Nonlinear Functional Analysis and Applications
Vol. 24, No. 1 (2019), pp. 61-71
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa
Copyright © 2019 Kyungnam University Press

OPERATOR SOLUTIONS OF GENERALIZED EQUILIBRIUM PROBLEMS IN HAUSDORFF TOPOLOGICAL VECTOR SPACES

Tirth Ram[1], Parshotam Lal[2] and Jong Kyu Kim[3]

[1]Department of Mathematics University of Jammu
Jammu 180006, India
e-mail: `tir1ram2@yahoo.com`

[2]Department of Mathematics University of Jammu
Jammu 180006, India

[3]Department of Mathematics Education
Kyungnam University, Changwon Gyeongnam, 51767, Korea
e-mail: `jongkyuk@kyungnam.ac.kr`

Abstract. In this paper, we consider the generalized equilibrium problems with operator solutions and derive some existence results for generalized operator equilibrium problems in Hausdorff topological vector spaces by employing KKM theorem. The results presented of this paper generalize and unify the corresponding results of the several authors.

1. Introduction

Domokos and Kolumban [4] gave nice interpretation of variational inequalities (VI) and vector variational inequalities (VVI) in Banach spaces setting in terms of variational inequalities with operator solutions (OVVI). These operator variational inequalities includes not only scalar and vector variational inequalities as special cases, see, for example, Chen [3], Hadjsavas and Schaible [9], Yu and Yao [19], Fu [6] but also have sufficient evidence for their importance to study, see, Domokos and Kolumban [4], Kum [15]. Inspired by the

[0]Received May 22, 2018. Revised October 1, 2018.
[0]2010 Mathematics Subject Classification: 49J40, 90C33.
[0]Keywords: Generalized operator equilibrium problems, P-convex mapping, P-monotone, upper semicontinuous, KKM- mapping.
[0]Corresponding author: T. Ram(`tir1ram2@yahoo.com`).

work of Domokos and Kolumban [4], Kum [15], developed generalized vector variational inequality with operator solutions and obtained some existence results.

On the other hand, Blum and Oettli [2] intensively studied equilibrium problems where they proposed it as a generalization of optimization and variational inequality problems. In 2005, Kazmi and Raouf [11], considered and study a class of operator equilibrium problems (OEP) and derive some existence results for the solution of (OEP) by employing KKM theorems, which extend the notion of operator variational inequalities (OVVI) to operator equilibrium problem (OEP). Recently Kim and Raouf [14] studied the generalized equilibrium problems and derive the existence results by using Fan KKM theorems.

Motivated and inspired by the work of Kum [15], Kim and Raouf [14], Kazmi and Raouf [11], we introduce generalized operator equilibrium problem (in short, GOEP) which generalize operator equilibrium problem into multi-valued operator equilibrium problems. Next, by using Fan [5] lemma, we will prove some new existence results on generalized operator equilibrium problems (GOEP) which are generalizations and unifications of many well-known results in the literature, see for example ([1], [8], [14]-[18]).

Throughout the paper unless otherwise stated, let X and Y be two Hausdorff topological vector spaces and $L(X, Y)$ be a space of all continuous linear operators from X to Y. Let $K \subset L(X, Y)$ be a nonempty convex set and let $C : K \to 2^Y$ be a set-valued mapping such that for each $f \in K$, $C(f)$ is a solid convex open cone and $0 \notin C(f)$ and let $P = \bigcap_{f \in K} C(f)$.

Let Y be an ordered topological vector space. It is clear that the cone $C(f)$, for each $f \in K$, define on Y a partial ordering $\leq_{C(f)}$ by $g \leq_{C(f)} h$ iff $h - g \in C(f)$.

Consider the following *generalized operator equilibrium problem (for short, GOEP)*. Find $f_0 \in K$ such that

$$G(f_0, g) + H(f_0, g) \not\subset -C(f), \text{ for all } g \in K, \tag{1.1}$$

where $G, H : K \times K \longrightarrow 2^Y$ be two bi-operators.

