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SOME RESULTS IN FIXED POINT THEORY CONCERNING RECTANGULAR b-METRIC SPACES

Z. Mostefaoui[1], M. Bousselsal[2] and Jong Kyu Kim[3]

[1]Department of Mathematics, Faculty of Sciences, King Kalied University
Abha, Saudi Arabia
e-mail: z.mostefaoui26@yahoo.fr

[2]Department of Mathematics, E.N.S
B.P. 92 Vieux Kouba 16050 Algiers, (Algeria)
e-mail: bousselsal55@gmail.com

[3]Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. In this paper, we proved some fixed point results with different types of contraction in Rectangular b-metric space. Our results extend very recent results of Fora et. al. [7] and extend and generalize many existing results in the literature.

1. Introduction

In 2000, Branciari [2] introduced a concept of generalized metric space where the triangle inequality of a metric space has been replaced by an inequality involving three terms instead of two. As such, any metric space is a generalized metric space but the converse is not true [2]. He proved the Banach's fixed point theorem in such a space. After that, many fixed point results were established for this interesting space. For more, the reader can refer to [10, 3]. It is also known that common fixed point theorems are generalizations of fixed point theorems. Recently, there have been many researchers who have

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[0]Corresponding author: M. Bousselsal(bousselsal55@gmail.com).

interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems.

George et. al. [9] introduced the concept of rectangular b-metric space, which is not necessarily Hausdorff and which generalizes the concepts of metric space, rectangular metric space and b-metric space. Note that spaces with non-Hausdorff topology plays an importnat role in Tarskian approach to programming language semantics used in computer science (For some details see [17]). An analog of the Banach contraction principle as well as the Kannan type fixed point theorem in rectangular b-metric spaces are also proved in [9].

2. Preliminaries

The following definitions are introduced in [1, 2, 4, 9] and [15], respectively.

Definition 2.1. ([1, 4]) Let X be a nonempty set and $s \geq 1$ be a given real number. A functional $d : X \times X \longrightarrow \mathbb{R}^+$ is called a b- metric if for $x, y, z \in X$, the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ (b-triangular inequality).

A pair (X, d) is called a b-metric space (with constant s).

Definition 2.2. ([2]) Let X be a nonempty set. A functional $d : X\times X \longrightarrow \mathbb{R}^+$ is called a rectangular metric if for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

A pair (X, d) is called a rectangular metric space or generalized metric spaces (g.m.s.) or Branciari's space.

For all properties and definitions of notions in Branciari's spaces see [2, 6, 8, 11, 12, 13, 15].

Definition 2.3. ([9, 15]) Let X be a nonempty set, $s \geq 1$ be a given real number and $d : X \times X \longrightarrow \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$ each distinct from x and y:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ (b-rectangular inequality).

Then (X, d) is called a rectangular b-metric space (with constant s) or a b-generalized metric space (RbMS).

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse of the above implication is not necessarily true (See Examples 1.4 and 1.5 [9]).

The following gives some easy examples of RbMS's.

Example 2.4. Let $X = \mathbb{N}$, define $d : X \times X \longrightarrow \mathbb{R}^+$ by

$$d(x, y) = \begin{cases} 0 & if\ x = y, \\ 4\alpha & if\ x, y \in \{1, 2\}\ and\ x \neq y, \\ \alpha & if\ x\ or\ y \notin \{1, 2\}\ and\ x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (X, d) is not a rectangular metric space, as

$$d(1, 2) = 4\alpha > 3\alpha = d(1, 3) + d(3, 4) + d(4, 2).$$

Example 2.5. Let (X, ρ) be a g.m.s., and $p \geq 1$ be a real number. Let $d(x, y) = (\rho(x, y))^p$. Evidently, from the convexity of function $f(x) = x^p$ for $x \geq 0$ and by Jensen inequality we have

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p)$$

for nonnegative real numbers a, b, c. So, it is easy to obtain that (X, d) is a b-g.m.s with $s \leq 3^{p-1}$.

Note that every b-metric space with coefficient s is a RbMS with coefficient s^2 but the converse is not necessarily true. (See Example 1.7 [9]).

For any $x \in X$ we define the open ball with center x and radius $r > 0$ by $B_r(x) = \{y \in X : d(x, y) < r\}$. The open balls in RbMS are not necessarily open (See Example 1.7 [9]). Let U be the collection of all subsets A of X satisfying the condition that for each $x \in A$ there exist $r > 0$ such that $B_r(x) \subseteq A$. Then U defines a topology for the RbMS (X, d), which is not necessarily Hausdorff (See Example 1.7 [9]).

