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COMMON STATIONARY POINTS OF MULTI-VALUED F-CONTRACTIONS WITH δ-DISTANCE

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Abstract. Three common stationary point theorems for some multi-valued F-contractions with δ-distance in bounded complete metric spaces are proved. The results obtained in this paper are extended or are different from several results in the literature. Three nontrivial examples are given.

1. Introduction and preliminaries

It is well known that one of the fundamental results in fixed point theory is the Banach fixed point theorem. Because of its importance in mathematical theory, this result has been extended and generalized in many directions for single-valued and multi-valued cases. Fixed point theorems for multi-valued contractive mappings were studied by using both Hausdorff metric H ([10, 16, 17]) and δ-distance ([6, 12-14, 21]).

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In 1969, Nadler [17] introduced the multi-valued contraction mapping by using the Huasdorff metric and proved the following result.

Theorem 1.1. ([17]) *Let (X, d) be a complete metric space, $CB(X)$ be the family of all nonempty closed and bounded subsets of X, and $S : X \to CB(X)$ be a mapping satisfying*

$$H(Sx, Sy) \leq rd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where $r \in [0, 1)$ is a constant. Then S has a fixed point.

Making use of δ-distance, Fisher [6] obtained the following common fixed point theorem for a pair of multi-valued contractive mappings.

Theorem 1.2. ([6]) *Let (X, d) be a bounded complete metric space, $B(X)$ be the family of all nonempty bounded subsets of X, and $S, T : X \to B(X)$ be commuting mappings satisfying for all $x, y \in X$,*

$$\delta(Sx, Ty) \leq r \max\{\delta(x, Sx), \delta(y, Ty), \delta(x, Ty), \delta(y, Sx), d(x, y)\}, \tag{1.2}$$

where $r \in [0, 1)$ is a constant. Then S and T have a common fixed point.

Let $\mathcal{F}$ be the set of all functions $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the following conditions:

(F1) F is strictly increasing;
(F2) For each sequence $\{\alpha_n\}_{n\geq 1}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.3. ([20]) Let (X, d) be a metric space. A mapping $f : X \to X$ is called F-contraction if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)), \quad \forall x, y \in X \text{ with } d(fx, fy) > 0.$$

One of the most interesting generalizations of the Banach fixed point theorem was given by Wardowski [20] in 2012. He proved a new fixed point theorem for F-contraction. Afterwards, a few researchers [1–4, 11,15,18–20] introduced new F-contractions for single-valued and multi-valued mappings and proved the existence of fixed points for these F-contractions. In particular, Acar and Altun [3] and Acar et al. [4] proved the following fixed point theorems.

Theorem 1.4. ([3]) *Let (X, d) be a complete metric space and $S : X \to B(X)$ be a multi-valued mapping. Assume that $F \in \mathcal{F}$, F is continuous and Sx is* *closed for all $x \in X$ and there exists $\tau > 0$ satisfying*

$$\begin{aligned} &\tau + F(\delta(Sx, Sy)) \le F(M(x, y)), \\ &\quad \forall x, y \in X \text{ with } \min\{\delta(Sx, Sy), d(x, y)\} > 0, \end{aligned} \tag{1.3}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Sy), \frac{1}{2}[d(x, Sy) + d(y, Sx)]\right\}. \tag{1.4}$$

Then S has a fixed point.

Theorem 1.5. ([4]) *Let (X, d) be a complete metric space, $K(X)$ be the family of all nonempty compact subsets of X, and $S : X \to K(X)$ be a multi-valued mapping. Assume that $F \in \mathcal{F}$, F or S is continuous and there exists $\tau > 0$ satisfying*

$$\tau + F(H(Sx, Sy)) \le F(M(x, y)), \quad \forall x, y \in X \text{ with } H(Sx, Sy) > 0, \tag{1.5}$$

where $M(x, y)$ is defined by (1.4). *Then S has a fixed point.*

Motivated and inspired by the results in [1–21], in this paper we introduce a few multi-valued F-contractions (2.1), (2.13) and (2.14) with δ-distance and establish the existence and uniqueness of common stationary point for these multi-valued F-contractions. Three examples are included to illustrate that the results obtained are extended or are different from results in [3, 4, 6, 12, 17].

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and all real numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}^+ = [0, +\infty)$. Let (X, d) be a metric space. It is clear that $\emptyset \ne K(X) \subseteq CB(X) \subseteq B(X)$. The Hausdorff metric $H : CB(X) \times CB(X) \to [0, +\infty)$ is defined by

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}, \quad \forall A, B \subseteq CB(X),$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. For $A, B \subseteq X$, define

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\} \text{ and } \delta(A, A) = \delta(A).$$

If A is singleton {a}, we write $\delta(A, B) = \delta(a, B)$. Let $S, T : X \to B(X)$ and $f : X \to X$. A point $x \in X$ is called a stationary point of S if $Sx = \{x\}$. Note that every stationary point of S is a fixed point of S, but not conversely. A point $x \in X$ is called a common stationary point of S and T if $Sx = Tx = \{x\}$. S and T are said to be commuting if $STx = TSx$ for all $x \in X$. S and f are said to be commuting if $Sfx = fSx$ for all $x \in X$. Define

$$C_f = \{g : g : X \to X \text{ satisfies that } g \text{ and } f \text{ are commuting}\}$$

and

$$C_S = \{G : G : X \to B(X) \text{ satisfies that } G \text{ and } S \text{ are commuting}\}.$$

It is clear that $C_S \supseteq \{S^n : n \in \mathbb{N}_0\}$ and $C_f \supseteq \{f^n : n \in \mathbb{N}_0\}$, where $S^0 = f^0 = i_X$ and i_X denotes the identity mapping in X.

Let $F : (0, +\infty) \to \mathbb{R}$ and $\eta : (0, +\infty) \to (0, +\infty)$ be two mappings, Λ be the set of all pairs (F, η) satisfying the following:

(λ1) F is upper semicontinuous and strictly increasing;

(λ2) $\lim_{n\to\infty} t_n = 0$ for each positive sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} F(t_n) = -\infty$;

(λ3) η is lower semicontinuous nonincreasing and $\eta(t_n) \nrightarrow 0$ for each strictly decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$.

Definition 1.6. ([7]) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets in $B(X)$ and $A \in B(X)$. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to converge to the set A if

(1) each point $a \in A$ is the limit of some convergent sequence $\{a_n\}_{n\in\mathbb{N}}$, where $a_n \in A_n$ for $n \in \mathbb{N}$;

(2) for arbitrary $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $A_n \subseteq A_\varepsilon$ for $n > k$, where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 1.7. ([5]) *If $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n, B_n)\}_{n\in\mathbb{N}}$ converges to $\delta(A, B)$.*

2. Main results

In this section, we prove stationary point theorems for the multi-valued F-contractions (2.1), (2.13) and (2.14) below with δ-distance.