2. Preliminaries

In this section, we need the following definitions and results which are needed for establishment of main results.

Definition 2.1. ([5]) A set-valued mapping $F : B \subset X \longrightarrow 2^B$ is said to be a $KKM-$ mapping if, for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of B,

$$con\,\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where *con* $\{x_1, x_2, \cdots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \cdots, x_n\}$.

The following well- known Fan theorem will play a crucial role in proving the existence results for the solution of GOEP in our paper.

Theorem 2.2. ([5]) *Let E be a topological vector space, K be a non-empty subset of E and let $F : K \to 2^E$ be a $KKM-$ map such that $F(x)$ is closed for each $x \in K$ and is compact for at least one $x \in K$, then* $\bigcap_{x\in K} F(x) \neq \phi$.

Definition 2.3. Let $F : B \subset X \longrightarrow 2^Y$ be a set-valued mapping. Then

(i) F is said to be *upper semicontinuous* on B, If for each $x_0 \in B$ and any open set V of Y containing $F(x_0)$, there exists an open neighborhood U of x_0 in B such that $F(x) \subset V$ for all $x \in U$.

(ii) The *graph* of F, denoted by $G(F)$, is

$$G(F) = \{(x, z) \in B \times Y : x \in B, z \in F(x)\}.$$

(iii) The *inverse* F^{-1} of F is the set-valued mapping from $R(F)$, range of F, to B defined by

$$x \in F^{-1}(y) \text{ if and only if } y \in F(x).$$

Definition 2.4. ([14]) Let B be a subset of K. A set-valued mapping $C : K \longrightarrow 2^Y$ is said to have a *closed graph with respect to* B if for every net $\{f_\alpha\}_{\alpha\in\Gamma} \subset K$ and $\{g_\alpha\}_{\alpha\in\Gamma} \subset Y$ such that $g_\alpha \in C(f_\alpha)$, $\{f_\alpha\}$ converges to $f \in B$ with respect to the pointwise convergence and $\{g_\alpha\}$ converges to $g \in Y$, then $g \in C(f)$.

Definition 2.5. ([14]) Let $K \subset L(X, Y)$ be a convex set, $C : K \longrightarrow 2^Y$ be a set-valued mapping and $C(f)$ be an open convex solid cone with $0 \notin C(f)$, for all $f \in K$. Then we define the following ordering relationship on sets, $A, B \subset Y$,

(i) $B - A \subseteq C(f) \Longleftrightarrow A \leq_{C(f)} B \Longleftrightarrow a \leq_{C(f)} b$, for all $a \in A, b \in B$.

(ii) $B - A \nsubseteq C(f) \Longleftrightarrow A \nleq_{C(f)} B \Longleftrightarrow a \nleq_{C(f)} b$, for all $a \in A, b \in B$.

Definition 2.6. ([2]) Let K and D be two convex subsets of $L(X, Y)$, with $K \subset D$. Then $Core_D K$, *the core of K relative to D*, is defined through $h \in Core_D K$ iff $h \in K$ and $K \cap (0, 1] \neq \phi$, for all $g \in D \setminus K$, where

$$(f, g] = \{h \in L(X, Y) : h = tf + (1 - t)g, \text{for } t \in [0, 1)\}.$$

Definition 2.7. Let X and Y be two Hausdorff topological vector spaces and $L(X,Y)$ be the space of all linear operators from X to Y and let $K \subset L(X,Y)$. Let $F : K \longrightarrow 2^Y$ and $G : K \times K \longrightarrow 2^Y$ be given. nThen

(i) F is said to be P-*convex* if for any $f, g \in K, t \in [0,1]$,

$$tF(f) + (1-t)F(g) \subset F(tf + (1-t)g) + P.$$

(ii) F is said to be P-*concave* if for any $f, g \in K, t \in [0,1]$,

$$F(tf + (1-t)g) \subset tF(f) + (1-t)F(g) + P.$$

(iii) G is said to be P-*monotone* if for any $f, g \in K$,

$$G(f,g) + G(g,f) \subset -P.$$

3. Main results

In this section, we prove some existence results for the solution of $GOEP$. Throughout this section, let X and Y be two Hausdorff topological vector spaces, D a non-empty, convex closed subset of X.

