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DIFFERENTIAL EQUATIONS ASSOCIATED WITH MAHLER AND SHEFFER-MAHLER POLYNOMIALS

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Abstract. In this paper, we study linear differential equations arising from the generating functions of Mahler and Sheffer-Mahler polynomials. We give explicit identities for the Mahler and Sheffer-Mahler polynomials. In addition, we investigate the zeros of the Sheffer-Mahler polynomials with numerical methods.

1. Introduction

Recently, nonlinear differential equations arising from the generating functions of special polynomials are studied by Kim and Kim in order to give explicit identities for special polynomials(see [1, 4]). In this paper, since the

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Mahler polynomial is not a Sheffer sequence, we introduce a new Sheffer-Mahler polynomials that become sheffer sequence. Next, we study linear differential equations arising from the generating functions of the Mahler and Sheffer-Mahler polynomials. We give explicit identities for the Mahler and Sheffer-Mahler polynomials. In order to study the Mahler and Sheffer-Mahler polynomials, we must understand the structure of the Mahler and Sheffer-Mahler polynomials. Therefore, using computer, a realistic study for the Mahler and Sheffer-Mahler polynomials is very interesting. Finally, we observe an interesting phenomenon of 'scattering' of the zeros of the the Mahler and Sheffer-Mahler polynomials in complex plane.

2. Preliminaries

Mahler polynomials, $g_n(x)(n \geq 0)$, were introduced by Mahler in his work on the zeros of the incomplete gamma functions(see [5]). They are defined by the generating function (see [2, 5, 8]):

$$F = F(t,x) = \sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!} = e^{x(1+t-e^t)}. \tag{2.1}$$

The first few examples of Mahler polynomials are

$$\begin{aligned}
&g_0(x) = 1, \quad g_1(x) = 0, \quad g_2(x) = -x, \quad g_3(x) = -x,\\
&g_4(x) = -x + 3x^2, \quad g_5(x) = -x + 10x^2, \quad g_6(x) = -x + 25x^2 - 15x^3,\\
&g_7(x) = -x + 56x^2 - 105x^3, \quad g_8(x) = -x + 119x^2 - 490x^3 + 105x^4,\\
&g_9(x) = -x + 246x^2 - 1918x^3 + 1260x^4,\\
&g_{10}(x) = -x + 501x^2 - 6825x^3 + 9450x^4 - 945x^5.
\end{aligned} \tag{2.2}$$

We observe here that, as $1 + t - e^t = -\frac{1}{2!}t^2 - \frac{1}{3!}t^3 - \cdots$ is not a delta series, the Mahler polynomials $g_n(x)$ are not a Sheffer sequence. To remedy this, we introduce, what we call, the Sheffer-Mahler polynomials $M_n(x)$ given by the generating function:

$$G = G(t,x) = \sum_{n=0}^{\infty} M_n(x)\frac{t^n}{n!} = e^{x(1-t-e^t)}. \tag{2.3}$$

The first few of them are

$$\begin{aligned}
&M_0(x) = 1, \quad M_1(x) = -2x, \quad M_2(x) = -x + 4x^2,\\
&M_3(x) = -x + 6x^2 - 8x^3, \quad M_4(x) = -x + 11x^2 - 24x^3 + 16x^4,\\
&M_5(x) = -x + 20x^2 - 70x^3 + 80x^4 - 32x^5,\\
&M_6(x) = -x + 37x^2 - 195x^3 + 340x^4 - 240x^5 + 64x^6.
\end{aligned} \tag{2.4}$$

As $1-t-e^t = -2t - \frac{1}{2!}t^2 - \frac{1}{3!}t^3 - \cdots$ is a delta series, $M_n(x)$ is a Sheffer sequence. In Section 4, we will display the curves of $M_n(x)(-1 \leq x \leq 1)$, for $n = 1, 2, 3, \ldots, 10$, and study the zeros of $M_n(x)$, for some values of n.

As is well known, the Bell polynomials $Bel_n(x)$ are given by the generating function (see [3]):

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}.$$

From (2.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!} &= e^{x(1+t-e^t)}\\ &= e^{-x(e^t-1)}e^{xt}\\ &= \left(\sum_{k=0}^{\infty} Bel_k(-x)\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} x^m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}Bel_k(-x)x^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus we get

$$g_n(x) = \sum_{k=0}^{n}\binom{n}{k}Bel_k(-x)x^{n-k} \quad (n \geq 0). \tag{2.5}$$

In the same way, we have

$$M_n(x) = \sum_{k=0}^{n}\binom{n}{k}Bel_k(-x)(-x)^{n-k} \quad (n \geq 0). \tag{2.6}$$

It is not difficult to show that

$$Bel_n(-x) = \sum_{m=0}^{n} S_2(n,m)(-x)^m,$$

where $S_2(n,m)$ is the Stirling number of the second kind given by (see [8])

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_t.$$

Here, $(x)_l = x(x-1)\cdots(x-l+1), (l \geq 1)$, and $(x)_0 = 1$.

