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COMMON COUPLED COINCIDENCE POINT RESULTS FOR TWO MAPS IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS

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Abstract. In this paper, we obtain several common coupled coincidence point results and coupled fixed point results for two mappings satisfying certain contractive condition on complete cone b-metric spaces over Banach algebra, using the properties of spectral radiuses. Also we give two simple examples and obtain the existence and uniqueness of solution for some equations by using our results.

1. Introduction

In 2007, Huang and Zhang [4] introduced the concept of cone metric space which is the generalization of metric space by replacing the set of real numbers with an ordering Banach space, and proved some fixed point theorems for contractive mappings on these spaces. Recently, in ([1], [3], [4], [8]-[10], [12]-[15]), some common fixed point theorems have been proved for contractive maps on cone metric spaces. Gnana Bhaskar and Lakshmikantham [2] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is used in various directions and also extended in various spaces like metric space, partially ordered metric space, fuzzy metric space, cone metric space, etc [7].

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Very recently, Liu and Xu [6] introduced the concept of cone metric space over Banach algebra by replacing Banach space with Banach algebra and proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constant k by means of spectral radius and pointed out that it is significant to introduce this concept because it can be proved that cone metric spaces over Banach algebras are not equivalent to metric spaces in terms of the existence of the fixed points of the generalized Lipschitz mappings. In the past three years, some researchers started to study the existence problems of (coupled) fixed points for some contractions in cone metric spaces over Banach algebras (see [5], [6], [11], [13], [14]).

Motivated by the above works, in this paper, we obtain several common coupled coincidence point results and coupled fixed point results for two mappings satisfying certain contractive condition on complete cone b-metric spaces over Banach algebra, using the properties of spectral radiuses. Also we give two simple examples and obtain the existence and uniqueness of solution for some equations by using our results. Our results generalize the corresponding results in cone metric spaces obtained by Nashie et. al. [7], Liu et. al. [6] or Xu et. al. [10].

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that A is a real Banach algebra with a unit (i.e., a multiplicative identity) e. An element $x \in A$ is said to be *invertible* if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Let A be a real Banach algebra with a unit e and θ the zero element of A. A nonempty closed subset P of Banach algebra A is called a *cone* if

(1) $\{\theta, e\} \subset P$;
(2) $\alpha P + \beta y P \subset P$ for all nonnegative real numbers α, β ;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \neq \emptyset$ then P is called a *solid cone*.

Definition 1.1. Let X be a nonempty set, $s \geq 1$ be a constant and A be a real Banach algebra. Suppose the mapping $d : X \times X \to A$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a *cone b-metric* on X, and (X, d) is called a *cone b-metric space* over Banach algebra A.

Example 1.2. Let $A = C[a, b]$ be the set of continuous functions on $[a, b]$ with the supremum. Define the multiplication in the usual way. Then A is a Banach algebra with a unit 1. Set $P = \{x \in A : x(t) \geq 0,\ t \in [a, b]\}$ and $X = \mathbb{R}$. We define a mapping $d : X \times X \to A$ by $d(x, y)(t) = |x - y|^p e^t$ for all $x, y \in X$ and for each $t \in [a, b]$, where $p > 1$ is a constant. This makes (X, d) into a cone b-metric space over Banach algebra A with the coefficient $s = 2^{p-1}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

In the following, we always assume that (X, d) is a cone b-metric space over Banach algebra A.

Definition 1.3. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$.

(1) If for every $c \in A$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x, and the point x is the *limit* of $\{x_n\}$. We denote this by
$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$
(2) If for all $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Definition 1.4. Let E be a real Banach space with a solid cone P. A sequence $\{x_n\} \subset P$ is called a *c−sequence* if for any $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $x_n \ll c$ for all $n \geq N$.

Lemma 1.5. ([8]) *Let E be a real Banach space with a cone P. Then*

(1) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(2) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*
(3) *If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.*

(4) *If* $\{x_n\}, \{y_n\}$ *are sequences in* E *such that* $x_n \to x$, $y_n \to y$ *and* $x_n \preceq y_n$ *for all* $n \geq 1$, *then* $x \preceq y$.

Lemma 1.6. ([10]) *Let* A *ba a real Banach algebra with a unit* e *and* P *be a solid cone in* A. *We define the spectral radius* $r(x)$ *of* $x \in A$ *by*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

(1) *If* $0 \leq r(x) < 1$, *then* $e - x$ *is invertible,*

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i \quad \text{and} \quad r((e-x)^{-1}) \leq \frac{1}{1-r(x)}.$$

(2) *If* $r(x) < 1$, *then* $\|x^n\| \to 0$ *as* $n \to \infty$.
(3) *If* $x \in P$ *and* $r(x) < 1$, *then* $(e-x)^{-1} \in P$.
(4) *If* $k, u \in P$, $r(k) < 1$ *and* $u \preceq ku$, *then* $u = \theta$.
(5) $r(x) \leq \|x\|$ *for all* $x \in A$.
(6) *If* $x, y \in A$ *and* x, y *commute, then the following holds:*
 (a) $r(xy) \leq r(x)r(y)$,
 (b) $r(x+y) \leq r(x) + r(y)$ *and*
 (c) $|r(x) - r(y)| \leq r(x-y)$.

Lemma 1.7. ([8], [10]) *Let* (X, d) *be a complete cone* b*-metric space over Banach algebra* A *and let* P *be a solid cone in* A. *Let* $\{x_n\}$ *be a sequence in* X. *Then*

(1) *If* $\|x_n\| \to 0$ *as* $n \to \infty$, *then* $\{x_n\}$ *is a* $c-$*sequence.*
(2) *If* $k \in P$ *is any vector and* $\{x_n\}$ *is* $c-$*sequence in* P, *then* $\{kx_n\}$ *is a* $c-$*sequence.*
(3) *If* $x, y \in A$, $a \in P$ *and* $x \preceq y$, *then* $ax \preceq ay$.
(4) *If* $\{x_n\}$ *converges to* $x \in X$, *then* $\{d(x_n, x)\}$, $\{d(x_n, x_{n+p})\}$ *are* c*-sequences for any* $p \in \mathbb{N}$.

