a_1 - a_2 \end{pmatrix}$$

is satisfying

$$\begin{vmatrix} -\lambda & 1 \\ -b & -a_1 - a_2 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + (a_1 + a_2)\lambda + b = 0.$$

Consider $p = -a_1 - a_2$, $q = b$ and $\Delta = p^2 - 4q$. According to Kreyszig [5], the criterion for critical points are given as follows:

(i) the critical point P_0 of the system of differential equation is a
   (a) *node* if $q > 0$ and $\Delta \geq 0$,
   (b) *saddle point* if $q < 0$,
   (c) *center* if $p = 0$ and $q > 0$,
   (d) *spiral point* if $p \neq 0$ and $\Delta < 0$,
(ii) the stability criterion for critical point P_0 of the system of differential equation is a:
   (a) *stable and attractive* if $p < 0$ and $q > 0$,

(b) *stable* if $p \leq 0$ and $q > 0$,
(c) *unstable* if $p > 0$ or $q < 0$.

From the system of the differential equations 2.2, we have the values of $p = -a_1 - a_2$, $q = b$ and $\Delta = p^2 - 4q$. Thus the critical point $P_0 = P(0,0)$ of the system of differential equation is a

(a) *node* if $b > 0$ and $(a_1 + a_2)^2 \geq 4b$,
(b) *saddle point* if $b < 0$,
(c) *center* if $a_1 + a_2 = 0$ and $b > 0$,
(d) *spiral point* if $a_1 + a_2 \neq 0$ and $(a_1 + a_2)^2 < 4b$.

The critical point P_0 of the system of differential equation is

(a) *stable and attractive* if $a_1 + a_2 > 0$ and $b > 0$,
(b) *stable* if $a_1 + a_2 \geq 0$ and $b > 0$,
(c) *unstable* if $a_1 + a_2 < 0$ or $b < 0$.

This completes the proof of the theorem. □

In this portion, two examples are given for the solution of the existence theorems.

Example 3.3. Consider the IVP with second order differential equation at infinity

$$e^{-2x}y'' + \left(-3e^{-x} + 2e^{-2x}\right) y' - 6y = 0 \tag{3.1}$$

with given initial conditions $y(0) = 0$ and $y'(0) = 1$. Here $\phi(e^{-x}, e^{-2x})$ is the function of linear combination of e^{-x} and e^{-2x} given by

$$\phi(e^{-x}, e^{-2x}) = a_1 e^{-x} + a_2 e^{-2x}$$

where $a_1 = -3$, $a_2 = 2$ and $b = -6$. We have $\lim\limits_{x \to 0} \phi(e^{-x}, e^{-2x}) = a_1 + a_2 = -1$.

The equation (3.1) is modeled as

$$\begin{aligned} y_1' &= 0 \cdot y_1 + y_2 \\ y_2' &= -be^{2x} y_1 - (a_1 e^x + a_2)\, y_2 \end{aligned} \tag{3.2}$$

using matrix the system can be represented as

$$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6e^{2x} & 3e^x - 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A_x Y.$$

The eigenvalues obtained from the pre-characteristic equation

$$\lim_{x \to 0} \det(A_x - \lambda I) = 0$$

are $\lambda_1 = 3$ and $\lambda_2 = -2$ whose corresponding eigenvectors are $X_1 = \begin{pmatrix} 1 & 3 \end{pmatrix}^T$ and $X_2 = \begin{pmatrix} 1 & -2 \end{pmatrix}^T$ respectively. The general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \exp(3e^x) \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \exp(-2e^x) \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

that is,

$$\begin{aligned} y_1 &= C_1 \exp(3e^x) + C_2 \exp(-2e^x) \\ y_2 &= 3C_1 \exp(3e^x) - 2C_2 \exp(-2e^x). \end{aligned}$$

Using the initial condition we have the values of the constants are $C_1 = \frac{1}{5}e^{-3}$ and $C_2 = \frac{-1}{5}e^2$. Now the particular solution of the system of differential equation are

$$y_1 = \frac{1}{5}e^{-3} \exp(3e^x) - \frac{1}{5}e^2 \exp(-2e^x) \tag{3.3}$$

$$y_2 = \frac{3}{5}e^{-3} \exp(3e^x) + \frac{2}{5}e^2 \exp(-2e^x). \tag{3.4}$$

Though $p = -4 < 0$, $q = 4 > 0$ and $\Delta = p^2 - 4q = 0$, the system is unstable as $a_1 + a_2 = -1 < 0$ or $b = -6 < 0$.