Theorem 2.1. *Let (X, d) be a bounded complete metric space and $S, T : X \to B(X)$ be continuous and commuting mappings satisfying*

$$F(\delta(S^pT^qx, S^iT^jy)) \le F(\delta(\cup_{D\in C_{ST}} D\{x, y\})) - \eta(\delta(\cup_{D\in C_{ST}} D\{x, y\})), \quad \forall x, y \in X \text{ with } \delta(S^pT^qx, S^iT^jy) > 0, \tag{2.1}$$

where $(F, \eta) \in \Lambda$ and $p, q, i, j \in \mathbb{N}_0$ with $p, j \in \mathbb{N}$ or $q, i \in \mathbb{N}$. Then

(i) *S and T have a unique common stationary point $z \in X$;*

(ii) *The sequence $\{S^nT^nx\}_{n\in\mathbb{N}}$ converges to $\{z\}$ for all $x \in X$.*

Proof. Let $k = \max\{p, q\} + \max\{i, j\}$, $X_n = S^nT^nX$ and $\delta_n = \delta(X_n)$ for each $n \in \mathbb{N}_0$. Clearly,

$$X_{n+1} \subseteq X_n \text{ and } \delta_{n+1} \leq \delta_n, \quad \forall n \in \mathbb{N}_0 \tag{2.2}$$

and

$$DX_n = DS^nT^nX = S^nT^nDX \subseteq S^nT^nX = X_n, \forall (n, D) \in \mathbb{N}_0 \times C_{ST}. \tag{2.3}$$

Let A, $B \subseteq X$. It follows from (2.1) and $(F, \eta) \in \Lambda$ that for all $(a, b) \in A \times B$ with $\delta(S^pT^qa, S^iT^jb) > 0$

$$\begin{aligned}&F(\delta(S^pT^qa, S^iT^jb))\\&\leq F(\delta(\cup_{D\in C_{ST}} D\{a, b\})) - \eta(\delta(\cup_{D\in C_{ST}} D\{a, b\}))\\&\leq F(\delta(\cup_{D\in C_{ST}} (DA \cup DB))) - \eta(\delta(\cup_{D\in C_{ST}} (DA \cup DB))),\end{aligned}$$

which yields that

$$\begin{aligned}&F(\delta(S^pT^qA, S^iT^jB))\\&\leq F(\delta(\cup_{D\in C_{ST}} (DA \cup DB))) - \eta(\delta(\cup_{D\in C_{ST}} (DA \cup DB)))\end{aligned} \tag{2.4}$$

for all $A, B \subseteq X$ with $\delta(S^pT^qA, S^iT^jB) > 0$.

Assume that there exists $n_0 \in \mathbb{N}$ such that $\delta_{n_0} = 0$. It follows that $S^{n_0}T^{n_0}X = \{z\}$ for some $z \in X$. (2.3) means that $Tz = Sz = \{z\}$. That is, $z \in X$ is common stationary point of S and T. Assume that $\delta_n > 0$ for all $n \in \mathbb{N}_0$. In light of (2.1)-(2.4) and $(F, \eta) \in \Lambda$, we deduce that

$$\begin{aligned}F(\delta_k) &= F(\delta(S^pT^qS^{k-p}T^{k-q}X, S^iT^jS^{k-i}T^{k-j}X))\\&\leq F(\delta(\cup_{D\in C_{ST}} D(S^{k-p}T^{k-q}X \cup S^{k-i}T^{k-j}X)))\\&\quad - \eta(\delta(\cup_{D\in C_{ST}} D(S^{k-p}T^{k-q}X \cup S^{k-i}T^{k-j}X)))\\&\leq F(\delta(X)) - \eta(\delta(X))\\&= F(\delta_0) - \eta(\delta_0)\end{aligned}$$

and

$$\begin{aligned}F(\delta_{2k}) &= F(\delta(S^pT^qS^{k-p}T^{k-q}X_k, S^iT^jS^{k-i}T^{k-j}X_k))\\&\leq F(\delta(\cup_{D\in C_{ST}} D(S^{k-p}T^{k-q}X_k \cup S^{k-i}T^{k-j}X_k)))\\&\quad - \eta(\delta(\cup_{D\in C_{ST}} D(S^{k-p}T^{k-q}X_k \cup S^{k-i}T^{k-j}X_k)))\\&\leq F(\delta_k) - \eta(\delta_k).\end{aligned}$$

Repeating this process, we conclude that

$$\begin{aligned}F(\delta_{kn}) &= F(\delta(S^pT^qS^{k-p}T^{k-q}X_{k(n-1)}, S^iT^jS^{k-i}T^{k-j}X_{k(n-1)}))\\&\leq F(\delta(\cup_{D\in C_{ST}} D(S^{k-p}T^{k-q}X_{k(n-1)} \cup S^{k-i}T^{k-j}X_{k(n-1)})))\\&\quad - \eta(\delta(\cup_{D\in C_{ST}} D(S^{k-p}T^{k-q}X_{k(n-1)} \cup S^{k-i}T^{k-j}X_{k(n-1)})))\\&\leq F(\delta_{k(n-1)}) - \eta(\delta_{k(n-1)}), \quad \forall n \in \mathbb{N}.\end{aligned} \tag{2.5}$$

By virtue of (2.5), we get that

$$\eta(\delta_{k(n-1)}) \le F(\delta_{k(n-1)}) - F(\delta_{kn}), \quad \forall n \in \mathbb{N}. \tag{2.6}$$

In terms of $\eta(\delta_{k(n-1)}) > 0$ for each $n \in \mathbb{N}$, we have $F(\delta_{k(n-1)}) > F(\delta_{kn})$. It follows from $(\lambda 1)$ that $\{\delta_{kn}\}_{n\in\mathbb{N}_0}$ is a strictly decreasing positive sequence, which implies that there exists a constant $c \ge 0$ with $\lim_{n\to\infty} \delta_{kn} = c$.

Next, we show that $c = 0$. By means of (2.6), we conclude immediately that

$$\begin{aligned} F(\delta_{kn}) &\le F(\delta_{k(n-1)}) - \eta(\delta_{k(n-1)}) \\ &\le F(\delta_{k(n-2)}) - \eta(\delta_{k(n-2)}) - \eta(\delta_{k(n-1)}) \\ &\le \cdots \\ &\le F(\delta_0) - \eta(\delta_0) - \cdots - \eta(\delta_{k(n-2)}) - \eta(\delta_{k(n-1)}), \end{aligned}$$

that is,

$$\Sigma_{i=0}^{n-1} \eta(\delta_{ki}) \le F(\delta_0) - F(\delta_{kn}), \quad \forall n \in \mathbb{N}. \tag{2.7}$$

Note that $\{\delta_{kn}\}_{n\in\mathbb{N}_0}$ is strictly decreasing. Making use of $(\lambda 3)$, we arrive at $\eta(\delta_{kn}) \nrightarrow 0$, which gives that $\Sigma_{i=0}^{\infty}\eta(\delta_{ki}) = +\infty$. It follows from (2.7) that $\lim_{n\to\infty} F(\delta_{kn}) = -\infty$. In light of $(\lambda 2)$, we have $\lim_{n\to\infty} \delta_{kn} = 0$, which together with (2.2) yields that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \delta_{kn} = 0. \tag{2.8}$$

Choose $x_n \in X_n$ for each $n \in \mathbb{N}$. In view of (2.2), we infer that

$$d(x_n, x_m) \le \delta(X_n, X_m) \le \delta_n, \quad \forall m, n \in \mathbb{N} \text{ with } m > n.$$

Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence by (2.8). Since X is complete, it follows that there exists a point z in X such that $\lim_{n\to\infty} x_n = z$. From (2.2), we have

$$\begin{aligned} \delta(z, X_n) &\le d(z, x_m) + \delta(x_m, X_n) \\ &\le d(z, x_m) + \delta(X_m, X_n) \\ &\le d(z, x_m) + \delta_n, \quad \forall m, n \in \mathbb{N} \text{ with } m > n. \end{aligned} \tag{2.9}$$

Letting m tend to infinity in (2.9), we obtain that

$$\delta(z, X_n) \le \delta_n, \quad \forall n \in \mathbb{N}. \tag{2.10}$$

Since S and T are continuous and $\lim_{n\to\infty} x_n = z$, it follows that $\{Sx_n\}_{n\in\mathbb{N}}$ and $\{Tx_n\}_{n\in\mathbb{N}}$ converge to $\{Sz\}$ and $\{Tz\}$, respectively. Note that

$$\begin{aligned} Sx_n &\subseteq SS^nT^nX = S^nT^nSX \subseteq X_n, \quad \forall n \in \mathbb{N}, \\ Tx_n &\subseteq TS^nT^nX = S^nT^nTX \subseteq X_n, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.11}$$

In view of (2.8), (2.10) and (2.11), we deduce that

$$\max\{\delta(z, Sx_n), \delta(z, Tx_n)\} \le \delta(z, X_n) \le \delta_n \to 0 \quad \text{ as } n \to \infty,$$

which together with Lemma 1.7 yields that

$$\max\{\delta(z, Sz), \delta(z, Tz)\} = 0.$$

That is, $Sz = Tz = \{z\}$.

Suppose that S and T have a second common stationary point $\omega \in X - \{z\}$. Obviously, $\{u\} = S^n T^n u \subseteq X_n$ for each $u \in \{z, \omega\}$ and $n \in \mathbb{N}$. In view of (2.8), we infer that

$$d(z, \omega) \leq \delta_n \to 0 \quad \text{as } n \to \infty,$$

which means that $z = \omega$. Hence S and T have a unique common stationary point z.

Choose $y_n \in S^n T^n x$ for each $(x, n) \in X \times \mathbb{N}$. By means of (2.10), we have

$$d(y_n, z) \leq \delta(S^n T^n x, z) \leq \delta(X_n, z) \leq \delta_n. \tag{2.12}$$

It follows from (2.8), (2.12) and Definition 1.6 that $\{S^n T^n x\}_{n \in \mathbb{N}}$ converges to $\{z\}$. This completes the proof. □

As in the arguments of Theorem 2.1, we conclude similarly the following result and omit its proof.

Theorem 2.2. *Let (X, d) be a bounded complete metric space, $(F, \eta) \in \Lambda$, $S : X \to B(X)$ be a continuous mapping satisfying*

$$\begin{gathered} F(\delta(S^p x, S^i y)) \leq F(\delta(\cup_{D \in C_S} D\{x, y\})) - \eta(\delta(\cup_{D \in C_S} D\{x, y\})), \\ \forall x, y \in X \text{ with } \delta(S^p x, S^i y) > 0, \end{gathered} \tag{2.13}$$

where $p, i \in \mathbb{N}$. Then

(i) *S has a unique stationary point $z \in X$;*

(ii) *The sequence $\{S^n x\}_{n \in \mathbb{N}}$ converges to $\{z\}$ for all $x \in X$.*

Now we give a common fixed point theorem for two pairs of single-valued and multi-valued F-contractions.

Theorem 2.3. *Let (X, d) be a bounded complete metric space, $S, T : X \to B(X)$ be commuting and $f, g : X \to X$ be continuous, $f, g \in C_S \cap C_T$ and*

$$\begin{aligned} &F(\delta(S^p x, T^q y)) \\ &\leq F(\max\{\delta(fx, S^p x), \delta(gy, T^q y), \delta(fx, T^q y), \delta(gy, S^p x), d(fx, gy)\}) \\ &\quad - \eta(\max\{\delta(fx, S^p x), \delta(gy, T^q y), \delta(fx, T^q y), \delta(gy, S^p x), d(fx, gy)\}), \\ &\quad \forall x, y \in X \text{ with } \delta(S^p x, T^q y) > 0, \end{aligned} \tag{2.14}$$

where $p, q \in \mathbb{N}$ and $(F, \eta) \in \Lambda$. Then

(i) *The sequence $\{S^n T^n x\}_{n \in \mathbb{N}}$ converges to $\{z\}$ for all $x \in X$;*

(ii) *f, g, S^p and T^q have a unique common fixed point $z \in X$ with $S^p z = T^q z = \{z\}$, which is also a unique common fixed point of $C_S \cap C_T \cap C_f \cap Cg$.*

Proof. Let $k = p + q$, $X_n = S^n T^n X$ and $\delta_n = \delta(X_n)$ for each $n \in \mathbb{N}_0$. Clearly, (2.2) holds and

$$\begin{aligned} & hX_n = hS^nT^nX = S^nT^nhX \subseteq S^nT^nX = X_n, \\ & \forall (n, h) \in \mathbb{N}_0 \times (C_S \cap C_T). \end{aligned} \tag{2.15}$$

As in the proof of Theorem 2.1, we infer that by (2.14) and $(F, \eta) \in \Lambda$

$$\begin{aligned} & F(\delta(S^pA, T^qB)) \\ & \leq F(\max\{\delta(fA, S^pA), \delta(gB, T^qB), \delta(fA, T^qB), \\ & \quad \delta(gB, S^pA), d(fA, gB)\}) \\ & \quad - \eta(\max\{\delta(fA, S^pA), \delta(gB, T^qB), \delta(fA, T^qB), \\ & \quad \delta(gB, S^pA), d(fA, gB)\}), \\ & \quad \forall A, B \subseteq X \text{ with } \delta(S^pA, T^qB) > 0. \end{aligned} \tag{2.16}$$

Assume that there exists $n_0 \in \mathbb{N}$ such that $\delta_{n_0} = 0$. It follows that

$$S^{n_0}T^{n_0}X = \{z\}$$

for some $z \in X$ and

$$hz = hS^{n_0}T^{n_0}X = S^{n_0}T^{n_0}hX \subseteq S^{n_0}T^{n_0}X = \{z\},$$

for all $h \in \{f, g, S^p, T^q\}$. That is, $z \in X$ is a common fixed point of f, g, S^p and T^q. Assume that $\delta_n > 0$ for all $n \in \mathbb{N}_0$. In light of (2.2), (2.14)-(2.16) and $(F, \eta) \in \Lambda$, we deduce that

$$\begin{aligned} F(\delta_k) &= F(\delta(S^pS^qT^kX, T^qS^kT^pX)) \\ &\leq F(\max\{\delta(fS^qT^kX, S^pS^qT^kX), \delta(gS^kT^pX, T^qS^kT^pX), \\ &\qquad \delta(fS^qT^kX, T^qS^kT^pX), \delta(gS^kT^pX, S^pS^qT^kX), \\ &\qquad \delta(fS^qT^kX, gS^kT^pX)\}) \\ &\quad - \eta(\max\{\delta(fS^qT^kX, S^pS^qT^kX), \delta(gS^kT^pX, T^qS^kT^pX), \\ &\qquad \delta(fS^qT^kX, T^qS^kT^pX), \delta(gS^kT^pX, S^pS^qT^kX), \\ &\qquad \delta(fS^qT^kX, gS^kT^pX)\}) \\ &\leq F(\delta(X)) - \eta(\delta(X)) \\ &= F(\delta_0) - \eta(\delta_0) \end{aligned}$$