Lemma 1.8. ([10]) *Let* P *be a solid cone in a real Banach algebra* A *and* $k \in P$. *If* $r(k) < 1$,*then the following assertions hold true:*

(1) *If* $u \in P$ *and* $u \preceq ku$, *then* $u = \theta$.
(2) *If* $k \succeq \theta$,*then* $(e-k)^{-1} \succeq \theta$.

Definition 1.9. ([2], [10]) Let (X, d) be a cone b-metric space over Banach algebra A.

(1) An element $(x, y) \in X \times X$ is called a *coupled fixed point* of $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

(2) An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called coupled point of coincidence;
(3) An element $(x, y) \in X \times X$ is called a *common coupled fixed point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.
(4) The mappings $F : X \times X \to X$ and $g : X \times X$ are called *weakly compatible* if $g(F(x, y)) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

2. Main results

In this section, we give common coupled coincidence point results and coupled fixed point results for two mappings $S, T : X \times X \to X$ under some natural and certain contractive condition given by fixed mapping g defined on a complete cone b-metric space X over Banach algebra. The following results generalize the corresponding results in cone metric spaces obtained by Nashie et. al. [7].

Theorem 2.1. *Let (X, d) be a complete cone b-metric space over Banach algebra A and let P be a solid cone in A. Suppose that $S, T : X \times X \to X$ and $g : X \to X$ are mappings satisfying the condition:*

$$\begin{aligned} d(S(x,y),T(u,v)) \;\preceq\; & a_1 d(gx,gu) + a_2 d(gy,gv) \\ & + a_3 d(S(x,y),gx) + a_4 d(T(u,v),gu) \\ & + a_5 d(S(x,y),gu) + a_6 d(T(u,v),gx) \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $(i = 1, 2, 3, 4, 5, 6)$ and

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + s^2 r(a_5) + (s^2 + s)r(a_6) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a common coupled coincidence point in X.

Proof. Let x_0 and y_0 be two arbitrary elements in X. Since $S(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = S(x_0, y_0)$ and $gy_1 = S(y_0, x_0)$. Again noting $T(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = T(x_1, y_1)$ and $gy_2 = T(y_1, x_1)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{2n+1} = S(x_{2n}, y_{2n}), gy_{2n+1} = S(y_{2n}, x_{2n}), gx_{2n+2} = T(x_{2n+1}, y_{2n+1})$ and $gy_{2n+2} = T(y_{2n+1}, x_{2n+1})$. For each $n \in \mathbb{N}$, by the given conditions, we have

$$\begin{aligned}
d(gx_{2k+1}, gx_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\
&\preceq a_1 d(gx_{2k}, gx_{2k+1}) + a_2 d(gy_{2k}, gy_{2k+1}) \\
&+ a_3 d(S(x_{2k}, y_{2k}), gx_{2k}) + a_4 d(T(x_{2k+1}, y_{2k+1}), gx_{2k+1}) \\
&+ a_5 d(S(x_{2k}, y_{2k}), gx_{2k+1}) + a_6 d(T(x_{2k+1}, y_{2k+1}), gx_{2k}) \\
&= a_1 d(gx_{2k}, gx_{2k+1}) + a_2 d(gy_{2k}, gy_{2k+1}) \\
&+ a_3 d(gx_{2k+1}, gx_{2k}) + a_4 d(gx_{2k+2}, gx_{2k+1}) \\
&+ a_5 d(gx_{2k+1}, gx_{2k+1}) + a_6 d(gx_{2k+2}, gx_{2k}) \\
&\preceq a_1 d(gx_{2k}, gx_{2k+1}) + a_2 d(gy_{2k}, gy_{2k+1}) \\
&+ a_3 d(gx_{2k+1}, gx_{2k}) + a_4 d(gx_{2k+2}, gx_{2k+1}) + a_5 \cdot \theta \\
&+ s a_6 [d(gx_{2k}, gx_{2k+1}) + d(gx_{2k+1}, gx_{2k+2})],
\end{aligned}$$

which implies that

$$\begin{aligned}
(e - a_4 - s a_6) d(gx_{2k+1}, gx_{2k+2}) &\preceq (a_1 + a_3 + s a_6) d(gx_{2k}, gx_{2k+1}) \\
&\quad + a_2 d(gy_{2k}, gy_{2k+1}).
\end{aligned}$$

Since $r(a_4 + s a_6) < 1$, by hypothesis and Lemma 1.6, $e - (a_4 + s a_6)$ is invertible. Putting

$$\alpha = (e - a_4 - s a_6)^{-1}(a_1 + a_3 + s a_6), \ \beta = (e - a_4 - s a_6)^{-1} a_2,$$

we have

$$d(gx_{2k+1}, gx_{2k+2}) \preceq \alpha d(gx_{2k}, gx_{2k+1}) + \beta d(gy_{2k}, gy_{2k+1}). \tag{2.2}$$