Definition 2.6. Let (X, d) be a rectangular b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent to $x \in X$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim\limits_{n\to+\infty} x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow +\infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy in X, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0$, or equivalently, if

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0.$$

(c) (X, d) is said to be a complete rectangular b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Note that limit of sequence in a rectangular b-metric space (the same as in a rectangular metric space (g.m.s)) is not necessarily unique and also every rectangular b-metric convergent sequence in a rectangular b-metric space is not necessarily rectangular b-metric-Cauchy (See [9], Example 2.7).

Lemma 2.7. ([15]) *Let (X, d) be a rectangular b-metric space with $s \geq 1$ and let $\{x_n\}$ be a rectangular Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

Let $T : X \to X$ be a mapping where (X, d) is RbMS. For each $x \in X$ let $O(x) = \{x, Tx, T^2x, T^3x, \cdots\}$ which will be called the orbit of T at x. $O(x)$ is called T-orbitally complete if and only if every Cauchy sequence in $O(x)$ converges to a point in X.

Let Φ denote the class of all nondecreasing upper semicontinuous functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{n=1}^{+\infty} s^n \varphi^n(t) < +\infty$ for all $t > 0$ where φ^n is the n^{th} iterate of φ. Since $\sum_{n=1}^{+\infty} s^n \varphi^n(t) < +\infty$ and $\varphi^n(t) \leq s^n \varphi^n(t)$ for all $t \geq 0$, so $\sum_{n=1}^{+\infty} \varphi^n(t) < +\infty$.

Lemma 2.8. ([7]) *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that the sequence $\{\varphi^n(t)\}$ converges to 0 for all $t > 0$. Then*

(i) *$\varphi(t) < t$ for all $t > 0$;*
(ii) *$\varphi(0) = 0$.*

3. Main results

Theorem 3.1. *Let (X, d) be a rectangular b-metric space with coefficient $s \geq 1$ and let $T : X \longrightarrow X$ be a mapping such that:*

$$d(Tx, Ty) \leq \varphi(\max\{d(x, y),\, d(x, Tx),\, \frac{1}{s} d(y, Ty),\, d(y, Tx)\}) \tag{3.1}$$

where $\varphi \in \Phi$. If there exists $x \in X$ such that $O(x)$ is orbitally complete, then T has a unique fixed point in X.

Proof. Define the sequence $\{x_n\}$ inductively as follows: $x_0 = x$, $x_n = Tx_{n-1} = T^n x$ for all $n \in \mathbb{N}$. Setting $d(x_n, x_{n+1}) = d_n$, it follows from (3.1) that

$$\begin{aligned} d(T^n x, T^{n+1}x) \quad \leq \quad & \varphi(\max\{d(T^{n-1}x, T^n x),\, d(T^{n-1}x, T^n x), \\ & \tfrac{1}{s} d(T^n x, T^{n+1}x),\, d(T^n x, T^n x)\}) \end{aligned}$$

which implies that

$$d_n = d(T^n x, T^{n+1}x) \leq \varphi(d(T^{n-1}x, T^n x)) = \varphi(d_{n-1}). \tag{3.2}$$

Then, for all $n \in \mathbb{N}$,

$$d_n = d(T^n x, T^{n+1}x) \leq \varphi^n(d(x, Tx)). \tag{3.3}$$

If there exists $n < m$ such that $x_n = x_m$, let $y = T^n x$ then $T^k y = y$ where $k = m - n$. Since $k \geq 1$, we have

$$d(y, Ty) = d(T^k y, T^{k+1}y) \leq \varphi^k(d(y, Ty)).$$

Since $\varphi(t) < t$ for all $t > 0$, so $d(y, Ty) = 0$ and hence y is a fixed point of T.

Assume that $x_n \neq x_m$ for all $n \neq m$, so we have

$$d(Tx, T^3x) \leq \varphi\big(\max\{d(x, T^2x),\, d(x, Tx),\, \frac{1}{s} d(T^2x, T^3x),\, d(T^2x, Tx)\}\big).$$

This implies that

$$d(Tx, T^3x) \leq \varphi(M)$$

where $M = \max\{d(x, T^2x), d(x, Tx)\}$. In general, if n is a positive integer, then

$$d(T^n x, T^{n+2}x) \leq \varphi^n(M). \tag{3.4}$$

For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases.