Lemma 3.1. ([14]) *Let X and Y be two Hausdorff topological vector spaces and $K \subset L(X,Y)$ be a non-empty convex set. Let $(Y, C(f))$ be an ordered topological vector space with convex solid cone $C(f)$, $0 \notin C(f)$, for each $f \in K$. Then for all $f, g \in K$, we have*

(i) *$g - f \subseteq C(f)$ and $g \nsubseteq C(f)$ implies $f \nsubseteq C(f)$.*

(ii) *$g - f \subseteq -C(f)$ and $g \nsubseteq -C(f)$ implies $f \nsubseteq -C(f)$.*

Theorem 3.2. *Let $G, H : D \times D \longrightarrow 2^Y$ be two set-valued maps such that*

(i) *for all $f \in D$, $G(f,f) = \{0\}$, $H(f,f) = \{0\}$,*

(ii) *G is P-monotone,*

(iii) *for all $f \in D$, the graph of $W(f) := Y \setminus (-C(f))$ is closed in Y,*

(iv) *for any $f, g \in K$, $g(t) := G(tg + (1-t)f, g)$, where $t \in [0,1]$, is upper semicontinuous with respect to $C(f)$ at t=0,*

(v) *for any $f \in D$, $G(f,.), H(f,.) : K \longrightarrow 2^Y$ are P-convex,*

(vi) *for any $f \in D$, $G(f,g)$ is continuous in g with respect to $C(f)$ on D and for any fixed $g \in D, H(f,g)$ is continuous in f with respect to $C(f)$ on D,*

(vii) *there exists a non-empty convex compact set $K \subseteq D$ such that, for any $f \in K \setminus core_D K$, one can find a point $h \in core_D K$ such that*

$$G(f,h) + H(f,h) \subset -P.$$

Then there exists $f_0 \in K$ such that

$$G(f_0, g) + H(f_0, g) \not\subset -C(f_0), \text{ for all } g \in D.$$

To prove the above theorem, we shall first prove the following lemmas:

Lemma 3.3. *Let $G, H : K \times K \longrightarrow 2^Y$ be two set-valued maps and $K \subseteq L(X,Y)$ be a compact subset such that following conditions hold:*

(i) *For all $f \in K$, $G(f,f) = \{0\}$, $H(f,f) = \{0\}$.*
(ii) *The graph of $W(f) = Y \setminus (-C(f))$ is closed for all $f \in K$.*
(iii) *G is P-monotone.*
(iv) *For any fixed $f, g \in K$, $t \longrightarrow G(tg + (1-t)f, g)$, where $t \in [0,1]$, is upper semicontinuous at t=0.*
(v) *for any $f \in K$, $G(f,.), H(f,.) : K \longrightarrow 2^Y$ are P-convex.*
(vi) *For fixed $f \in K$, $G(f,.)$ is continuous with respect to $C(f)$ on K, and for fixed $g \in K$, $H(.,g)$ is continuous with respect to $C(f)$ on K.*