Similarly, we have

$$\begin{aligned}
d(gy_{2k+1}, gy_{2k+2}) &= d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) \\
&\preceq a_1 d(gy_{2k}, gy_{2k+1}) + a_2 d(gx_{2k}, gx_{2k+1}) \\
&+ a_3 d(S(y_{2k}, y_{2k}), gy_{2k}) + a_4 d(T(y_{2k+1}, x_{2k+1}), gy_{2k+1}) \\
&+ a_5 d(S(y_{2k}, x_{2k}), gy_{2k+1}) + a_6 d(T(y_{2k+1}, x_{2k+1}), gy_{2k}) \\
&= a_1 d(gy_{2k}, gy_{2k+1}) + a_2 d(gx_{2k}, gx_{2k+1}) \\
&+ a_3 d(gy_{2k+1}, gy_{2k}) + a_4 d(gy_{2k+2}, gy_{2k+1}) \\
&+ a_5 d(gy_{2k+1}, gy_{2k+1}) + a_6 d(gy_{2k+2}, gy_{2k}) \\
&\preceq a_1 d(gy_{2k}, gy_{2k+1}) + a_2 d(gx_{2k}, gx_{2k+1}) \\
&+ a_3 d(gy_{2k+1}, gy_{2k}) + a_4 d(gy_{2k+2}, gy_{2k+1}) + a_5 \cdot \theta \\
&+ s a_6 [d(gy_{2k}, gy_{2k+1}) + d(gy_{2k+1}, gy_{2k+2})],
\end{aligned}$$

which implies that

$$d(gy_{2k+1}, gy_{2k+2}) \preceq \alpha d(gy_{2k}, gy_{2k+1}) + \beta d(gx_{2k}, gx_{2k+1}). \tag{2.3}$$

Adding both inequalities, we have

$$\begin{aligned} d(gx_{2k+1}, gx_{2k+2}) &+ d(gy_{2k+1}, gy_{2k+2}) \\ &\preceq (\alpha+\beta)[d(gx_{2k}, gx_{2k+1}) + d(gy_{2k}, gy_{2k+1})] \\ &= h[d(gx_{2k}, gx_{2k+1}) + d(gy_{2k}, gy_{2k+1})] \end{aligned}$$

where $h = \alpha + \beta = (e - a_4 - sa_6)^{-1}(a_1 + a_2 + a_3 + sa_6)$.

Also we have

$$d(gx_{2k+2}, gx_{2k+3}) + d(gy_{2k+2}, gy_{2k+3}) = h[d(gx_{2k+1}, gx_{2k+2}) + d(gy_{2k+1}, gy_{2k+2})].$$

Therefore

$$\begin{aligned} d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) &\preceq h[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\ &\vdots \\ &\preceq h^n[d(gx_0, gx_1) + d(gy_0, gy_1)]. \end{aligned}$$

By hypothesis and Lemma 1.6, we have

$$\begin{aligned} r(h) &\leq r((e - a_4 - sa_6)^{-1})r(a_1 + a_2 + a_3 + sa_6) \\ &\leq \frac{r(a_1) + r(a_2) + r(a_3) + sr(a_6)}{1 - r(a_4) - sr(a_6)}, < \frac{1}{s} \end{aligned}$$

which means that $e - h$ is invertible, $(e - h)^{-1} = \sum_{i=0}^{\infty} h^n$ and $\|h^n\| \to 0$ as $n \to \infty$.

Now if $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$, then the above relation implies

$$\delta_n \preceq h\delta_{n-1} \preceq \cdots \preceq h^n\delta_0.$$

For $m > n$, we have

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\preceq s[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\ &+ s[d(gy_n, gy_{n+1}) + d(gy_{n+1}, dy_m)] \\ &\preceq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) \\ &+ s^2 d(gx_{n+2}, gx_m) + sd(gy_n, gy_{n+1}) \\ &+ s^2 d(gy_{n+1}, gy_{n+2}) + s^2 d(gy_{n+2}, gy_m) \\ &\preceq \cdots \\ &\preceq s\delta_n + s^2\delta_{n+1} + \cdots + s^{m-n}\delta_{m-1} \\ &\preceq s(h^n + sh^{n+1} + \cdots + s^{m-n-1}h^{m-1})\delta_0 \\ &= sh^n[e + sh + (sh)^2 + \cdots + (sh)^{m-n-1}]\delta_0 \\ &\preceq sh^n(\sum_{i=0}^{\infty}(sh)^i)\delta_0 \\ &= (e - sh)^{-1}sh^n\delta_0, \end{aligned}$$

since $r(sh) < 1$ and P is closed. Since $r(h) < 1$, $\|h^n\| \to 0$ as $n \to \infty$ and so

$$\|(e - sh)^{-1} sh^n \delta_0\| \leq \|(e - sh)^{-1} s\| \|h^n\| \|\delta_0\| \to 0.$$

Thus for any $c \in A$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$d(gx_n, gx_m) + d(gy_n, gy_m) \preceq (e - h)^{-1} sh^n \delta_0 \ll c.$$

Thus $\{d(gx_n, gx_m) + d(gy_n, gy_m)\}$ is a c-sequence in P. Since

$$\theta \preceq d(gx_n, gx_m), d(gy_n, gy_m) \preceq d(gx_n, gx_m) + d(gy_n, gy_m),$$

$\{d(gx_n, gx_m)\}$ and $\{d(gy_n, gy_m)\}$ are c-sequences and so Cauchy sequence in X. Since X is complete, there exists $x \in X$ and $y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$. Now we show that $gx = S(x, y)$ and $gy = S(y, x)$. Then

$$\begin{aligned}
d(gx, S(x,y)) &\preceq sd(gx, gx_{2k+2}) + sd(gx_{2k+2}, S(x,y)) \\
&= sd(gx, gx_{2k+2}) + sd(T(x_{2k+1}, gy_{2k+1}), S(x,y)) \\
&\preceq sd(gx, gx_{2k+2}) + sa_1 d(gx, gx_{2k+1}) + sa_2 d(gy, gy_{2k+1}) \\
&\quad + sa_3 d(S(x,y), gx) + sa_4 d(T(x_{2k+1}, y_{2k+1}), gx_{2k+1}) \\
&\quad + sa_5 d(S(x,y), gx_{2k+1}) + sa_6 d(T(x_{2k+1}, y_{2k+1}), gx) \\
&= sd(gx, gx_{2k+2}) + sa_1 d(gx, gx_{2k+1}) + sa_2 d(gy, gy_{2k+1}) \\
&\quad + sa_3 d(S(x,y), gx) + sa_4 d(gx_{2k+2}, gx_{2k+1}) \\
&\quad + sa_5 d(S(x,y), gx_{2k+1}) + sa_6 d(gx_{2k+2}, gx),
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x, S(x,y)) &\preceq s(e + a_6) d(gx, gx_{2k+2}) + sa_1 d(gx, gx_{2k+1}) \\
&\quad + sa_2 d(gy, gy_{2k+1}) + sa_3 d(gx, S(x,y)) \\
&\quad + sa_4 d(gx_{2k+2}, gx_{2k+1}) + sa_5 d(S(x,y), gx_{2k+1}).
\end{aligned}$$