and

$$\begin{aligned}F(\delta_{2k}) &= F(\delta(S^pS^qT^kX_k, T^qS^kT^pX_k))\\ &\le F(\max\{\delta(fS^qT^kX_k, S^pS^qT^kX_k), \delta(gS^kT^pX_k, T^qS^kT^pX_k),\\ &\qquad \delta(fS^qT^kX_k, T^qS^kT^pX_k), \delta(gS^kT^pX_k, S^pS^qT^kX_k),\\ &\qquad \delta(fS^qT^kX_k, gS^kT^pX_k)\})\\ &\quad - \eta(\max\{\delta(fS^qT^kX_k, S^pS^qT^kX_k), \delta(gS^kT^pX_k, T^qS^kT^pX_k),\\ &\qquad \delta(fS^qT^kX_k, T^qS^kT^pX_k), \delta(gS^kT^pX_k, S^pS^qT^kX_k),\\ &\qquad \delta(fS^qT^kX_k, gS^kT^pX_k)\})\\ &\le F(\delta_k) - \eta(\delta_k).\end{aligned}$$

Repeating this process, we obtain that

$$\begin{aligned}&F(\delta_{kn})\\ &= F(\delta(S^pS^qT^kX_{k(n-1)}, T^qS^kT^pX_{k(n-1)}))\\ &\le F(\max\{\delta(fS^qT^kX_{k(n-1)}, S^pS^qT^kX_{k(n-1)}),\\ &\qquad \delta(gS^kT^pX_{k(n-1)}, T^qS^kT^pX_{k(n-1)}), \delta(fS^qT^kX_{k(n-1)}, T^qS^kT^pX_{k(n-1)}),\\ &\qquad \delta(gS^kT^pX_{k(n-1)}, S^pS^qT^kX_{k(n-1)}), \delta(fS^qT^kX_{k(n-1)}, gS^kT^pX_{k(n-1)})\})\\ &\quad - \eta(\max\{\delta(fS^qT^kX_{k(n-1)}, S^pS^qT^kX_{k(n-1)}),\\ &\qquad \delta(gS^kT^pX_{k(n-1)}, T^qS^kT^pX_{k(n-1)}), \delta(fS^qT^kX_{k(n-1)}, T^qS^kT^pX_{k(n-1)}),\\ &\qquad \delta(gS^kT^pX_{k(n-1)}, S^pS^qT^kX_{k(n-1)}), \delta(fS^qT^kX_{k(n-1)}, gS^kT^pX_{k(n-1)})\})\\ &\le F(\delta_{k(n-1)}) - \eta(\delta_{k(n-1)}), \quad \forall n \in \mathbb{N}.\end{aligned} \tag{2.17}$$

It follows from (2.17) that

$$\eta(\delta_{k(n-1)}) \le F(\delta_{k(n-1)}) - F(\delta_{kn}), \quad \forall n \in \mathbb{N}.$$

Proceeding as in the proof of Theorem 2.1, we obtain that (2.8) holds and $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, it is clear that there exists a point z in X such that $\lim_{n\to\infty} x_n = z$. For each $n \in \mathbb{N}$, choose a point $x_n \in X_n$. It follows that

$$fx_n \in fS^nT^nX = S^nT^nfX \subseteq S^nT^nX, \quad \forall n \in \mathbb{N}. \tag{2.18}$$

Similarly, $gx_n \in S^nT^nX$ for each $n \in \mathbb{N}$. The continuity of f and g ensures that $fx_n \to fz$ and $gx_n \to gz$ as $n \to \infty$. Consequently, by means of (2.18), we have

$$\begin{aligned}0 \le d(fz, gz) &\le d(fz, fx_n) + d(fx_n, gx_n) + d(gx_n, gz)\\ &\le d(fz, fx_n) + \delta_n + d(gx_n, gz), \quad \forall n \in \mathbb{N}.\end{aligned}$$

Letting n tend to infinity and using (2.8), we obtain that $d(fz, gz) = 0$, that is, $fz = gz$.

We next show that z is a common fixed point of f, g, S^p and T^q. It follows from (2.18) that

$$\begin{aligned} 0 \le d(z, fz) &\le d(z, x_n) + d(x_n, fx_n) + d(fx_n, fz) \\ &\le d(z, x_n) + \delta_n + d(fx_n, fz), \quad \forall n \in \mathbb{N}. \end{aligned}$$

As $n \to \infty$ we conclude that $d(z, fz) = 0$, that is, $z = fz = gz$.

We now assert that $\delta(z, T^q z) = 0$. Otherwise $\delta(z, T^q z) > 0$. By virtue of (2.2) and (2.18), we get that

$$\begin{aligned} \delta(z, T^q z) &\le d(z, gx_m) + \delta(gx_m, T^q z) \\ &\le d(z, gx_m) + \delta(S^m T^m X, T^q z) \\ &\le d(z, gx_m) + \delta(S^n T^n X, T^q z), \quad \forall m, n \in \mathbb{N} \text{ with } m > n. \end{aligned}$$

Letting m tend to infinity, we obtain that

$$\delta(z, T^q z) \le \delta(S^n T^n X, T^q z), \quad \forall n \in \mathbb{N}. \tag{2.19}$$

It follows from (2.14)-(2.16), (2.19), $(F, \eta) \in \Lambda$ and $gz = z$ that

$$\begin{aligned} \delta(fS^{n-p}T^n X, T^q z) &\le \delta(S^{n-p}T^{n-p} fT^p X, gx_{n-p}) + d(gx_{n-p}, z) + \delta(z, T^q z) \\ &\le \delta_{n-p} + d(gx_{n-p}, z) + \delta(z, T^q z), \end{aligned}$$

$$\delta(gz, S^p S^{n-p} T^n X) \le d(gz, gx_n) + \delta(gx_n, S^n T^n X) \le d(z, gx_n) + \delta_n,$$

$$\begin{aligned} \delta(fS^{n-p}T^n X, gz) &\le \delta(S^{n-p}T^{n-p} fT^p X, gx_{n-p}) + d(gx_{n-p}, gz) \\ &\le \delta_{n-p} + d(gx_{n-p}, z) \end{aligned}$$

and $\forall n > p$

$$\begin{aligned} &F(\delta(z, T^q z)) \\ &\le F(\delta(S^n T^n X, T^q z)) = F(\delta(S^p S^{n-p} T^n X, T^q z)) \\ &\le F(\max\{\delta(fS^{n-p}T^n X, S^p S^{n-p} T^n X), \delta(gz, T^q z), \\ &\qquad \delta(fS^{n-p}T^n X, T^q z), \delta(gz, S^p S^{n-p} T^n X), \delta(fS^{n-p}T^n X, gz)\}) \\ &\quad - \eta(\max\{\delta(fS^{n-p}T^n X, S^p S^{n-p} T^n X), \delta(gz, T^q z), \\ &\qquad \delta(fS^{n-p}T^n X, T^q z), \delta(gz, S^p S^{n-p} T^n X), \delta(fS^{n-p}T^n X, gz)\}) \\ &\le F(\max\{\delta_{n-p}, \delta(z, T^q z), \delta_{n-p} + d(gx_{n-p}, z) + \delta(z, T^q z), \\ &\qquad d(z, gx_n) + \delta_n, \delta_{n-p} + d(gx_{n-p}, z)\}) \\ &\quad - \eta(\max\{\delta_{n-p}, \delta(z, T^q z), \delta_{n-p} + d(gx_{n-p}, z) + \delta(z, T^q z), \\ &\qquad d(z, gx_n) + \delta_n, \delta_{n-p} + d(gx_{n-p}, z)\}) \\ &= F(\max\{\delta_{n-p} + d(gx_{n-p}, z) + \delta(z, T^q z), d(z, gx_n) + \delta_n\}) \\ &\quad - \eta(\max\{\delta_{n-p} + d(gx_{n-p}, z) + \delta(z, T^q z), d(z, gx_n) + \delta_n\}). \end{aligned} \tag{2.20}$$