Then there exists $f_0 \in K$ such that

$$G(f_0, g) + H(f_0, g) \not\subset -C(f), \text{ for all } g \in K.$$

Proof. For any $g \in K$, define $S : K \longrightarrow 2^K$ by

$$S(g) := \{f \in K : G(g,f) - H(f,g) \not\subset C(f)\}.$$

Claim 1: $S(g)$ is closed, for all $g \in K$. For this, let $\{h_i\} \subset S(g)$ be such that $h_i \longrightarrow h$ with respect to the pointwise convergence. Then $h \in K$. Since, for all i, $h_i \in S(g)$,

$$G(g, h_i) - H(h_i, g) \not\subset C(h_i),$$

this implies that

$$G(g, h_i) - H(h_i, g) \subset Y \setminus C(h_i).$$

Also, $G(g,.) - H(.,g)$ is continuous and $Y \setminus C(f)$ is closed, for all $f \in K$, we have

$$G(g,h) - H(h,g) \subset Y \setminus C(h),$$

this means that $h \in S(g)$. Hence, for all $g \in K$, $S(g)$ is closed. Since, $S(g) \subset K$ and K is compact, $S(g)$ is compact, for all $g \in K$.

Claim 2: S is a KKM-mapping. i.e. for every finite set $\{g_1, g_2, \cdots, g_n\}$ of K, we have

$$conv\,\{g_1, g_2, \cdots, g_n\} \subseteq \bigcup_{i=1}^{n} S(g_i).$$

If possible, let us suppose that there exists a finite set $\{g_1, g_2, \cdots, g_n\}$ of K and non-negative real numbers $t_1, t_2, \cdots, t_n$ with $\sum_{i=1}^{n} t_i = 1$ such that

$$conv\,\{g_1, g_2, \cdots, g_n\} \not\subseteq \bigcup_{i=1}^{n} S(g_i).$$

Then, there exists $h \in conv\{g_1, g_2, \cdots, g_n\}$ such that

$$h \notin \bigcup_{i=1}^{n} S(g_i),$$

this implies that

$$h \notin S(g_i), \forall, i = 1, 2, \cdots, n.$$

Hence we have

$$G(g_i, h) - H(h, g_i) \subset C(h),\ \forall i = 1, 2, \cdots, n.$$

This implies that

$$\sum_{i=1}^{n} t_i G(g_i, h) - H(h, g_i) \subset C(h). \tag{3.1}$$

Next, Since G is P-monotone,

$$G(g_i, h) + G(h, g_i) \subset -P \subset -C(h),$$

it implies that

$$G(g_i, h) \subset -G(h, g_i) - C(h).$$

Hence we have,

$$\begin{aligned}\sum_{i=1}^{n} G(g_i, h) &\subset -\sum_{i=1}^{n} G(h, g_i) - C(h)\\ &\subset -[G(h, h) + P] - C(h)\\ &= -P - C(h)\\ &\subset -C(h) - C(h)\\ &\subset -C(h).\end{aligned} \tag{3.2}$$

Also,

$$\sum_{i=1}^{n} t_i H(h, g_i) \subset H(h, h) + P \subset C(h). \tag{3.3}$$

From (3.2) and (3.3), we get

$$\sum_{i=1}^{n} t_i\left[G(g_i, h) - H(h, g_i)\right] \subset -C(h) - C(h) \subseteq -C(h). \tag{3.4}$$

From (3.1) and (3.4), we get

$$\sum_{i=1}^{n} t_i\left[G(g_i, h) - H(h, g_i)\right] \subset C(h) \cap (-C(h)) = \Phi,$$

which is a contradiction. Hence, S is a KKM-mapping. So, we have

$$conv\{g_1, g_2 \cdots, g_n\} \subseteq \bigcup_{i=1}^{n} S(g_i).$$

Therefore by using Ky Fan Lemma, we have

$$\bigcap_{g\in K} S(g) \neq \phi.$$

Hence, there exists $f_0 \in K$ such that

$$G(g, f_0) - H(f_0, g) \not\subset C(f_0), \text{ for all } g \in K.$$

$\square$

Lemma 3.4. *Let $G, H : D \times D \longrightarrow 2^Y$ be two set-valued maps and $D \subset L(X, Y)$ be a non-empty, closed and convex subset such that*