Since $d(S(x,y), gx_{2k+1}) \preceq sd(S(x,y), gx) + sd(gx, gx_{2k+1})$, we have

$$\begin{aligned}
(e - sa_3 - s^2 a_5) d(gx, S(x,y)) &\preceq (sa_1 + s^2 a_5) d(gx, x_{2k+1}) \\
&\quad + (se + a_6) d(gx, x_{2k+2}) \\
&\quad + sa_4 d(gx_{2k+1}, gx_{2k+2}) + sa_2 d(gy, y_{2k+1})
\end{aligned} \tag{2.4}$$

Since $r(sa_3 + s^2 a_5) < 1$, $e - sa_3 - s^2 a_5$ is invertible. By Lemma 1.6 and Lemma 1.7, the right-hand side of (2.4) is a c-sequence and so $d(x, S(x,y)) = \theta$. Therefore $gx = S(x,y)$. Similarly

$$\begin{aligned}
(e - sa_3 - s^2 a_5) d(gy, S(x,y)) &\preceq (sa_1 + s^2 a_5) d(gy, gy_{2k+1}) \\
&\quad + (se + a_6) d(gy, y_{2k+2}) \\
&\quad + sa_4 d(gy_{2k+1}, gy_{2k+2}) + sa_2 d(gx, gx_{2k+1})
\end{aligned} \tag{2.5}$$

and so we can prove that $gy = S(y, x)$. It follows similarly that

$$gx = T(x, y) \quad \text{and} \quad gy = T(y, x).$$

Therefore (x, y) is a common coupled coincidence point of S and T.

In order to prove the uniqueness, let $(x', y') \in X \times X$ be another common coupled coincidence point of S and T. Then

$$\begin{array}{rcl} d(gx, gx') & = & d(S(x,y), T(x',y')) \\ & \preceq & a_1 d(gx, gx') + a_2 d(gy, gy') + a_3 d(S(x,y), gx) \\ & + & a_4 d(T(x',y'), gx') + a_5 d(S(x,y), gx') + a_6 d(T(x',y'), gx) \\ & = & a_1 d(gx, gx') + a_2 d(gy, gy') + a_3 d(gx, gx) \\ & + & a_4 d(gx', gx') + a_5 d(gx, gx') + a_6 d(gx', gx) \\ & = & (a_1 + a_5 + a_6) d(gx', gx) + a_2 d(gy, gy') \end{array}$$

which implies that

$$(e - a_1 - a_5 - a_6) d(gx, gx') \preceq a_2 d(gy, gy').$$

Since $r(a_1 + a_5 + a_6) < 1$, $e - (a_1 + a_5 + a_6)$ is invertible and

$$d(gx, gx') \preceq (e - a_1 - a_5 - a_6)^{-1} a_2 d(gy, gy').$$

Similarly we can prove that

$$d(gy, gy') \preceq (e - a_1 - a_5 - a_6)^{-1} a_2 d(gx, gx').$$

Adding both sides, we get

$$d(gx, gx') + d(y, y') \preceq (e - a_1 - a_5 - a_6)^{-1} a_2 [d(gx, gx') + d(gy, gy')].$$

Since $r((e - a_1 - a_5 - a_6)^{-1} a_2) < 1$, by Lemma 1.8, we have

$$d(gx, gx') + d(gy, gy') = \theta.$$

Therefore $gx = gx'$ and $gy = gy'$. $\square$

Corollary 2.2. *Let (X, d) be a complete cone metric space over Banach algebra A and let P be a solid cone in A. Suppose that $S, T : X \times X \to X$ and $g : X \to X$ are mappings satisfying the condition*

$$\begin{array}{rcl} d(S(x,y), T(u,v)) & \preceq & a_1 d(gx, gu) + a_2 d(gy, gv) \\ & + & a_3 d(S(x,y), gx) + a_4 d(T(u,v), gu) \\ & + & a_5 d(S(x,y), gu) + a_6 d(T(u,v), gx) \end{array}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, 3, 4, 5, 6$ and

$$r(a_1) + r(a_2) + r(a_3) + r(a_4) + r(a_5) + 2r(a_6) < 1.$$

If $S(X \times X), T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S, T and g have a common coupled coincidence point in X.

Proof. Taking $s = 1$ in Theorem 2.1, we get the required result. $\square$

The following results generalize the corresponding results in cone metric spaces obtained by Nashie, Rohen and Thokchom [7].

Corollary 2.3. *Let (X, d) be a complete b-cone metric space over Banach algebra A and let P be a solid cone in A. Suppose that $S, T : X \times X \to X$ are mappings satisfying the condition:*

$$\begin{aligned} d(S(x,y), T(u,v)) &\preceq a_1 d(x,u) + a_2 d(y,v) + a_3 d(S(x,y), x) \\ &+ a_4 d(T(u,v), u) + a_5 d(S(x,y), u) + a_6 d(T(u,v), x) \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, 3, 4, 5, 6$ and

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + s^2 r(a_5) + (s^2 + s) r(a_6) < 1.$$

Then S and T have a common coupled fixed point in X.