Letting n tend to infinity in (2.20) and using (2.8) and $(F,\eta)\in\Lambda$, we get that

$$\begin{aligned}F(\delta(z,T^qz)) &\le F(\delta(z,T^qz))-\eta(\delta(z,T^qz))\\ &< F(\delta(z,T^qz)),\end{aligned}$$

which is a contradiction. Hence $\delta(z,T^qz)=0$. Consequently, $T^qz=\{z\}$. Similarly, $S^pz=\{z\}$. That is, $S^pz=T^qz=\{z\}$.

For each $(n,x)\in\mathbb{N}\times X$, choose $y_n\in S^nT^nx$. It is clear that

$$d(y_n,z)\le\delta(S^nT^nx,z)\le\delta(X_n,z)\le\delta_n,\quad\forall(n,x)\in\mathbb{N}\times X.$$

Letting n tend to infinity and using (2.8) and Definition 1.6, we conclude that $\{S^nT^nx\}_{n\in\mathbb{N}}$ converges to $\{z\}$.

We next show that z is the unique common fixed point of f, g, S^p and T^q with

$$S^pz=T^qz=\{fz\}=\{gz\}=\{z\}. \tag{2.21}$$

Suppose that f, g, S^p and T^q have a second common fixed point $\omega\in X-\{z\}$. If $\delta(S^p\omega,T^q\omega)>0$, it follows from (2.14), (2.15) and $(F,\eta)\in\Lambda$ that

$$\begin{aligned}&F(\delta(S^p\omega,T^q\omega))\\ &\le F(\max\{\delta(f\omega,S^p\omega),\delta(g\omega,T^q\omega),\delta(f\omega,T^q\omega),\delta(g\omega,S^p\omega),d(f\omega,g\omega)\})\\ &\quad-\eta(\max\{\delta(f\omega,S^p\omega),\delta(g\omega,T^q\omega),\delta(f\omega,T^q\omega),\delta(g\omega,S^p\omega),d(f\omega,g\omega)\})\\ &\le F(\delta(T^q\omega,S^p\omega))-\eta(\delta(T^q\omega,S^p\omega))\\ &< F(\delta(S^p\omega,T^q\omega)),\end{aligned}$$

which is a contradiction. Therefore $\delta(S^p\omega,T^q\omega)=0$. Note that $\omega\in S^p\omega\cap T^q\omega$. Consequently, $S^p\omega=T^q\omega=\{\omega\}$. Using (2.14), (2.15) and $(F,\eta)\in\Lambda$ again, we get that

$$\begin{aligned}&F(\delta(z,\omega))\\ &=F(\delta(S^pz,T^q\omega))\\ &\le F(\max\{\delta(fz,S^pz),\delta(g\omega,T^q\omega),\delta(fz,T^q\omega),\delta(g\omega,S^pz),\delta(fz,g\omega))\})\\ &\quad-\eta(\max\{\delta(fz,S^pz),\delta(g\omega,T^q\omega),\delta(fz,T^q\omega),\delta(g\omega,S^pz),\delta(fz,g\omega))\})\\ &=F(\delta(z,\omega))-\eta(\delta(z,\omega))\\ &< F(\delta(z,\omega)),\end{aligned}$$

which is impossible. Therefore z is the unique common fixed point of f, g, S^p and T^q with (2.21).

Finally we prove that z is also a unique common fixed point of $C_S\cap C_T\cap C_f\cap C_g$. For each $h\in C_S\cap C_T\cap C_f\cap C_g$, we infer that by (2.21)

$$S^phz=hS^pz=\{hz\}=hT^qz=T^qhz=h\{fz\}=\{fhz\}=h\{gz\}=\{ghz\},$$

which means that hz is a common fixed point of f, g, S^p and T^q. Thus the uniqueness of the common fixed point of f, g, S^p and T^q yields that $hz = z$. This completes the proof. □

3. Examples

In this section, we give three examples to show that Theorem 2.1 is different from Theorems 1.1, 1.4 and 1.5 in the first section and Theorem 2.3 extends indeed Theorem 1.2.

Remark 3.1. The following example manifests that Theorem 2.1 differs from Theorems 1.1 and 1.4 in the first section.

Example 3.2. Let $X = [0,1] \cup \{2\}$ be endowed with the Euclidean metric $d = |\cdot|$. Define $S, T : X \to K(X)$, $F : (0,+\infty) \to \mathbb{R}$ and $\eta : (0,+\infty) \to (0,+\infty)$ by

$$Sx = \begin{cases} [0, \frac{x}{2}], & \forall x \in [0,1], \\ \{1\}, & x = 2, \end{cases} \qquad Tx = \{x\}, \quad \forall x \in X,$$

$$F(t) = \ln t + t \quad \text{and} \quad \eta(t) = 1, \quad \forall t \in (0,+\infty).$$

Take $p = j = 2$, $q = i = 3$. Obviously, (X, d) is a bounded complete metric space, S and T are continuous and commuting, $(F, \eta) \in \Lambda$ and

$$S^2x = \cup_{y\in Sx} Sy = \cup_{y\in[0,\frac{x}{2}]}\left[0, \frac{y}{2}\right] = \left[0, \frac{x}{4}\right], \quad S^3x = \left[0, \frac{x}{8}\right], \quad \forall x \in [0,1],$$

$$S^2 2 = S1 = \left[0, \frac{1}{2}\right], \quad S^3 2 = \left[0, \frac{1}{4}\right].$$

Put $x, y \in X$. In order to verify (2.1), we need to consider four possible cases as follows:

Case 1. $(x, y) \in [0,1] \times [0,1] - (0,0)$. It follows that

$$\begin{aligned}
&F(\delta(S^2T^3x, S^3T^2y)) \\
&= F\left(\delta\left(\left[0, \frac{x}{4}\right], \left[0, \frac{y}{8}\right]\right)\right) = F\left(\max\left\{\frac{x}{4}, \frac{y}{8}\right\}\right) \\
&= \ln\left(\max\left\{\frac{x}{4}, \frac{y}{8}\right\}\right) + \max\left\{\frac{x}{4}, \frac{y}{8}\right\} \le \ln(\max\{x,y\}) + \max\{x,y\} - 1 \\
&= \ln\left(\delta\left(\left[0, \frac{x}{4}\right] \cup \left[0, \frac{y}{4}\right] \cup \{x,y\}\right)\right) + \delta\left(\left[0, \frac{x}{4}\right] \cup \left[0, \frac{y}{4}\right] \cup \{x,y\}\right) - 1 \\
&\le \ln(\delta(\cup_{D\in C_{ST}} D\{x,y\})) + \delta(\cup_{D\in C_{ST}} D\{x,y\}) - 1 \\
&= F(\delta(\cup_{D\in C_{ST}} D\{x,y\})) - \eta(\delta(\cup_{D\in C_{ST}} D\{x,y\}));
\end{aligned}$$