(i) *for all $f \in D$, $G(f, f) = \{0\}$, $H(f, f) = \{0\}$,*
(ii) *the graph of $W(f) = Y \setminus (-C(f))$ is closed for all $f \in D$,*
(iii) *G is P-monotone.*
(iv) *for any fixed $f, g, h \in D$, $t \longrightarrow G(tf + (1-t)g, h)$, where $t \in [0, 1]$, is upper semicontinuous with respect to $C(f)$ at t=0,*
(v) *for any $f \in D$, $G(f, .), H(f, .) : D \longrightarrow 2^Y$ are P-convex.*

Then the following statements are equivalent:

(a) *For $f \in D$, $G(g, f) - H(f, g) \not\subset C(f)$, for all $g \in D$.*
(b) *For $f \in D$, $G(f, g) + H(f, g) \not\subset -C(f)$, for all $g \in D$.*

Proof. $(a) \Rightarrow (b)$: Suppose that there exist $f \in D$ such that

$$G(g, f) - H(f, g) \not\subset C(f), \text{ for all } g \in D.$$

For any $g \in D$ and $t \in (0, 1]$, $f_t = tg + (1-t)f \in D$. Hence we have

$$G(f_t, f) - H(f, f_t) \not\subset C(f).$$

Since $G(f, .)$ is P-convex, it follows that

$$tG(f_t, g) + (1-t)G(f_t, f) \subseteq G(f_t, f_t) + P = P. \tag{3.5}$$

Also, $H(f, .)$ is P-convex, we have

$$tH(f, g) \subseteq tH(f, g) + (1-t)H(f, f) \subseteq H(f, f_t) + P,$$

that is,

$$tH(f, g) - H(f, f_t) \subseteq P. \tag{3.6}$$

Adding (3.5) and (3.6), we get

$$tG(f_t, g) + (1-t)G(f_t, f) + t(1-t)H(f, g) - (1-t)H(f, f_t) \subseteq P \subseteq C(f),$$

it implies that

$$(t-1)\left[G(f_t, f) - H(f, f_t)\right] - \left[tG(f_t, g) + t(1-t)H(f, g)\right] \subseteq P \subseteq C(f).$$

Hence we have,

$$G(f_t, f) + (1-t)H(f, g) \not\subset -C(f).$$

Letting $t \to 0^+$ and $f_t \to f$ with respect to pointwise convergence. Since $W(f)$ is closed and G is hemicontinuous in the first argument, we have

$$G(f,g) + H(f,g) \not\subset -C(f).$$

$(b) \Rightarrow (a)$: Suppose (b) holds. Then ther exists $f \in K$ such that

$$G(f,g) + H(f,g) \not\subset -C(f), \quad \forall g \in D.$$

If possible, suppose that there exists $g_0 \in D$ such that

$$G(g_0, f) - H(f, g_0) \subset C(f).$$

Then we have

$$H(f, g_0) \leq_{C(f)} G(g_0, f). \tag{3.7}$$

Also, since G is P-monotone

$$G(g_0, f) + G(f, g_0) \leq_P 0.$$

It implies that

$$G(g_0, f) + G(f, g_0) \subset -P.$$

Hence we have

$$G(g_0, f) + G(f, g_0) \subset -C(f),$$

that is,

$$G(g_0, f) \leq_{C(f)} -G(f, g_0). \tag{3.8}$$

From (3.7) and (3.8), we get

$$H(f, g_0) \leq_{C(f)} -G(f, g_0),$$

that is,

$$H(f, g_0) - G(f, g_0) \leq_{C(f)} 0.$$

Therefore, we have

$$H(f, g_0) - G(f, g_0) \subset -C(f),$$

which is a contradiction. Hence, there exists $f \in D$ such that

$$G(g, f) - H(f, g) \not\subset C(f), \quad \forall g \in D.$$

$\square$

Lemma 3.5. *Let $K \subset D$ be as in Theorem 3.2, and let $\Phi : D \longrightarrow 2^Y$ be given. Assume that*

(i) *Φ is P-convex,*
(ii) *$f \in Core_D K, \ \Phi(f) \subset -C(f)$,*
(iii) *$\forall g \in K, \ \Phi(g) \not\subset -C(f)$.*

Then $\Phi(g) \not\subset C(f), \ \forall g \in D$.