If p is odd say $2m + 1$ then using (3.2) and (3.3) we obtain

$$\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\
&\quad + d(x_{n+2}, x_{n+2m+1})] \\
&\leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\
&\quad + d(x_{n+4}, x_{n+2m+1})] \\
&\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\
&\quad + \cdots + s^m d_{n+2m} \\
&\leq s[\varphi^n(d(x,Tx)) + \varphi^{n+1}(d(x,Tx))] \\
&\quad + s^2[\varphi^{n+2}(d(x,Tx)) \quad + \varphi^{n+3}(d(x,Tx))] \\
&\quad + s^3[\varphi^{n+4}(d(x,Tx)) + \varphi^{n+5}(d(x,Tx))] \\
&\quad + \cdots + s^m \varphi^{n+2m}(d(x,Tx)) \\
&\leq s^n\varphi^n(d(x,Tx)) + s^{n+1}\varphi^{n+1}(d(x,Tx)) \\
&\quad + s^{n+2}\varphi^{n+2}(d(x,Tx)) \\
&\quad + s^{n+3}\varphi^{n+3}(d(x,Tx)) + s^{n+4}\varphi^{n+4}(d(x,Tx)) \\
&\quad + s^{n+5}\varphi^{n+5}(d(x,Tx)) \\
&\quad + \cdots + s^{n+2m}\varphi^{n+2m}(d(x,Tx)).
\end{aligned}$$

Therefore, we have

$$d(x_n, x_{n+2m+1}) \leq \sum_{k=n}^{k=n+2m} s^k \varphi^k (d(x,Tx). \tag{3.5}$$

If p is even say $2m$ then using (3.2) and (3.3) we obtain

$$\begin{aligned}
d(x_n, x_{n+2m}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
&\leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) \\
&\quad + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\
&\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\
&\quad + \cdots + s^{m-1}[d_{n+2m-4} + d_{n+2m-3}] \\
&\quad + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
&\leq s[\varphi^n(d(x,Tx)) + \varphi^{n+1}(d(x,Tx))] \\
&\quad + s^2[\varphi^{n+2}(d(x,Tx)) \quad + \varphi^{n+3}(d(x,Tx))] \\
&\quad + s^3[\varphi^{n+4}(d(x,Tx)) + \varphi^{n+5}(d(x,Tx))] \\
&\quad + \cdots + s^{m-1}[\varphi^{n+2m-4}(d(x,Tx)) + \varphi^{n+2m-3}(d(x,Tx))] \\
&\quad + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
&\leq s^n\varphi^n(d(x,Tx)) + s^{n+1}\varphi^{n+1}(d(x,Tx)) \\
&\quad + s^{n+2}\varphi^{n+2}(d(x,Tx)) + s^{n+3}\varphi^{n+3}(d(x,Tx)) \\
&\quad + s^{n+4}\varphi^{n+4}(d(x,Tx)) \\
&\quad + s^{n+5}\varphi^{n+5}(d(x,Tx)) \\
&\quad + \cdots + s^{n+2m-4}\varphi^{n+2m-4}(d(x,Tx)) \\
&\quad + s^{n+2m-3}\varphi^{n+2m-3}(d(x,Tx)) \\
&\quad + s^{m-1} d(x_{n+2m-2}, x_{n+2m}).
\end{aligned}$$

Using (3.4), for all $p > 0$ we obtain

$$d(x_n, x_{n+2m}) \leq \sum_{k=n}^{k=n+2m-3} s^k \varphi^k(d(x, Tx)) + s^{m-1}\varphi^{n+2m-2}(M). \tag{3.6}$$

Thus, by (3.3), (3.4), (3.5), and (3.6) we have

$$d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} s^k \varphi^k(M).$$

Since $\lim\limits_{n\to+\infty} \sum\limits_{k=n}^{+\infty} s^k \varphi^k(M) = 0$, we have

$$\lim_{n\to+\infty} d(x_n, x_{n+p}) = 0,$$

for all $p > 0$. Thus it is clearly verified that $\{x_n\}$ is a Cauchy sequence. Since $O(x)$ is T-orbitally complete, $\{x_n\}$ converges to $z \in X$.

The point z is a fixed point of T. To see this we have two cases under consideration.