3. Main Results

In this section, we study linear differential equations arising from the generating functions of Mahler polynomials.

Let

$$F = F(t,x) = e^{x(1+t-e^t)}. \tag{3.1}$$

Then, by (3.1), we get

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,x) = \frac{d}{dt}\left(e^{x(1+t-e^t)}\right) = e^{x(1+t-e^t)}x(1-e^t) \\ &= x(1-e^t)F, \end{aligned} \tag{3.2}$$

$$\begin{aligned} F^{(2)} &= \frac{d}{dt}F^{(1)} = x(-e^t)F + x(1-e^t)F^{(1)} \\ &= -xe^tF + x^2(1-e^t)^2F = (x^2 - (2x^2+x)e^t + x^2e^{2t})F, \end{aligned} \tag{3.3}$$

and

$$F^{(3)} = \frac{d}{dt}F^{(2)} = (x^3 - (3x^3+3x^2+x)e^t + (3x^3+3x^2)e^{2t} - x^3e^{3t})F.$$

Continuing this process, we are led to put

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N,x)e^{it}\right)F, \quad (N = 0,1,2,\ldots). \tag{3.4}$$

Taking the derivative with respect to t in (3.4), we have

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} = \left(\sum_{i=0}^{N} ia_i(N,x)e^{it}\right)F + \left(\sum_{i=0}^{N} a_i(N,x)e^{it}\right)F^{(1)} \\ &= \left(\sum_{i=0}^{N} ia_i(N,x)e^{it}\right)F + \left(\sum_{i=0}^{N} a_i(N,x)e^{it}\right)(x - xe^t)F \\ &= \left\{\sum_{i=0}^{N}(x+i)a_i(N,x)e^{it} - \sum_{i=0}^{N} xa_i(N,x)e^{(i+1)t}\right\}F \\ &= \left\{\sum_{i=0}^{N}(x+i)a_i(N,x)e^{it} - \sum_{i=1}^{N+1} xa_{i-1}(N,x)e^{it}\right\}F. \end{aligned} \tag{3.5}$$

On the other hand, by replacing N by $N+1$ in (3.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1,x)e^{it}\right)F. \tag{3.6}$$

Comparing the coefficients on both sides of (3.5) and (3.6), we obtain

$$a_0(N+1,x) = xa_0(N,x), \quad a_{N+1}(N+1,x) = -xa_N(N,x), \tag{3.7}$$

and

$$a_i(N+1,x) = -xa_{i-1}(N,x) + (x+i)a_i(N,x), (1 \le i \le N). \tag{3.8}$$

In addition, by (3.4), we get

$$F = F^{(0)} = a_0(0,x)F. \tag{3.9}$$

Thus, by (3.9), we get

$$a_0(0,x) = 1. \tag{3.10}$$

It is not difficult to show that

$$\begin{aligned}(x - xe^t)F = F^{(1)} &= \left(\sum_{i=0}^{1} a_i(1,x)e^{it}\right) F \\ &= \left(a_0(1,x) + a_1(1,x)e^t\right) F.\end{aligned} \tag{3.11}$$

Thus, by (3.11), we also get

$$a_0(1,x) = x, \quad a_1(1,x) = -x. \tag{3.12}$$

From (3.7), we note that

$$a_0(N+1,x) = xa_0(N,x) = x^2a_0(N-1,x) = \cdots = x^N a_0(1,x) = x^{N+1}, \tag{3.13}$$

and

$$\begin{aligned}a_{N+1}(N+1,x) &= -xa_N(N,x) = (-x)^2 a_{N-1}(N-1,x) \\ &= \cdots = (-x)^N a_1(1,x) = (-x)^{N+1}.\end{aligned} \tag{3.14}$$

For $i = 1, 2, 3$ in (3.8), we have

$$a_1(N+1,x) = -x\sum_{k=0}^{N}(x+1)^k a_0(N-k,x), \tag{3.15}$$

$$a_2(N+1,x) = -x\sum_{k=0}^{N-1}(x+2)^k a_1(N-k,x), \tag{3.16}$$

and

$$a_3(N+1,x) = -x\sum_{k=0}^{N-2}(x+3)^k a_2(N-k,x). \tag{3.17}$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x) = -x\sum_{k=0}^{N-i+1}(x+i)^k a_{i-1}(N-k,x). \tag{3.18}$$

Theorem 3.1. *For $N = 0, 1, 2, \ldots,$ the linear functional equations*

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N, x) e^{it} \right) F$$

have a solution

$$F = F(t, x) = e^{x(1+t-e^t)},$$

where

$$a_0(N, x) = x^N$$

and

$$a_i(N, x) = (-x)^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (x+l)^{k_l} \right) x^{N-i-\sum_{l=1}^{i} k_l},$$

$(1 \leq i \leq N)$.