Example 3.4. Consider the IVP of second order ODE at point of infinity

$$e^{-2x}y'' + \left(2e^{-x} + 2e^{-2x}\right) y' + 4y = 0 \tag{3.5}$$

with given initial conditions

$$y(0) = 0, \quad \text{and} \quad y'(0) = 1.$$

Here $\phi(e^{-x}, e^{-2x})$ is the function of linear combination of e^{-x} and e^{-2x} given by

$$\phi(e^{-x}, e^{-2x}) = 2e^{-x} + 2e^{-2x}.$$

The given differential equation (3.5) is modeled as

$$\begin{aligned} y_1' &= 0 \cdot y_1 + y_2 \\ y_2' &= -4e^{2x}y_1 - (2e^x + 2)\, y_2 \end{aligned} \tag{3.6}$$

using matrix the system can be represented as

$$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4e^{2x} & -2e^x - 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A_x Y.$$

The eigenvalues obtained from the pre-characteristic equation

$$\lim_{x \to 0} \det(A_x - \lambda I) = 0$$

are $\lambda_1 = -2$ and $\lambda_2 = -2$ whose corresponding eigenvectors are

$$\begin{aligned} X_1 &= \begin{pmatrix} 1 & -2 \end{pmatrix}^T, \\ X_2 &= \begin{pmatrix} 1 - t & 2t - 1 \end{pmatrix}^T \end{aligned}$$

where $X_2 = tX_1 + \beta$ satisfying the condition $(A - \lambda I)\beta = X_1$. The general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left[C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1-t \\ 2t-1 \end{pmatrix}\right] \exp(-2e^x).$$

that is,

$$\begin{aligned} y_1 &= [C_1 + (1-t)C_2] \exp(-2e^x) \\ y_2 &= [-2C_1 + (2t-1)C_2] \exp(-2e^x). \end{aligned}$$

Using the initial condition we obtain $C_1 = 0$ and $C_2 = e^2$. Hence the particular solution of the system of differential equation are

$$\begin{aligned} y_1 &= e^2(1-t) \exp(-2e^x) \\ y_2 &= e^2(2t-1) \exp(-2e^x). \end{aligned}$$

Since $p = -4 < 0$, $q = 4 > 0$ and $\Delta = p^2 - 4q = 0$, the critical point $P_0 = P(0,0)$ is a node and the system of differential equation is stable and attractive as $a_1 + a_2 = 4 > 0$ and $b = 6 > 0$.

4. Homogeneous logarithmic differential equations of higher order

In this section we consider n^{th} order homogeneous logarithmic differential equation.

Theorem 4.1. *The n^{th} order homogeneous logarithmic differential equation*

$$\begin{aligned} & e^{-nx}y^{(n)} + \phi_1(e^{-(n-1)x}, e^{-nx})y^{(n-1)} \\ & +\phi_2(e^{-(n-2)x}, e^{-(n-1)x}, e^{-nx})y^{(n-2)} \\ & +\phi_3(e^{-(n-3)x}, e^{-(n-2)x}, e^{-(n-1)x}, e^{-nx})y^{(n-3)} \\ & +\cdots + \phi_{n-1}(e^{-x}, e^{-2x}, \cdots, e^{-nx})y' + by = 0 \end{aligned} \tag{4.1}$$

with initial conditions

$$y(x_0) = z_0,\ y'(x_0) = z_1,\ y''(x_0) = z_2, \cdots, y^{(n-1)}(x_0) = z_{n-1},$$

where b is real and the primes are the order of derivatives of the entire function y with respect to x and the function $\phi_k\left(e^{-(n-k)x}, \ldots, e^{-nx}\right)$ with $k = 1, 2, \cdots, n-1$ is the linear combination of negative exponential function given by

$$\phi_k(e^{-(n-k)x}, \ldots, e^{-nx}) = \sum_{j=1}^{k+1} a_j e^{-(n-k+j-1)x}$$

with each a_k *independent of* a_j *for* $k \neq j$ *where*

$$\lim_{x\to 0} \phi_k(e^{-(n-k)x}, \dots, e^{-nx}) = \sum_{j=1}^{k+1} a_j$$

exists for finite values of a_k*'s is solvable.*

Proof. The equation (4.1) can be rewritten as a system of differential equations given by