Case 1. If $\{x_n\}$ does not converge to Tz, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq Tz$ for all $k \in \mathbb{N}$. Hence

$$d(z, Tz) \leq s\big[d(z, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, Tz)\big].$$

If $k \to +\infty$, we get

$$d(z, Tz) \leq s. \limsup_{k\to+\infty} d(x_{n_k}, Tz). \tag{3.7}$$

On the other hand, we have from (3.1)

$$\begin{aligned} d(x_n, Tz) &= d(Tx_{n-1}, Tz) \\ &\leq \varphi(\max\{d(x_{n-1}, z),\, d(x_{n-1}, x_n),\, \tfrac{1}{s}d(z, Tz),\, d(z, x_n)\}). \end{aligned}$$

Let $n \to +\infty$, we get

$$\limsup_{n\to+\infty} d(x_n, Tz) \leq \varphi\Big(\frac{1}{s}d(z, Tz)\Big) < \frac{1}{s}d(z, Tz). \tag{3.8}$$

Hence, by (3.7) and (3.8), we have $d(z, Tz) = 0$ and $z = Tz$.

Case 2. Let $\{x_n\}$ be convergent to Tz. Suppose that $z \neq Tz$. Then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \in X - \{z, Tz\}$ for all $k \in \mathbb{N}$, hence

$$d(z, Tz) \leq s\big[d(z, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tz)\big]. \tag{3.9}$$

As $k \to +\infty$ in (3.9), we get $Tz = z$, a contradiction. Then in all cases z is a fixed point of T. For the uniqueness, assume that $w \neq z$ is also a fixed point of T. From (3.1),

$$d(z,w) = d(Tz,Tw) \leq \varphi\big(\max\{d(z,w),\, d(z,Tz),\, \frac{1}{s}d(w,Tw),\, d(w,Tz)\}\big)$$

which implies that

$$d(z,w) \leq \varphi(d(z,w)),$$

hence $z = w$, a contradiction. Therefore, T has a unique fixed point z. □

Corollary 3.2. *Let (X,d) be a RbMS and let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leq q\max\{d(x,y),\, d(x,Tx),\, \frac{1}{s}d(y,Ty),\, d(y,Tx)\}$$

where $0 \leq sq < 1$. If there exists $x \in X$ such that $O(x)$ is orbitally complete, then T has a unique fixed point in X.

Proof. Put $\varphi(t) = qt$ in Theorem 3.1. □

The condition "there exists $x \in X$ such that $O(x)$ is orbitally complete" is necessary; to see this, consider the next example.

Example 3.3. Let $X = (0,1]$, $d(x,y) = |x-y|$, and let $T : X \to X$ be a mapping such that $Tx = \frac{x}{2}$ for all $x \in X$. So, $O(a) = \{a, \frac{a}{2}, \frac{a}{2^2}, \cdots, \frac{a}{2^n}, \cdots\}$ for all $a \in X$. Let $x_n = \frac{a}{2^n}$, $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in $O(a)$, but $\{x_n\}$ does not converge. Hence $O(a)$ is not complete. Moreover, T satisfies the condition (3.1) where $\varphi(t) = \frac{1}{2}t$, and does not have a fixed point in X.

For the next result, let Ψ denote the class of all functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which are nondecreasing and $\sum_{n=1}^{+\infty} s^n\psi^n(t) < +\infty$ for all $t > 0$.

Theorem 3.4. *Let (X,d) be a RbMS and let $T : X \to X$ be a continuous mapping such that*

$$d(Tx,T^2x) \leq \psi(d(x,Tx)),\ d(Tx,T^3x) \leq \psi(d(x,T^2x)) \tag{3.10}$$

where $\psi \in \Psi$. If there exists $x \in X$ such that $O(x)$ is orbitally complete, then T has a fixed point in X.

Proof. Define the sequence $\{x_n\}$ inductively as follows: $x_0 = x$, $x_n = Tx_{n-1}$ for all $n \geq 1$. For all $n \in \mathbb{N}$, we have

$$d_n = d(T^nx, T^{n+1}x) \leq \psi^n(d(x,Tx)). \tag{3.11}$$

If $x_n = x_m$ for some $m > n$, then T^nx is a fixed point of T.

Now, assume that $x_n \neq x_m$ for all $n \neq m$. For all $n \in \mathbb{N}$, we have

$$d(T^nx, T^{n+2}x) \leq \psi^n(d(x,T^2x)). \tag{3.12}$$

For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases:

Case 1. If p is odd say $2m+1$ then using (3.10) and (3.11) we obtain

$d(x_n, x_{n+2m+1}) \leq s[d_n+d_{n+1}]+s^2[d_{n+2}+d_{n+3}]+s^3[d_{n+4}+d_{n+5}]+\cdots+s^m d_{n+2m}$.