Proof. Assume on the contrary, that there exists $g \in D \setminus K$ such that $\Phi(g) \supset -C(f)$. Let $h \in (f, g],\ h = tf + (1-t)g$, for some $t \in [0, 1)$. Then, since Φ is convex, we have

$$t\Phi(f) + (1-t)\Phi(g) \subseteq \Phi(h) + P.$$

Hence, we have

$$\begin{aligned} -\Phi(h) &\subseteq -t\Phi(f) + (1-t)\Phi(g) + P \\ &\subseteq tC(f) + (1-t)C(f) + P \\ &\subseteq C(f) + C(f) + C(f) \\ &\subset C(f), \end{aligned}$$

that is,

$$\Phi(h) \subseteq -C(f), \text{ for all } h \in (f, g].$$

Since $f \in Core_D K$, there exists $h_0 \in (f, g] \cap K$. Hence $\Phi(h_0) \subset -C(f)$. This contradicts $\Phi(g) \not\subset -C(f)$, for all $g \in K$ and the result is complete.

Proof of the Theorem 3.2: By Lemma 3.5, there exists $f \in K$ such that

$$G(g, f) - H(f, g) \not\subset C(f), \ \forall\ g \in K.$$

It follow from Lemma 3.5, that

$$G(f, g) + H(f, g) \not\subset -C(f), \ \forall g \in K.$$

Let $\Phi(g) = G(f, g) + H(f, g)$, for all $g \in D$. Then Φ is P-convex. Also from above, we have

$$\Phi(g) \not\subset -C(f), \ \forall g \in K.$$

If $f_0 \in Core_D K$, we set $f = f_0$, otherwise, we set $f = h$, where h is from assumption (vi). Then we always have $\Phi(f) \subset -C(f)$. Using Lemma 3.4, we conclude $\Phi(g) \not\subset -C(f)$, foa all $g \in D$. That is,

$$G(f_0, g) + H(f_0, g) \not\subset -C(f_0), \ \forall g \in D.$$

□

Remark 3.6. We can replace the assumption (vii) in Theorem 3.2 by the following assumption:

$(vii)'$ There exist a non-empty, convex compact subset $B \subset D$ such that, for all $f \in D \setminus B$, there exists $f_0 \in B$ such that

$$G(h, f) + H(f, h) \subset -C(f_0). \tag{3.9}$$

Theorem 3.7. Suppose the assumptions (i)- (vii) of Theorem 3.2 and $(vii)'$ hold.Then there exists $f_0 \in B$ such that

$$G(f_0, g) + H(f_0, g) \not\subset -C(f_0), \text{ for all } g \in D.$$

Proof. For $g \in D$, define

$$F(g) = \{f_0 \in B : G(f_0, g) - H(f_0, g) \not\subset -C(f_0),\ \forall g \in D\}. \tag{3.10}$$

Then F has the finite intersection property. In fact, let $K = conv(B \cup \{g_1, g_2, \cdots, g_n\} \subset D$. Then K is a compact convex subset of D. Applying the hypotheses of Lemma 3.3, there exists $f_0 \in K$ such that

$$G(g, f_0) - H(f_0, g) \not\subset -C(f_0),\ \forall g \in D.$$

If $f_0 \in K \setminus B$, then by the condition $(vii)'$, there exists $h \in K$ such that

$$G(h, f_0) - H(f_0, h) \subset C(f_0),$$

which is a contradiction to (3.10). We conclude that $f_0 \in B$. Hence $f_0 \in \bigcap_{i=1}^{n} F(g_i)$, that is, $F(g)$ has finite intersection property. Since B is compact, we have $f_0 \in \bigcap_{g \in D} F(g)$. Thus, $f_0 \in B$ and

$$G(g, f_0) - H(f_0, g) \not\subset C(f_0),\ \forall g \in D.$$

By Lemma 3.4, it follows that

$$G(f_0, g) + H(f_0, g) \not\subset -C(f_0),\ \forall g \in D.$$

Thia completes the proof. □

Remark 3.8.