$$\begin{aligned} y_1' &= 0\cdot y_1 + \ 1\cdot y_2 + 0\cdot y_3 + \cdots + 0\cdot y_n \\ y_2' &= 0\cdot y_1 + 0\cdot y_2 + 1\cdot y_3 + \cdots + 0\cdot y_n \\ &\vdots \\ y_n' &= -be^{nx}y_1 - e^{nx}\phi_{n-1}y_2 - e^{nx}\phi_{n-2}y_3 \\ &\quad - \cdots - e^{nx}\phi_1 y_n, \end{aligned} \tag{4.2}$$

using matrix the system can be represented as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -b & -\phi_{n-1} & -\phi_{n-2} & \phi_{n-3} & \cdots & -\phi_2 & -\phi_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix},$$

that is,,

$$Y' = A_x Y.$$

The transpose of the coefficient matrix A_x

$$A_x^T = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & -b \\ 1 & 0 & 0 & 0 & \dots & 0 & -\phi_{n-1} \\ 0 & 1 & 0 & 0 & \dots & 0 & -\phi_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & -\phi_2 \\ 0 & 0 & 0 & \dots & 0 & 1 & -\phi_1 \end{pmatrix}$$

is clearly a companion matrix. So the pre-characteristic equation and the minimal polynomial equation are same, i.e., $\lim_{x\to 0} \det(A_x - \lambda I) = 0$, implying

$$\begin{aligned} P(\lambda) &= \lambda^n + \lim_{x\to 0}\phi_1(x)\lambda^{n-1} + \lim_{x\to 0}\phi_2(x)\lambda^{n-2} + \lim_{x\to 0}\phi_3(x)\lambda^{n-3} \\ &\quad + \cdots + \lim_{x\to 0}\phi_{n-1}(x)\lambda + b = 0, \end{aligned}$$

that is,

$$P(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + c_{n-1}\lambda + b = 0$$

is characteristic equation, where the spectrals $\lambda_1, \lambda_2, \cdots, \lambda_n$ are can be obtained by using synthetic division. This completes the proof. □

According to the nature of the roots, we have to express the solution in above three cases.

Example 4.2. Consider the fourth order differential equation as

$$\begin{aligned} &e^{-4x}y^{(iv)} - 3\phi_1(e^{-3x}, e^{-4x})y^{'''} - 2\phi_2(e^{-2x}, e^{-3x}, e^{-4x})y^{''} \\ &+2\phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x})y^{'} + 12y = 0 \end{aligned}$$

where the primes are the order of derivatives of the entire function y with respect to x. Consider the functions $\phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x})$, $\phi_2(e^{-2x}, e^{-3x}, e^{-4x})$ and $\phi_1(e^{-3x}, e^{-4x})$ are the convex combination of e^{-x}, e^{-2x}, e^{-3x} and e^{-4x} given by

$$\begin{aligned} &\phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x}) = a_1e^{-x} + a_2e^{-2x} + a_3e^{-3x} + a_4e^{-4x}, \\ &\phi_2(e^{-2x}, e^{-3x}, e^{-4x}) = b_1e^{-2x} + b_2e^{-3x} + b_3e^{-4x}, \end{aligned}$$

and

$$\phi_1(e^{-3x}, e^{-4x}) = c_1e^{-3x} + c_2e^{-4x},$$

with independent choice of $a_i, i = 1, 2, 3, 4$; $b_i, i = 1, 2, 3$ and $c_i, i = 1, 2$ satisfying

$$\begin{aligned} &\lim_{x\to 0} \phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x}) = \sum_{i=1}^{4} a_i = 1; \\ &\lim_{x\to 0} \phi_2(e^{-2x}, e^{-3x}, e^{-4x}) = \sum_{i=1}^{3} b_i = 1; \\ &\lim_{x\to 0} \phi_1(e^{-3x}, e^{-4x}) = \sum_{i=1}^{2} c_i = 1. \end{aligned}$$

Therefore the differential equation is rewritten as the system of the differential equation given by

$$\begin{aligned} y_1^{'} &= 0\cdot y_1 + 1\cdot y_2 + 0\cdot y_3 + 0\cdot y_4 \\ y_2^{'} &= 0\cdot y_1 + 0\cdot y_2 + 1\cdot y_3 + 0\cdot y_4 \\ y_3^{'} &= 0\cdot y_1 + 0\cdot y_2 + 0\cdot y_3 + 1\cdot y_4 \\ y_4^{'} &= -12y_1 - 2\phi_3y_2 + 2\phi_2y_3 + 3\phi_1y_4 \end{aligned}$$

that is,

$$Y' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -2\phi_3 & 2\phi_2 & 3\phi_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = A_x Y.$$