Proof. Taking $g = I$ in Theorem 2.1, we get the required result. $\square$

Corollary 2.4. ([7]) *Let (X, d) be a complete metric space. Suppose that $S, T : X \times X \to X$ are two mappings satisfying the condition:*

$$\begin{aligned} d(S(x,y), T(u,v)) &\preceq a_1 d(x,u) + a_2 d(y,v) + a_3 d(S(x,y), x) \\ &+ a_4 d(T(u,v), u) + a_5 d(S(x,y), u) + a_6 d(T(u,v), x) \end{aligned}$$

for all $x, y, u, v \in X$, where a_i $(i = 1, 2, 3, 4, 5, 6)$ are non-negative real numbers such that $\sum_{i=1}^{5} a_i + 2a_6 < 1$. Then S and T have a common coupled fixed point in X.

Proof. Taking $A = \mathbb{R}$, $s = 1$ and $g = I$ in Theorem 2.1, we get the required result. $\square$

Corollary 2.5. *Let (X, d) be a complete cone b-metric space over the Banach algebra A and let P be a solid cone. Suppose that $T : X \times X \to X$ and $g : X \to X$ are mappings satisfying the condition:*

$$\begin{aligned} d(T(x,y), T(u,v)) &\preceq a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ a_3 d(T(x,y), gx) + a_4 d(T(u,v), gu) \\ &+ a_5 d(T(x,y), gu) + a_6 d(T(u,v), gx) \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ $(i = 1, 2, 3, 4, 5, 6)$ commute and

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + s^2 r(a_5) + (s^2 + s) r(a_6) < 1.$$

If $T(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then T and g have a coupled coincidence point in X.

Proof. Taking $S = T$ in Theorem 2.1, we get the required result. $\square$

Corollary 2.6. *Let (X,d) be a complete cone b-metric space over the Banach algebra A and let P be a solid cone. Suppose that $S : X \times X \to X$ and $g : X \to X$ are mappings satisfying the condition:*

$$d(S(x,y),S(u,v)) \preceq \alpha d(gx,gu) + \beta d(gy,gv)$$

for all $x,y,u,v \in X$, where α, $\beta \in P$ commute and $r(\alpha)+r(\beta) < \frac{1}{s}$. If $S(X\times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S and g have a coupled coincidence point in X.

Proof. Taking $T = S$ and $a_1 = \alpha, a_2 = \beta, a_3 = a_4 = a_5 = a_6 = \theta$ in Theorem 2.1, we get the required result. □

Corollary 2.7. *Let (X,d) be a complete cone b-metric space over the Banach algebra A and let P be a solid cone. Suppose that $S,T : X \times X \to X$ and $g : X \to X$ are mappings satisfying the condition:*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq ad(gx,gu) + bd(gy,gv) \\ &+ c[d(S(x,y),gx) + d(T(u,v),gu)] \\ &+ e[d(S(x,y),gu) + d(T(u,v),gx)] \end{aligned}$$

for all $x,y,u,v \in X$, where $a,b,c,e \in P$ commute and

$$s(r(a)+r(b)) + (s+1)r(c) + (2s^2+s)r(e) < 1.$$

If $S(X\times X), T(X\times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then S,T and g have a common coupled coincidence point in X.

Proof. Taking $a_1 = a, a_2 = b, a_3 = a_4 = c, a_5 = a_6 = d$ in Theorem 2.1, we get the required result. □

Corollary 2.8. *Let (X,d) be a complete cone metric space over the Banach algebra A and let P be a solid cone. Suppose that $T : X \times X \to X$ and $g : X \to X$ are mappings satisfying the condition:*

$$\begin{aligned} d(T(x,y),T(u,v)) &\preceq ad(gx,gu) + bd(gy,gv) \\ &+ c[d(T(x,y),gx) + d(T(u,v),gu)] \\ &+ e[d(T(x,y),gu) + d(T(u,v),gx)] \end{aligned}$$

for all $x,y,u,v \in X$, where $a,b,c,e \in P$ commute and

$$s(r(a)+r(b)) + (s+1)r(c) + (2s^2+s)r(e) < 1.$$

If $T(X\times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then T and g have a coupled coincidence point in X.

Proof. Taking $S = T$ in Corollary 2.7, we get the required result. □

Theorem 2.9. *Let (X,d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let $F,G : X \times X \to X$ and $g : X \to X$ be mappings which satisfy all the conditions of Theorem* 2.1*. If F,G and g are weakly compatible, then F,G and g have a unique common coupled fixed point. Moreover, common fixed point of F,G and g is of the form (u,u) for some $u \in X$.*

Proof. First we claim that coupled point of coincidence is unique. Suppose that $(x,y),(x^*,y^*) \in X \times X$ with

$$gx = F(x,y) = G(x,y), \quad gy = F(y,x) = G(y,x)$$

and

$$gx^* = F(x^*,y^*) = G(x^*,y^*), \quad gy^* = F(y^*,x^*) = G(y^*,x^*).$$

Using (2.2), we get

$$\begin{aligned} d(gx,gx^*) &= d(F(x,y),G(x^*,y^*)) \preceq a_1 d(gx,gx^*) + a_2 d(gy,gy^*) \\ &+ a_3 d(F(x,y),gx) + a_4 d(G(x^*,y^*),gx^*) \\ &+ a_5 d(F(x,y),gx^*) + a_6 d(G(x^*,y^*),gx) \\ &= (a_1 + a_5 + a_6) d(gx,gx^*) + a_2 d(gy,gy^*) \end{aligned}$$

and so

$$d(gx,gx^*) \preceq (a_1 + a_5 + a_6) d(gx,gx^*) + a_2 d(gy,gy^*). \tag{2.6}$$

Similarly, we have

$$d(gy,gy^*) \preceq (a_1 + a_5 + a_6) d(gy,gy^*) + a_2 d(gx,gx^*). \tag{2.7}$$

Thus

$$d(gx,gx^*) + d(gy,gy^*) \preceq (a_1 + a_2 + a_5 + a_6)(d(gx,gx^*) + d(gy,gy^*)).$$

Since $s \geq 1$ and $r(a_1) + r(a_2) + r(a_5) + r(a_6) < 1$, by Lemma 1.8(1), we have $d(gx,gx^*) + d(gy,gy^*) = \theta$, which implies that $gx = gx^*$ and $gy = gy^*$. Similarly we prove that $gx = gy^*$ and $gy = gx^*$. Thus $gx = gy$. Therefore (gx,gx) is a unique coupled point of coincidence of F,G and g.