Therefore,

$$d(x_n, x_{n+2m+1}) \leq \sum_{k=n}^{k=n+2m} s^k \psi^k(d(x, Tx)). \tag{3.13}$$

Case 2. If p is even say $2m$ then using (3.11) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) \quad \leq \quad & s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] + \cdots \\ & +s^{m-1}[d_{n+2m-4} + d_{n+2m-3}] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}). \end{aligned}$$

Using (3.12) we obtain

$$d(x_n, x_{n+2m}) \leq \sum_{k=n}^{k=n+2m-3} s^k \psi^k(d(x, Tx)) + s^{m-1}\psi^{n+2m-2}(d(x, T^2x)). \tag{3.14}$$

Thus, by (3.11), (3.12), (3.13), and (3.14) we have

$$d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} s^k \psi^k(R),$$

where $R = \max\{d(x, Tx),\, d(x, T^2x)\}$. Since $\sum_{k=1}^{+\infty} s^k \psi^k < +\infty$, $\{x_n\}$ is a Cauchy sequence. Since $O(x)$ is T-orbitally complete, $\{x_n\}$ converges to $z \in X$, and by the continuity of T, we have $\{x_n\}$ converges also to Tz. Hence z is a fixed point of T. □

Corollary 3.5. *Let (X, d) be a RbMS and let $T : X \to X$ be a continuous mapping such that*

$$\begin{gathered} \min\{d(Tx, Ty), \max\{d(x, Tx), d(y, Ty)\}\} \leq \psi(d(x, y)) \\ \textit{and} \\ d(x, T^2x) \leq d(x, Tx) \end{gathered} \tag{3.15}$$

where $\psi \in \Psi$. If there exists $x \in X$ such that $O(x)$ is orbitale complete, then T has a fixed point in X.

Proof. By setting $y = Tx$ in (3.15), we get

$$\min\{d(Tx, T^2x),\, \max\{d(x, Tx),\, d(Tx, T^2x)\}\} \leq \psi(d(x, Tx))$$

which implies that

$$d(Tx, T^2x) \leq \psi(d(x, Tx))$$

for all $x \in X$. Similarly, if we put $y = T^2x$ in (3.15), we get

$$\min\{d(Tx, T^3x),\ \max\{d(x, Tx),\ d(T^2x, T^3x)\}\} \le \psi(d(x, T^2x))$$

hence

$$\min\{d(Tx, T^3x), d(x, Tx)\} \le \psi(d(x, T^2x))$$

for all $x \in X$, which implies that

$$d(Tx, T^3x) \le \psi(d(x, T^2x)).$$

Then by Theorem 3.4, T has a fixed point. □

Example 3.6. Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2, 3, 4, 5\}\}$ and $B = [1, 2]$. Define $d : X \times X \to [0, 1)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$\begin{cases} d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0,03; \\ d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0,02; \\ d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0,6; \\ d(x, y) = |x - y|^2 \text{ otherwise.} \end{cases}$$

Then (X, d) is a rectangular b-metric space with coefficient $s = 4 > 1$. But (X, d) is neither a metric space nor a rectangular metric space. Let $T : X \to X$ be defined as :

$$Tx = \begin{cases} \frac{1}{4} & x \in A; \\ \frac{1}{5} & x \in B, \end{cases}$$

$\psi(t) = \frac{1}{5}t$ for all $t \in \mathbb{R}_+$, Then $\sum_{n=1}^{+\infty} s^n \psi^n(t) = \sum_{n=1}^{+\infty} (\frac{4}{5})^n t < +\infty$ for all $t \in \mathbb{R}_+$.
If $x \in A$ then $Tx = T^2x = T^3x = \frac{1}{4}$,

$$d(Tx, T^2x) = d(\frac{1}{4}, \frac{1}{4}) = 0 \le \psi(d(x, Tx))$$

and

$$d(Tx, T^3x) = 0 \le \psi(d(x, T^2x)).$$

If $x \in B$ then $Tx = \frac{1}{5}$, $T^2x = T^3x = \frac{1}{4}$,

$$d(Tx, T^2x) = d(\frac{1}{5}, \frac{1}{4}) = 0,03 \le \psi(d(x, Tx)) = \frac{1}{5}(x - \frac{1}{5})^2$$

because $\frac{1}{5}(x - \frac{1}{5})^2 \ge \frac{16}{125} > 0,03$ and

$$d(Tx, T^3x) = 0,03 \le \psi(d(x, T^2x)) = \frac{1}{5}(x - \frac{1}{4})^2$$

because $\frac{1}{5}(x - \frac{1}{4})^2 \ge \frac{9}{80} > 0,03$.
There exist $x = \frac{1}{2} \in X$ such that $O(x) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}, \ldots\}$ orbitale complete. Then T satisfies the condition of Theorem 3.4 and has a unique fixed point $x = \frac{1}{4}$.

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