Case 2. $x = y = 2$. It is clear that

$$\begin{aligned}
&F(\delta(S^2T^3x, S^3T^2y))\\
&= F\bigg(\delta\bigg(\bigg[0,\frac{1}{2}\bigg],\bigg[0,\frac{1}{4}\bigg]\bigg)\bigg) = F\bigg(\frac{1}{2}\bigg) = \ln\frac{1}{2}+\frac{1}{2} \le \ln 2 + 2 - 1\\
&= \ln\bigg(\delta\bigg(\bigg[0,\frac{1}{2}\bigg]\cup\{2\}\bigg)\bigg) + \delta\bigg(\bigg[0,\frac{1}{2}\bigg]\cup\{2\}\bigg) - 1\\
&\le \ln(\delta(\cup_{D\in C_{ST}}D\{x,y\})) + \delta(\cup_{D\in C_{ST}}D\{x,y\}) - 1\\
&= F(\delta(\cup_{D\in C_{ST}}D\{x,y\})) - \eta(\delta(\cup_{D\in C_{ST}}D\{x,y\}));
\end{aligned}$$

Case 3. $(x, y) \in [0, 1] \times \{2\}$. It is easy to see that

$$\begin{aligned}
&F(\delta(S^2T^3x, S^3T^2y))\\
&= F\bigg(\delta\bigg(\bigg[0,\frac{x}{4}\bigg],\bigg[0,\frac{1}{4}\bigg]\bigg)\bigg) = F\bigg(\frac{1}{4}\bigg) = \ln\frac{1}{4}+\frac{1}{4} \le \ln 2 + 2 - 1\\
&= \ln\bigg(\delta\bigg(\bigg[0,\frac{x}{4}\bigg]\cup\bigg[0,\frac{1}{2}\bigg]\cup\{x,2\}\bigg)\bigg) + \delta\bigg(\bigg[0,\frac{x}{4}\bigg]\cup\bigg[0,\frac{1}{2}\bigg]\cup\{x,2\}\bigg) - 1\\
&\le \ln(\delta(\cup_{D\in C_{ST}}D\{x,y\})) + \delta(\cup_{D\in C_{ST}}D\{x,y\}) - 1\\
&= F(\delta(\cup_{D\in C_{ST}}D\{x,y\})) - \eta(\delta(\cup_{D\in C_{ST}}D\{x,y\}));
\end{aligned}$$

Case 4. $(x, y) \in \{2\} \times [0, 1]$. It is easy to verify that

$$\begin{aligned}
&F(\delta(S^2T^3x, S^3T^2y))\\
&= F\bigg(\delta\bigg(\bigg[0,\frac{1}{2}\bigg],\bigg[0,\frac{y}{8}\bigg]\bigg)\bigg) = F\bigg(\frac{1}{2}\bigg) = \ln\frac{1}{2}+\frac{1}{2} \le \ln 2 + 2 - 1\\
&= \ln\bigg(\delta\bigg(\bigg[0,\frac{1}{2}\bigg]\cup\bigg[0,\frac{y}{4}\bigg]\cup\{2,y\}\bigg)\bigg) + \delta\bigg(\bigg[0,\frac{1}{2}\bigg]\cup\bigg[0,\frac{y}{4}\bigg]\cup\{2,y\}\bigg) - 1\\
&\le \ln(\delta(\cup_{D\in C_{ST}}D\{x,y\})) + \delta(\cup_{D\in C_{ST}}D\{x,y\}) - 1\\
&= F(\delta(\cup_{D\in C_{ST}}D\{x,y\})) - \eta(\delta(\cup_{D\in C_{ST}}D\{x,y\})).
\end{aligned}$$

Hence, (2.1) holds. That is, the conditions of Theorem 2.1 are satisfied. It follows from Theorem 2.1 that S and T have a unique common stationary point $0 \in X$ and the sequence $\{S^nT^nx\}_{n\in\mathbb{N}}$ converges to $\{0\}$ for all $x \in X$.

However, we don't invoke Theorems 1.1 and 1.4 in the first section to show the existence of fixed points of S in X. Suppose that S satisfies the conditions of Theorem 1.1. That is, there exists $r \in [0, 1)$ satisfying (1.1). By virtue of (1.1), we deduce that

$$1 = H\bigg(\bigg[0,\frac{1}{2}\bigg],\{1\}\bigg) = H(S1, S2) \le rd(1, 2) = r < 1,$$

which is a contradiction.

Suppose that S satisfies the conditions of Theorem 1.4. It follows that there exists $\tau > 0$ satisfying (1.3) and (1.4). In view of (1.3) and (1.4), we infer that

$$\begin{aligned} M(1,2) &= \max\left\{d(1,2), d(1,S1), d(2,S2), \frac{1}{2}[d(1,S2)+d(2,S1)]\right\} \\ &= \max\left\{d(1,2), d\left(1,\left[0,\frac{1}{2}\right]\right), d(2,1), \frac{1}{2}\left[d(1,1)+d\left(2,\left[0,\frac{1}{2}\right]\right)\right]\right\} \\ &= \max\left\{1,\frac{1}{2},1,\frac{1}{2}\left(0+\frac{3}{2}\right)\right\} = 1 \end{aligned}$$

and

$$\begin{aligned} \tau + F(1) &= \tau + F\left(\delta\left(\left[0,\frac{1}{2}\right],\{1\}\right)\right) = \tau + F(\delta(S1,S2)) \\ &\leq F(M(1,2)) = F(1), \end{aligned}$$

which is impossible.

Remark 3.3. The below example demonstrates that Theorem 2.1 is different from Theorem 1.5 in the first section.