Now, let $g(x) = u$. Then we have $u = g(x) = F(x,x) = G(x,x)$. By weak compatibility of F,G and g, we have

$$\begin{aligned} g(u) &= g(g(x)) = g(F(x,x)) = F(gx,gx) = F(u,u), \\ g(u) &= g(g(x)) = g(G(x,x)) = G(gx,gx) = G(u,u). \end{aligned}$$

Then (gu,gu) is a coupled point of coincidence of F,G and g. Consequently $gu = gx$. Therefore $u = gu = F(u,u) = G(u,u)$. Hence (u,u) is a unique common coupled fixed point of F,G and g. This completes the proof. $\square$

Now we give two examples as an application of the main result.

Example 2.10. Let $A = C^1_{\mathbb{R}}[0,1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in A$. Define multiplication in A as just pointwise multiplication. Then A is a real Banach algebra with unit $e = 1(e(t) = 1$ for all $t \in [0,1])$. The set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal.

Let $X = \{1,2,3\}$. Define $d : X \times X \to A$ by

$$\begin{aligned} d(1,2)(t) &= d(2,1)(t) = d(2,3)(t) = d(3,2)(t) = e^t, \\ d(1,3)(t) &= d(3,1)(t) = 2e^t, \quad d(x,x)(t) = \theta \end{aligned}$$

for all $t \in [0,1]$ and for each $x \in X$. Then (X,d) is a solid cone metric space over Banach algebra without normality [10].

Define two mappings $S, T : X \times X \to X$ by $S(x,y) = 1$ for any $(x,y) \in X \times X$, and

$$T(x,y) = \begin{cases} 2, & (x,y) = (3,1) \\ 1, & \text{otherwise.} \end{cases}$$

Let $g = I$ be the identity mapping and let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with

$$a_1(t) = a_2(t) = a_4(t) = 0.11,\ a_3(t) = a_6(t) = 0.1,\ a_5(t) = 0.35$$

for all $t \in [0,1]$. Then, by definition of spectral radius,

$$r(a_1) = r(a_2) = r(a_4) = 0.11, r(a_3) = r(a_6) = 0.1, r(a_5) = 0.35$$

and so $\sum_{i=1}^{5} r(a_i) + 2r(a_6) < 1$. Since

$$d(S(x,y), T(3,1))(t) = d(1,2)(t)) = e^t$$

for any $x, y \in X$, by careful calculations, we can get that for any $x, y, u, v \in X$, S and T satisfy the contractive condition (2.2) of Theorem 2.1. Hence the hypotheses are satisfied and so by Theorem 2.1, S and T have a common coupled fixed point in X. Since $S(1,1) = 1 = T(1,1)$, $(1,1)$ is a unique coupled fixed point of S and T.

Now we present an example showing that Corollary 2.5 is a proper extension of known results. In this example, the conditions of Corollary 2.5 are fulfilled.

Example 2.11. (The case of normal cone) Let $A = \mathbb{R}^2$ and define a norm on A by $\|(x_1,x_2)\| = |x_1| + |x_2|$ for $x = (x_1,x_2) \in A$. Define the multiplication in A by

$$(x_1,x_2)(y_1,y_2) = (x_1y_1, x_2y_2).$$

Put $P = \{x = (x_1,x_2) \in A : x_1, x_2 \geq 0\}$. Then P is a normal cone and A is a real Banach algebra with unit $e = (1,1)$.

Let $X = [0,\infty)$. Define a mapping $d : X \times X \to A$ by

$$d(x,y) = (|x-y|^2, |x-y|^2)$$

for each $x, y \in X$. Then (X, d) is a complete cone b-metric space over Banach algebra with the coefficient $s = 2$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Consider the mappings $S : X \times X \to X$ and $g : X \to X$ defined by

$$S(x, y) = x + \frac{|\sin y|}{2} \quad \text{and} \quad g(x) = 3x.$$

Then $S(X \times X) \subseteq g(X) = X$. Let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with

$$\begin{aligned} a_1 &= (\frac{2}{9}, \frac{2}{9}),\ a_2 = (\frac{1}{18}, \frac{1}{18}),\ a_3 = (\frac{1}{54}, \frac{1}{54}), \\ a_4 &= (\frac{1}{27}, \frac{1}{27}),\ a_5 = (\frac{1}{108}, \frac{1}{108}),\ a_6 = (\frac{111}{2,000}, \frac{111}{2,000}). \end{aligned}$$

Then, by definition of spectral radius,

$$\begin{aligned} r(a_1) &= \frac{2}{9},\ r(a_2) = \frac{1}{18},\ r(a_3) = \frac{1}{54}, \\ r(a_4) &= \frac{1}{27},\ r(a_5) = \frac{1}{108},\ r(a_6) = \frac{111}{2,000}, \end{aligned}$$

and so

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + s^2 r(a_5) + (s^2 + s) r(a_6) = 0.9996 < 1.$$

By careful calculations, it is easy to verify that for any $x, y, u, v \in X$, S and g satisfy the contractive condition of Corollary 2.5. Thus by Corollary 2.5, S and g have a coupled coincidence point in a complete cone b-metric space X over Banach algebra $A = \mathbb{R}^2$. Since $S(0, 0) = g0 = 0$, $(0, 0)$ is the common coupled coincidence point of F and g.