Clearly the transpose of the coefficient matrix A_x, i.e.,

$$A_x^T = \begin{pmatrix} 0 & 0 & 0 & -12 \\ 1 & 0 & 0 & -2\phi_3 \\ 0 & 1 & 0 & 2\phi_2 \\ 0 & 0 & 1 & 3\phi_1 \end{pmatrix}$$

is a companion matrix. So the pre-characteristic equation and the minimal polynomial equation are same, i.e.,

$$P(\lambda) = \lambda^4 - 3\lambda^3 \lim_{x\to 0} \phi_1(x) - 2\lambda^2 \lim_{x\to 0} \phi_2(x) + 2\lambda \lim_{x\to 0} \phi_3(x) + 12 = 0,$$

that is,

$$P(\lambda) = \lambda^4 - 3\lambda^3 - 2\lambda^2 + 2\lambda + 12 = 0$$

whose roots are $\lambda_1 = 2$, $\lambda_2 = 3$ $\lambda_3 = -(1+i)$ and $\lambda_4 = -(1-i)$ are the spectrals of A, and their corresponding eigenvectors are

$$X_1 = \begin{pmatrix} 1 & 2 & 4 & 8 \end{pmatrix}^T, \; X_2 = \begin{pmatrix} 1 & 3 & 9 & 27 \end{pmatrix}^T,$$
$$X_3 = \begin{pmatrix} -1 & 1+i & -2i & -2+2i \end{pmatrix}^T, \; X_4 = \begin{pmatrix} -1 & 1-i & 2i & -2-2i \end{pmatrix}^T$$

respectively. Hence the general solution is

$$\begin{aligned} Y &= C_1 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 3 \\ 9 \\ 27 \end{pmatrix} e^{3t} + C_3 \begin{pmatrix} -1 \\ 1-i \\ 2i \\ -2-2i \end{pmatrix} e^{(-1+i)t} \\ &\quad + C_4 \begin{pmatrix} -1 \\ 1+i \\ -2i \\ -2+2i \end{pmatrix} e^{(-1-i)t}, \end{aligned}$$

this implies

$$\begin{aligned} y_1 &= C_1 e^{2t} + C_2 e^{3t} - C_3 e^{(-1+i)t} - C_4 e^{(-1-i)t} \\ y_2 &= 2C_1 e^{2t} + 3C_2 e^{3t} + (1-i) C_3 e^{(-1+i)t} + (1+i) C_4 e^{(-1-i)t} \\ y_3 &= 4C_1 e^{2t} + 9C_2 e^{3t} + 2iC_3 e^{(-1+i)t} - 2iC_4 e^{(-1-i)t} \\ y_4 &= 8C_1 e^{2t} + 27C_2 e^{3t} + (-2-2i) C_3 e^{(-1+i)t} + (-2+2i) C_4 e^{(-1-i)t}. \end{aligned}$$

Example 4.3. Consider the fourth order differential equation

$$e^{-4x}y^{(iv)} - 5\phi_1(e^{-3x}, e^{-4x})y^{'''} + 6\phi_2(e^{-2x}, e^{-3x}, e^{-4x})y^{''} \\ +4\phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x})y^{'} - 8y = 0$$

where the primes are the order of derivatives of the entire function y with respect to x. Considering $\phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x})$, $\phi_2(e^{-2x}, e^{-3x}, e^{-4x})$ and $\phi_1(e^{-3x}, e^{-4x})$ are convex combination of e^{-x}, e^{-2x}, e^{-3x} and e^{-4x}, we have

$$\phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x}) = a_1e^{-x} + a_2e^{-2x} + a_3e^{-3x} + a_4e^{-4x}, \\ \phi_2(e^{-2x}, e^{-3x}, e^{-4x}) = b_1e^{-2x} + b_2e^{-3x} + b_3e^{-4x}, \text{ and} \\ \phi_1(e^{-3x}, e^{-4x}) = c_1e^{-3x} + c_2e^{-4x}$$

with independent choice of $a_i, i = 1, 2, 3, 4$; $b_i, i = 1, 2, 3$ and $c_i, i = 1, 2$ satisfying