Example 3.4. Let $X = [1, \frac{5}{2}]$ be endowed with the Euclidean metric $d = |\cdot|$. Define $S, T : X \to K(X)$, $F : (0,+\infty) \to \mathbb{R}$ and $\eta : (0,+\infty) \to (0,+\infty)$ by

$$Sx = \begin{cases} \left[1, \frac{x}{2}+\frac{1}{2}\right], & \forall x \in [1,2], \\ \left[1, x-\frac{1}{2}\right], & \forall x \in (2, \frac{5}{2}], \end{cases} \qquad Tx = \begin{cases} \left[1, \frac{x}{3}+\frac{2}{3}\right], & \forall x \in [1,2], \\ \left[1, \frac{2}{3}x\right], & \forall x \in (2, \frac{5}{2}], \end{cases}$$

$$F(t) = \ln t \quad \text{and} \quad \eta(t) = \ln\frac{4}{3}, \quad \forall t \in (0,+\infty).$$

Take $p = j = 1$ and $q = i = 0$. Obviously, (X, d) is a bounded complete metric space, S and T are continuous and commuting, $(F, \eta) \in \Lambda$ and

$$STx = \begin{cases} \left[1, \frac{x}{6}+\frac{5}{6}\right], & \forall x \in [1,2], \\ \left[1, \frac{x}{3}+\frac{1}{2}\right], & \forall x \in (2, \frac{5}{2}]. \end{cases}$$

Put $x, y \in X$. In order to verify (2.1), we need to consider four possible cases as follows:

Case 1. $(x,y)\in[1,2]\times[1,2]-(1,1)$. It follows that

$$\begin{aligned}
&F(\delta(Sx,Ty))\\
&=F\left(\delta\left(\left[1,\frac{x}{2}+\frac{1}{2}\right],\left[1,\frac{y}{3}+\frac{2}{3}\right]\right)\right)=\ln\left(\max\left\{\frac{1}{2}(x-1),\frac{1}{3}(y-1)\right\}\right)\\
&\le\ln(\max\{x-1,y-1\})-\ln\frac{4}{3}\\
&=\ln\left(\delta\left(\left[1,\frac{x}{6}+\frac{5}{6}\right]\cup\left[1,\frac{y}{6}+\frac{5}{6}\right]\cup\{x,y\}\right)\right)-\ln\frac{4}{3}\\
&\le\ln(\delta(\cup_{D\in C_{ST}}D\{x,y\}))-\ln\frac{4}{3}\\
&=F(\delta(\cup_{D\in C_{ST}}D\{x,y\}))-\eta(\delta(\cup_{D\in C_{ST}}D\{x,y\}));
\end{aligned}$$

Case 2. $(x,y)\in(2,\frac{5}{2}]\times(2,\frac{5}{2}]$. It is clear that

$$\begin{aligned}
&F(\delta(Sx,Ty))\\
&=F\left(\delta\left(\left[1,x-\frac{1}{2}\right],\left[1,\frac{2}{3}y\right]\right)\right)=\ln\left(\max\left\{x-\frac{3}{2},\frac{2}{3}y-1\right\}\right)\\
&\le\ln(\max\{x-1,y-1\})-\ln\frac{4}{3}\\
&=\ln\left(\delta\left(\left[1,\frac{x}{3}+\frac{1}{2}\right]\cup\left[1,\frac{y}{3}+\frac{1}{2}\right]\cup\{x,y\}\right)\right)-\ln\frac{4}{3}\\
&\le\ln(\delta(\cup_{D\in C_{ST}}D\{x,y\}))-\ln\frac{4}{3}\\
&=F(\delta(\cup_{D\in C_{ST}}D\{x,y\}))-\eta(\delta(\cup_{D\in C_{ST}}D\{x,y\}));
\end{aligned}$$

Case 3. $(x,y)\in[1,2]\times(2,\frac{5}{2}]$. It is easy to verify that

$$\begin{aligned}
&F(\delta(Sx,Ty))\\
&=F\left(\delta\left(\left[1,\frac{x}{2}+\frac{1}{2}\right],\left[1,\frac{2}{3}y\right]\right)\right)=\ln\left(\max\left\{\frac{x}{2}-\frac{1}{2},\frac{2}{3}y-1\right\}\right)\\
&\le\ln(y-1)-\ln\frac{4}{3}\\
&=\ln\left(\delta\left(\left[1,\frac{x}{6}+\frac{5}{6}\right]\cup\left[1,\frac{y}{3}+\frac{1}{2}\right]\cup\{x,y\}\right)\right)-\ln\frac{4}{3}\\
&\le\ln(\delta(\cup_{D\in C_{ST}}D\{x,y\}))-\ln\frac{4}{3}\\
&=F(\delta(\cup_{D\in C_{ST}}D\{x,y\}))-\eta(\delta(\cup_{D\in C_{ST}}D\{x,y\}));
\end{aligned}$$

Case 4. $(x, y) \in (2, \frac{5}{2}] \times [1, 2]$. It is easy to verify that

$$\begin{aligned}
&F(\delta(Sx, Ty)) \\
&= F\left(\delta\left(\left[x - \frac{1}{2}\right], \left[1, \frac{y}{3} + \frac{2}{3}\right]\right)\right) = \ln\left(x - \frac{3}{2}\right) \le \ln(x-1) - \ln\frac{4}{3} \\
&= \ln\left(\delta\left(\left[1, \frac{x}{3} + \frac{1}{2}\right] \cup \left[1, \frac{y}{6} + \frac{5}{6}\right] \cup \{x, y\}\right)\right) - \ln\frac{4}{3} \\
&\le \ln(\delta(\cup_{D\in C_{ST}} D\{x, y\})) - \ln\frac{4}{3} \\
&= F(\delta(\cup_{D\in C_{ST}} D\{x, y\})) - \eta(\delta(\cup_{D\in C_{ST}} D\{x, y\})).
\end{aligned}$$

Hence, (2.1) holds. That is, the conditions of Theorem 2.1 are satisfied. It follows from Theorem 2.1 that S and T have a unique common stationary point $1 \in X$ and the sequence $\{S^n T^n x\}_{n\in\mathbb{N}}$ converges to $\{1\}$ for all $x \in X$.

Now we claim that Theorem 1.5 in the first section is useless in proving the existence of fixed points of S in X. Suppose that S satisfies the conditions of Theorem 1.5. It follows that there exists $\tau > 0$ satisfying (1.4) and (1.5). By means of (1.4) and (1.5), we have

$$\begin{aligned}
M\left(2, \frac{5}{2}\right) &= \max\left\{d\left(2, \frac{5}{2}\right), d(2, S2), d\left(\frac{5}{2}, S\frac{5}{2}\right), \frac{1}{2}\left[d\left(2, S\frac{5}{2}\right) + d\left(\frac{5}{2}, S2\right)\right]\right\} \\
&= \max\left\{d\left(2, \frac{5}{2}\right), d\left(2, \left[1, \frac{3}{2}\right]\right), d\left(\frac{5}{2}, [1, 2]\right),\right. \\
&\qquad \left.\frac{1}{2}\left[d(2, [1, 2]) + d\left(\frac{5}{2}, \left[1, \frac{3}{2}\right]\right)\right]\right\} \\
&= \max\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}(0 + 1)\right\} = \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\tau + F\left(\frac{1}{2}\right) &= \tau + F\left(H\left(\left[1, \frac{3}{2}\right], [1, 2]\right)\right) = \tau + F\left(H\left(S2, S\frac{5}{2}\right)\right) \\
&\le F\left(M\left(2, \frac{5}{2}\right)\right) = F\left(\frac{1}{2}\right),
\end{aligned}$$

which is impossible.