Theorem 2.12. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f, h : X \to X$ and $g : X \to X$ satisfy*

$$\begin{aligned} d(fx, hu) \quad &\preceq \quad a_1 d(gx, gu) + a_2 d(gy, gv) + a_3 d(fx, gx) \\ &+ \quad a_4 d(hu, gu) + a_5 d(fx, gu) + a_6 d(hu, gx) \end{aligned} \tag{2.8}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, 3, 4, 5, 6$ and

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + s^2 r(a_5) + (s^2 + s) r(a_6) < 1.$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point.

Moreover, if f, h and g are weakly compatible, then f, h and g have a unique common coupled fixed point.

Proof. Let $f, h : X \to X$ be mappings satisfying the hypotheses. Define the mappings $S, T : X \times X \to X$ by

$$S(x, y) = fx, \quad T(x, y) = hx, \quad x, y \in X.$$

From (2.8), we get

$$\begin{aligned} d(S(x,y), T(u,v)) &\preceq a_1 d(gx, gu) + a_2 d(gy, gv) \\ &+ a_3 d(S(x,y), gx) + a_4 d(T(u,v), gu) \\ &+ a_5 d(S(x,y), gu) + a_6 d(T(u,v), gx) \end{aligned}$$

for all $x, y, u, v \in X$. Thus the contractive condition (2.2) is satisfied.

On the other hand, from the definitions of S and T, we have $S(X \times X) = f(X) \subseteq g(X)$ and $T(X \times X) = h(X) \subseteq g(X)$. Also, $g(X)$ is a complete subspace of (X, d). Now, applying Theorem 2.1, we obtain that S, T and g have a coupled coincidence point in X, that is, there exists $(x, y) \in X \times X$ such that $gx = S(x, y) = T(x, y)$ and $gy = S(y, x) = T(y, x)$. From the definitions of S and T, this implies that $gx = fx = hx$, that is, x is a coincidence point of f, h and g.

If f, h and g are weakly compatible, then S, T and g are weakly compatible. By Theorem 2.9, S, T and g have a unique common coupled fixed point and so f, h and g have a unique common coupled fixed point. □

Corollary 2.13. *Let (X, d) be a complete cone metric space over Banach algebra A with the underlying solid cone P. Let the mappings $f, h : X \to X$ and $g : X \to X$ satisfy*

$$\begin{aligned} d(fx, hu) &\preceq a_1 d(gx, gu) + a_2 d(gy, gv) + a_3 d(fx, gx) \\ &+ a_4 d(hu, gu) + a_5 d(fx, gu) + a_6 d(hu, gx) \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i \in P$ commute for $i = 1, 2, 3, 4, 5, 6$ and

$$r(a_1) + r(a_2) + r(a_3) + r(a_4) + r(a_5) + 2r(a_6) < 1.$$

If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point.

Proof. The proof follows by taking $s = 1$ in Theorem 2.12. □

Corollary 2.14. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f : X \to X$ and $g : X \to X$ satisfy*

$$\begin{aligned} d(fx, fu) &\preceq a_1 d(gx, gu) + a_2 d(gy, gv) + a_3 d(fx, gx) \\ &+ a_4 d(fu, gu) + a_5 d(fx, gu) + a_6 d(fu, gx) \end{aligned}$$

for all $x, y, u, v \in X$, *where* $a_i \in P$ *commute for* $i = 1, 2, 3, 4, 5, 6$ *and*

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + s^2 r(a_5) + (s^2 + s) r(a_6) < 1.$$

If $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a coupled coincidence point.*

Proof. The proof follows by taking $h = f$ in Theorem 2.12. □

Corollary 2.15. *Let* (X, d) *be a complete cone b-metric space over Banach algebra* A *with the coefficient* $s \geq 1$ *and the underlying solid cone* P. *Let the mappings* $f, h : X \to X$ *satisfy*

$$\begin{aligned} d(fx, hu) \quad &\preceq \quad a_1 d(x, u) + a_2 d(y, v) + a_3 d(fx, x) \\ &+ \quad a_4 d(hu, u) + a_5 d(fx, u) + a_6 d(hu, x) \end{aligned}$$

for all $x, y, u, v \in X$, *where* $a_i \in P$ *commute for* $i = 1, 2, 3, 4, 5, 6$ *and*

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + s^2 r(a_5) + (s^2 + s) r(a_6) < 1.$$

Then f *and* h *have a common coupled fixed point.*

Proof. The proof follows by taking $g = I$ in Theorem 2.12. □

The following Corollary is a generalization of Theorem 2.1 of Liu et. al. [6] or Theorem 3.1 of Xu et. al. [10] or Corollary 2.10 of Huang and Radenovic [5].

Corollary 2.16. *Let* (X, d) *be a complete cone b-metric space over Banach algebra* A *with the coefficient* $s \geq 1$ *and the underlying solid cone* P. *Let the mappings* $f, h : X \to X$ *and* $g : X \to X$ *satisfy*

$$d(fx, hy) \quad \preceq \quad k d(gx, gy)$$

for all $x, y \in X$, *where* $k \in P$ *and* $r(k) < \frac{1}{s}$. *If* $f(X), h(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* f, h *and* g *have a common coupled coincidence point.*

Proof. The proof follows by taking $a_1 = k, a_2 = a_3 = a_4 = a_5 = a_6 = \theta$ in Theorem 2.12. □

If $g = I$ in the above Corollary, then f and h have a common fixed point.

Corollary 2.17. *Let* (X, d) *be a complete cone b-metric space over Banach algebra* A *with the coefficient* $s \geq 1$ *and the underlying solid cone* P. *Let* $f : X \to X$ *be a self-map of* X *satisfying*

$$d(fx, fy) \quad \preceq \quad k[d(x, f^m z) + d(y, f^m z)] \tag{2.9}$$

for some $m \in \mathbb{N}$ and all $x, y, z \in X$, where $k \in P$ and $r(k) < \frac{1}{s}$. Then f has a unique fixed point in X.