Proof. First, we give explicit expressions for $a_i(N+1, x)$. By (3.13)-(3.18), we get

$$\begin{aligned} a_1(N+1, x) &= -x \sum_{k_1=0}^{N} (x+1)^{k_1} a_0(N-k_1, x) \\ &= -x \sum_{k_1=0}^{N} (x+1)^{k_1} x^{N-k_1}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} a_2(N+1, x) &= -x \sum_{k_2=0}^{N-1} (x+2)^{k_2} a_1(N-k_2, x) \\ &= (-x)^2 \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-1-k_2} (x+2)^{k_2} (x+1)^{k_1} x^{N-k_2-k_1-1} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &a_3(N+1, x) \\ &= -x \sum_{k_3=0}^{N-2} (x+3)^{k_3} a_2(N-k_3, x) \\ &= (-x)^3 \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-2-k_3} \sum_{k_1=0}^{N-2-k_3-k_2} (x+3)^{k_3} (x+2)^{k_2} (x+1)^{k_1} x^{N-k_3-k_2-k_1-2}. \end{aligned} \tag{3.21}$$

Continuing this process, we have

$$\begin{aligned}&a_i(N+1,x)\\&=(-x)^i\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-i+1-k_i}\cdots\sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2}\left(\prod_{l=1}^{i}(x+l)^{k_l}\right)x^{N-i+1-\sum_{l=1}^{i}k_l}.\end{aligned}\tag{3.22}$$

This complete the proof. □

Theorem 3.2. *For $k, N = 0, 1, 2, \ldots,$ we have*

$$g_{k+N}(x)=\sum_{i=0}^{N}\sum_{m=0}^{k}\binom{k}{m}i^{k-m}a_i(N,x)g_m(x),\tag{3.23}$$

where

$$\begin{aligned}&a_0(N,x)=x^N,\\&a_i(N,x)=(-x)^i\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-i-k_i}\cdots\sum_{k_1=0}^{N-i-k_i-\cdots-k_2}\left(\prod_{l=1}^{i}(x+l)^{k_l}\right)x^{N-i-\sum_{l=1}^{i}k_l},\\&(1\le i\le N).\end{aligned}$$

Proof. From (2.1), we note that

$$F^{(N)}=\left(\frac{d}{dt}\right)^N F(t,x)=\sum_{k=0}^{\infty}g_{k+N}(x)\frac{t^k}{k!}.\tag{3.24}$$

From Theorem 3.1 and (3.24), we can derive the following equation:

$$\begin{aligned}\sum_{k=0}^{\infty}g_{k+N}(x)\frac{t^k}{k!}&=F^{(N)}=\left(\sum_{i=0}^{N}a_i(N,x)e^{it}\right)F\\&=\sum_{i=0}^{N}a_i(N,x)\left(\sum_{l=0}^{\infty}i^l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}g_m(x)\frac{t^m}{m!}\right)\\&=\sum_{i=0}^{N}a_i(N,x)\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}i^{k-m}g_m(x)\frac{t^k}{k!}\right)\\&=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{m=0}^{k}\binom{k}{m}i^{k-m}a_i(N,x)g_m(x)\right)\frac{t^k}{k!}.\end{aligned}\tag{3.25}$$

By comparing the coefficients on both sides of (3.25), we get assertion (3.23). □

Corollary 3.3. *For $N = 0, 1, 2, \ldots,$ we have*

$$g_N(x) = \sum_{i=0}^{N} a_i(N, x).$$

Proof. Let us take $k = 0$ in (3.23). This complete the proof of Corollary 3.3. □

Here we consider the Sheffer-Mahler polynomials $M_n(x)$ given by the generating function

$$G = G(t, x) = e^{x(1-t-e^t)} = \sum_{n=0}^{\infty} M_n(x) \frac{t^n}{n!}. \tag{3.26}$$

As was noted in Section 1, while the Mahler polynomials $g_n(x)$ are not Sheffer polynomials, the newly introduced Sheffer-Mahler polynomials $M_n(x)$ are. Clearly, they are Sheffer polynomials associated to the delta series $f(t)$, with the compositional inverse $\bar{f}(t)$ of $f(t)$ given by

$$\bar{f}(t) = 1 - t - e^t. \tag{3.27}$$

We will be as brief as possible and leave the details to the reader, as everything can be carried out analogously to Section 2.