Remark 3.5. $(a1)$ if $r = 0$ in Theorem 1.2, it follows from (1.2) that

$$\begin{aligned}
\delta(Sx, Ty) &\le r \max\{\delta(x, Sx), \delta(y, Ty), \delta(x, Ty), \delta(y, Sx), d(x, y)\} \\
&= 0, \quad \forall x, y \in X,
\end{aligned}$$

which implies that there exists some $a \in X$ with $Sx = Tx = \{a\}$, which yields that (1.2) holds for each $r \in (0, 1)$, that is, in Theorem 1.2, "$r \in [0, 1)$ is a constant" is equivalent to "$r \in (0, 1)$ is a constant";

$(a2)$ (1.2) is equivalent to

$$\delta(Sx,Ty) \le r\max\{\delta(x,Sx),\delta(y,Ty),\delta(x,Ty),\delta(y,Sx),d(x,y)\}, \quad \forall x,y\in X \text{ with } \delta(Sx,Ty)>0. \tag{3.1}$$

In fact, it is easy to see that (1.2) implies (3.1). Conversely, for any $x, y \in X$, if $\delta(Sx,Ty) > 0$, it follows from (3.1) that

$$\delta(Sx,Ty) \le r\max\{\delta(x,Sx),\delta(y,Ty),\delta(x,Ty),\delta(y,Sx),d(x,y)\}; \tag{3.2}$$

if $\delta(Sx,Ty) = 0$, it is clear that

$$\delta(Sx,Ty) = 0 \le r\max\{\delta(x,Sx),\delta(y,Ty),\delta(x,Ty),\delta(y,Sx),d(x,y)\}. \tag{3.3}$$

Thus (1.2) follows from (3.2) and (3.3).

Remark 3.6. If $p = q = 1$, $f = g = i_X$, $F(t) = \ln t$, $\eta(t) = \ln\frac{1}{r}$, $\forall t \in (0,+\infty)$, where $r \in (0,1)$ is a constant, in Theorem 2.3, it is easy to verify that (2.14) implies (3.1). It follows from Remark 3.5 that Theorem 2.3 extends Theorem 1.2.

Remark 3.7. The following example reveals that Theorem 2.3 generalizes indeed Theorem 1.2 in the first section and differs from Theorem 1 in [12].

Example 3.8. Let $X = \{1,2,5,7,9\}$ be endowed with the Euclidean metric $d = |\cdot|$. Define $f,g : X \to X$, $S,T : X \to B(X)$, $F : (0,+\infty) \to \mathbb{R}$ and $\eta : (0,+\infty) \to (0,+\infty)$ by

$$\begin{gathered} f1 = f2 = f5 = f7 = 2, \quad f9 = 1, \quad gx = x, \quad \forall x \in X, \\ S1 = S2 = S7 = \{2\}, \quad S5 = \{2,7\}, \quad S9 = \{5\}, \quad T = S, \\ F(t) = \ln t \quad \text{and} \quad \eta(t) = \ln\frac{11+t}{10+t}, \quad \forall t \in (0,+\infty). \end{gathered}$$

Take $p = 2$, $q = 3$. Obviously, (X,d) is a bounded complete metric space, $(F,\eta) \in \Lambda$, S and T are commuting, f and g are continuous and belong to $C_S \cap C_T$.

Put $x, y \in X$. Clearly, $\delta(S^2x, T^3y) = |2-2| = 0$ for all $(x,y) \in \{1,2,5,7\} \times \{1,2,5,7,9\}$. In order to verify (2.14), we need to consider two possible cases as follows:

Case 1. $x = 9$, $y \in \{1, 2, 5, 7\}$. It follows that

$$\begin{aligned}
&F(\delta(S^29, T^3y)) \\
&= F(\delta(\{2,7\}, 2)) = F(5) = \ln 5 < \ln \frac{96}{17} = F(6) - \eta\left(\frac{11+6}{10+6}\right) \\
&= F(\max\{\delta(1, \{2,7\}), \delta(y, 2), \delta(1, 2), \delta(y, \{2,7\}), d(1, y)\}) \\
&\quad - \eta(\max\{\delta(1, \{2,7\}), \delta(y, 2), \delta(1, 2), \delta(y, \{2,7\}), d(1, y)\}) \\
&= F(\max\{\delta(f9, S^29), \delta(gy, T^3y), \delta(f9, T^3y), \delta(gy, S^29), d(f9, gy)\}) \\
&\quad - \eta(\max\{\delta(f9, S^29), \delta(gy, T^3y), \delta(f9, T^3y), \delta(gy, S^29), d(f9, gy)\});
\end{aligned}$$

Case 2. $x = y = 9$. Notice that

$$\begin{aligned}
&F(\delta(S^29, T^39)) \\
&= F(\delta(\{2,7\}, 2)) = F(5) = \ln 5 < \ln \frac{144}{19} = F(8) - \eta\left(\frac{11+8}{10+8}\right) \\
&= F(\max\{\delta(1, \{2,7\}), \delta(9, 2), \delta(1, 2), \delta(9, \{2,7\}), d(1, 9)\}) \\
&\quad - \eta(\max\{\delta(1, \{2,7\}), \delta(9, 2), \delta(1, 2), \delta(9, \{2,7\}), d(1, 9)\}) \\
&= F(\max\{\delta(f9, S^29), \delta(g9, T^39), \delta(f9, T^39), \delta(g9, S^29), d(f9, g9)\}) \\
&\quad - \eta(\max\{\delta(f9, S^29), \delta(g9, T^39), \delta(f9, T^39), \delta(g9, S^29), d(f9, g9)\}).
\end{aligned}$$

Hence, (2.14) holds. That is, the conditions of Theorem 2.3 are fulfilled. It follows from Theorem 2.3 that f, g, S^2 and T^3 have a unique common fixed point $2 \in X$.

However, Theorem 1.2 in the first section and Theorem 1 in [12] cannot be used to prove the existence of stationary points of S, common stationary points of S and T in X and common fixed points of f, g, S and T in X, respectively. Suppose that S and T satisfy the conditions of Theorem 1.2. That is, there exists $r \in [0, 1)$ satisfying (1.2). By virtue of (1.2), we deduce that

$$\begin{aligned}
5 &= \delta(\{2,7\}, \{2\}) = \delta(S5, T2) \\
&\le r \max(\delta(5, S5), \delta(2, T2), \delta(5, T2), \delta(2, S5), d(5, 2)) \\
&= r \max\{\delta(5, \{2,7\}), \delta(2, \{2\}\}, \delta(5, \{2\}), \delta(2, \{2,7\}), d(5, 2)) \\
&= r \max\{3, 0, 3, 5, 3\} = 5r < 5,
\end{aligned}$$

which is impossible.

Suppose that f, g, S and T satisfy the conditions of Theorem 1 in [12]. That is, there exists $\varphi \in \Phi$ satisfying $\forall x, y \in X$,

$$\delta(Sx, Ty) \le \varphi(\delta(fx, Sx), \delta(gy, Ty), \delta(fx, Ty), \delta(gy, Sx), d(fx, gy)\}, \quad (3.4)$$

where $\Phi = \{\varphi : \varphi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is upper semicontinuous, nondecreasing in each coordinate variable and $\varphi(t,t,t,t,t) < t$ for any $t > 0\}$.

It follows from (3.4) that

$$\begin{aligned} 5 &= \delta(\{2,7\},\{2\}) = \delta(S5,T2) \\ &\le \varphi(\delta(f5,S5),\delta(g2,T2),\delta(f5,T2),\delta(g2,S5),d(f5,g2)) \\ &= \varphi(\delta(2,\{2,7\}),\delta(2,\{2\}),\delta(2,\{2\}),\delta(2,\{2,7\}),d(2,2)) \\ &= \varphi(5,0,0,5,0) \\ &\le \varphi(5,5,5,5,5) \\ &< 5, \end{aligned}$$

which is a contradiction.

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