Proof. Let $u \in X$. In (2.9), put $x = f^{m-1}u, y = f^m u, z = u$. Then we have $d(T^m u, T^{m+1}u) \preceq kd(T^{m-1}u, T^m u)$ which is a special case of Corollary 2.16. Thus f has a fixed point p in X. Condition (2.9) implies uniqueness. □

Corollary 2.18. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f, h : X \to X$ and $g : X \to X$ satisfy*

$$d(fx, hu) \preceq k[d(gx, gu) + d(gy, gv)]$$

for all $x, y, u, v \in X$, where $k \in P$ and $r(k) < \frac{1}{2s}$. If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point.

Proof. The proof follows by taking $a_1 = a_2 = k, a_3 = a_4 = a_5 = a_6 = \theta$ in Theorem 2.12. □

The following Corollary is a generalization of Theorem 2.3 of Liu et. al. [6] or Theorem 3.3 of Xu et. al. [10] or Corollary 2.11 of Huang and Radenovic [5].

Corollary 2.19. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f, h : X \to X$ and $g : X \to X$ satisfy*

$$d(fx, hy) \preceq k[d(fx, gx) + d(hy, gy)]$$

for all $x, y \in X$, where $k \in P$ and $r(k) < \frac{1}{s+1}$. If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point.

Proof. The proof follows by taking $a_1 = a_2 = a_5 = a_6 = \theta, a_3 = a_4 = k$ in Theorem 2.12. □

Corollary 2.20. ([5]) *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f : X \to X$ and $g : X \to X$ satisfy*

$$d(fx, fy) \preceq k[d(fx, gx) + d(fy, gy)]$$

for all $x, y \in X$, where $k \in P$ and $r(k) < \frac{1}{s+1}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a coupled coincidence point.

Proof. The proof follows by taking $h = f$ in Corollary 2.19. □

If $g = I$ in the above Corollary, then f has a fixed point.

The following corollary is a generalization of Theorem 2.2 of Liu et. al. [6] or Theorem 3.2 of Xu et. al. [10].

Corollary 2.21. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f, h : X \to X$ and $g : X \to X$ satisfy*

$$d(fx, hy) \preceq k[d(fx, gy) + d(hy, gx)]$$

for all $x, y \in X$, where $k \in P$ and $(2s^2 + s)r(k) < 1$. If $f(X), h(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, h and g have a common coupled coincidence point.

Proof. The proof follows by taking $a_1 = a_2 = a_3 = a_4 = \theta, a_5 = a_6 = k$ in Theorem 2.12. □

Corollary 2.22. *Let (X, d) be a complete cone b-metric space over Banach algebra A with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ satisfy*

$$d(fx, fy) \preceq k[d(fx, y) + d(fy, x)]$$

for all $x, y \in X$, where $k \in P$ and $(2s^2 + s)r(k) < 1$. Then f has a fixed point.

Proof. The proof follows by taking $h = f$ and $g = I$ in Corollary 2.21. □

3. Applications

In this section, we shall apply the obtained conclusions to deal with the existence and uniqueness of solution for some equations. First of all, we refer to the following coupled equations:

$$\begin{cases} F(x, y) = 0, \\ G(x, y) = 0, \end{cases} \tag{3.1}$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}$ are two mappings.

Theorem 3.1. *For (3.1), if there exists $0 < L < \frac{1}{2}$ such that for all the pairs $(x_1 y_1), (x_2, y_2) \in \mathbb{R}^2$, it satisfies that*

$$|F(x_1, y_1) - F(x_2, y_2) + x_1 - x_2| \leq \sqrt{L}|x_1 - x_2|,$$

$$|G(x_1, y_1) - G(x_2, y_2) + y_1 - y_2| \leq \sqrt{L}|y_1 - y_2|.$$

Then the coupled equation (3.1) has a unique common solution in $\mathbb{R}^2$.

Proof. Let $A = \mathbb{R}^2$ with the norm $\|(u_1, u_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1v_1, u_2v_2).$$

Let $P = \{u = (u_1, u_2) \in A : u_1, u_2 \geq 0\}$. It is clear that P is a normal cone and A is a Banach algebra with a unit $e = (1, 1)$. Put $X = \mathbb{R}^2$ and define a mapping $d : X \times X \to A$ by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|^2, |y_1 - y_2|^2).$$

It is easy to see that (X, d) is a complete cone b-metric space over Banach algebra A with the coefficient $s = 2$. Now define the mappings $S, T : X \to X$ by

$$S(x, y) = (x, y), \quad T(x, y) = (F(x, y) + x, G(x, y) + y).$$

Then

$$\begin{aligned}
d(T(x_1, y_1), T(x_2, y_2)) &= d((F(x_1, y_1) + x_1, G(x_1, y_1) + y_1), \\
&\quad (F(x_2, y_2) + x_2, G(x_2, y_2) + y_2)) \\
&= (|F(x_1, y_1) - F(x_2, y_2) + x_1 - x_2|^2, \\
&\quad |G(x_1, y_1) - G(x_2, y_2) + y_1 - y_2|^2) \\
&\preceq (L|x_1 - x_2|^2, L|y_1 - y_2|^2) \\
&= (L, L)(|x_1 - x_2|^2, |y_1 - y_2|^2) \\
&= (L, L)d(S(x_1, y_1), S(x_2, y_2)).
\end{aligned}$$

Since

$$\|(L, L)^n\|^{1/n} = \|(L^n, L^n)\|^{1/n} = (L^n + L^n)^{1/n} = 2^{1/n}L \to L < \frac{1}{2}$$

as $n \to \infty$, we have $r((L, L)) < \frac{1}{2}$. Now if we choose $k = (L, L)$, then all conditions of Corollary 2.16 are satisfied. Hence, by Corollary 2.16, S and T have a unique common fixed point in X. In other words, the coupled equation (3.1) has a unique common solution in $\mathbb{R}^2$. □

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