(i) When G and H are single-valued and $K \subset X$, then Theorem 3.2 and Theorem 3.7 reduces to Theorem 7 and Theorem 11, respectively of Kazmi [10].

(ii) When G and H are single valued, then Theorem 3.2 and Theorem 3.7 reduces to Theorem 3.1 and Theorem 3.2, respectively of Kazmi and Raouf [12].

Acknowledgments: This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

References

[1] Q.H. Ansari, E. Kobis and J.C. Yao, *Vector variational inequalities and vector optimization Theory and applications*, Springer international Publishing AG, Switzerland, 2018.

[2] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145.

[3] G.Y. Chen, *Existence of solutions for a vector variational inequality: An extension of the Hartmann-Stampacchia theorem*, J. Optim. Theory Appl., **74** (1992), 445–456.

[4] A. Domokos and J. Kolumban, *Variational inequalities with operator solutions*, J. Glob. Optim., **23** (2002), 99–110.

[5] K. Fan, *A minimax inequality and application, Inequalities III*, Shisha, Academic Press, 1972.

[6] J.Y. Fu, *Generalized vector quasi-equilibrium problems*, Math. Meth. of Operat. Research, **52** (2000), 57–64.

[7] J.Y. Fu, *Vector equilibrium problems. Existence theorems and convexity of solution set*, J. Global Optim., **31** (2005), 109–119.

[8] F. Giannessi, *Vector Variational Inequality and Vector Equilibria: Mathematical Theories*, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 2000.

[9] N. Hadjisavas and S. Schaible, *From scalar to vector equilibrium problems in the quasi-monotone case*, J. Optim. Theory Appl., **96** (1998), 297–309.

[10] K.R. Kazmi, *On vector equilibrium problem*, Proc. Indian Acad. Sci. (Math. Sci.,) **2** (2000), 213–223.

[11] K.R. Kazmi and A. Raouf, *A class of operator equilibrium problems*, J. Math. Anal. Appl., **308** (2005), 554–564.

[12] K.R. Kazmi and A. Raouf, *On perturbed equilibrium problems with operator solutions*, South East Asian J. Math. and Math. Sci., **8** (2009), 91–100.

[13] G. Kassay and M. Miholca, *Existence results for vector equilibrium problems given by a sum of two functions*, J. Glob Optim., **63** (2015), 195–211.

[14] J.K. Kim and A. Raouf, *A class of generalized operator equilibrium problems*, Filomat, **31**(1) (2017), 1–8.

[15] S. Kum , *A variant of the generalized vector variational inequalities with operator solutions*, Commun. Kor. Math. Soc., **21**(4) (2006), 665–673.

[16] T. Ram, *On existence of operator solutions of generalized vector quasi- variational inequalities*, Commun. in Optim. Theory, 2015:1, (2015), 1–8.

[17] T. Ram and A.K. Khanna, *On perturbed quasi- equilibrium problems with operator solutions*, Nonlinear Funct. Anal. Appl., **22**(2) (2017), 385–394.

[18] N.X. Tan and P.N. Tinh, *On the existence of equilibrium points of vector functions*, Numer.Funct. Anal. Optim., **19**(1-2) (1998), 141–156.

[19] S.J. Yu and J.C. Yao, *On vector variational inequalities*, J. Optim. Theory Appl., **89** (1996), 749–769.