$$\lim_{x\to 0} \phi_3(e^{-x}, e^{-2x}, e^{-3x}, e^{-4x}) = \sum_{i=1}^{4} a_i = 1,$$

$$\lim_{x\to 0} \phi_2(e^{-2x}, e^{-3x}, e^{-4x}) = \sum_{i=1}^{3} b_i = 1,$$

$$\lim_{x\to 0} \phi_1(e^{-3x}, e^{-4x}) = \sum_{i=1}^{2} c_i = 1.$$

Therefore the differential equation is rewritten as the system of the differential equation given by

$$\begin{aligned} y_1' &= 0 \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3 + 0 \cdot y_4 \\ y_2' &= 0 \cdot y_1 + 0 \cdot y_2 + 1 \cdot y_3 + 0 \cdot y_4 \\ y_3' &= 0 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3 + 1 \cdot y_4 \\ y_4' &= 8 \cdot y_1 - 4\phi_3 y_2 - 6\phi_2 y_3 + 5\phi_1 y_4, \end{aligned}$$

this implies

$$Y' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & -4\phi_3 & -6\phi_2 & 5\phi_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = A_xY.$$

Clearly the transpose of the coefficient matrix of A_x, i.e.,

$$A_x^T = \begin{pmatrix} 0 & 0 & 0 & 8 \\ 1 & 0 & 0 & -4\phi_3 \\ 0 & 1 & 0 & -6\phi_2 \\ 0 & 0 & 1 & 5\phi_1 \end{pmatrix}$$

is clearly a companion matrix. So the pre-characteristic equation and the minimal polynomial equation are same, i.e.,

$$P(\lambda) = \lambda^4 - 5\lambda^3 \lim_{x\to 0} \phi_1(x) + 6\lambda^2 \lim_{x\to 0} \phi_2(x) + 4\lambda \lim_{x\to 0} \phi_3(x) - 8 = 0,$$

that is,

$$P(\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 + 4\lambda - 8 = 0,$$

whose roots are $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 2$ and $\lambda_4 = 2$ are the spectrals of A and their corresponding eigenvectors are

$$\begin{aligned} &X_1 = \begin{pmatrix} -1 & 1 & -1 & 1 \end{pmatrix}^T,\ X_2 = \begin{pmatrix} 1 & 2 & 4 & 8 \end{pmatrix}^T, \\ &X_3 = \begin{pmatrix} t - \frac{3}{2} & 2t-2 & 4t-2 & 8t \end{pmatrix}^T \text{ and} \\ &X_4 = \begin{pmatrix} \frac{t^2}{2} - \frac{3}{2}t & t^2 - 2t + \frac{3}{2} & 2t^2 - 2t + 1 & 4t^2 \end{pmatrix}^T \end{aligned}$$

respectively where $X_3 = tX_2 + \beta$, $X_4 = \frac{t^2}{2}X_2 + \beta t + \gamma$ satisfying the condition that $(A - \lambda I)\beta = X_2$ and $(A - \lambda I)\gamma = \beta$. Using the eigenvalues, the general solution is obtained as

$$Y = C_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} e^{-t} + \left[C_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + C_3 \begin{pmatrix} t - \frac{3}{2} \\ 2t-2 \\ 4t-2 \\ 8t \end{pmatrix} + C_4 \begin{pmatrix} \frac{t^2}{2} - \frac{3}{2}t \\ t^2 - 2t + \frac{3}{2} \\ 2t^2 - 2t + 1 \\ 4t^2 \end{pmatrix} \right] e^{2t}.$$

Hence the general solution of the system of differential equation is

$$\begin{aligned} y_1 &= -C_1 e^{-t} + \left[C_2 + \left(t - \frac{3}{2}\right) C_3 + \left(\frac{t^2}{2} - \frac{3}{2}t\right) C_4 \right] e^{2t} \\ y_2 &= C_1 e^{-t} + \left[2C_2 + (2t-2)C_3 + \left(t^2 - 2t + \frac{3}{2}\right) C_4 \right] e^{2t} \\ y_3 &= -C_1 e^{-t} + \left[4C_2 + (4t-2)C_3 + (2t^2 - 2t + 1)C_4 \right] e^{2t} \\ y_4 &= C_1 e^{-t} + \left[8C_2 + 8tC_3 + 4t^2 C_4 \right] e^{2t}. \end{aligned}$$

Acknowledgments: The authors are thankful to the learned referee for his/her deep observations.

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