By taking the derivatives of G in (3.26), we get

$$G^{(1)} = \frac{d}{dt} G(t, x) = (-x - xe^t)G \tag{3.28}$$

and

$$G^{(2)} = \left(\frac{d}{dt}\right)^2 G(t, x) = (x^2 + (2x^2 - x)e^t + x^2 e^{2t})G. \tag{3.29}$$

From (3.28) and (3.29), we can guess that

$$G^{(N)} = \left(\frac{d}{dt}\right)^N G(t, x) = \left(\sum_{i=0}^{N} b_i(N, x) e^{it}\right) G. \tag{3.30}$$

Taking the derivative of (3.30) with respect to t gives

$$G^{(N+1)} = \left\{\sum_{i=0}^{N} (i - x) b_i(N, x) e^{it} - \sum_{i=1}^{N+1} x b_{i-1}(N, x) e^{it}\right\} G. \tag{3.31}$$

On the other hand, by replacing N by $N + 1$ in (3.30), we obtain

$$G^{(N+1)} = \left(\sum_{i=0}^{N+1} b_i(N + 1, x) e^{it}\right) G. \tag{3.32}$$

Comparing (3.31) and (3.32), we have the following recurrence relations:

$$b_0(N+1,x) = -xb_0(N,x), \quad b_{N+1}(N+1,x) = -xb_N(N,x), \tag{3.33}$$

and

$$b_i(N+1,x) = -xb_{i-1}(N,x) + (i-x)b_i(N,x), \quad (1 \le i \le N). \tag{3.34}$$

In addition, from (3.30) with $N = 0, 1$ and (3.28), we easily get

$$b_0(0,x) = 1, \quad b_0(1,x) = b_1(1,x) = -x. \tag{3.35}$$

From (3.33) and (3.34), we obtain

$$b_0(N+1,x) = (-x)^{N+1}, \quad b_{N+1}(N+1,x) = (-x)^{N+1}. \tag{3.36}$$

Also, proceeding just as in Section 2, from (3.34) we can deduce that, for $1 \le i \le N$,

$$b_i(N+1,x) = -x \sum_{k=0}^{N-i+1} (i-x)^k b_{i-1}(N-k,x). \tag{3.37}$$

In turn, from (3.37) we can get the following explicit expressions for $b_i(N+1,x)$.

$$\begin{aligned} &b_i(N+1,x) \\ &= (-x)^i \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (-x+l)^{k_l} \right) (-x)^{N-i+1-\sum_{l=1}^{i} k_l}, \\ &\quad (1 \le i \le N+1). \end{aligned} \tag{3.38}$$

Theorem 3.4. *For $N = 0, 1, 2, \dots,$ the family of linear functional equations*

$$G^{(N)} = \left(\sum_{i=0}^{N} b_i(N,x) e^{it} \right) G$$

have a solution $G = G(t,x) = e^{x(1-t-e^t)}$, where

$$b_0(N,x) = (-x)^N,$$

$$b_i(N,x) = (-x)^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (-x+l)^{k_l} \right) (-x)^{N-i-\sum_{l=1}^{i} k_l},$$

$(1 \le i \le N)$.

Proof. Note here that (3.38) is also valid for $i = N+1$(see (3.36)). Thus, from (3.33), (3.38), and (3.30), this complete the proof of Theorem 3.4. □

Theorem 3.5. *For $k, N = 0, 1, 2, \ldots,$ we have*

$$M_{k+N}(x) = \sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} i^l b_i(N, x) M_{k-l}(x).$$

In particular, for $k = 0$,

$$M_N(x) = \sum_{i=0}^{N} b_i(N, x), \quad (N = 0, 1, 2, \ldots).$$

Proof. From (3.26) we get

$$G^{(N)} = \left(\frac{d}{dt}\right)^N G = \sum_{k=0}^{N} M_{k+N}(x) \frac{t^k}{k!}. \tag{3.39}$$

On the other hand, from Theorem 3.4 we have

$$\begin{aligned} G^{(N)} &= \left(\sum_{i=0}^{N} b_i(N, x) e^{it} \right) G \\ &= \sum_{i=0}^{N} b_i(N, x) \sum_{l=0}^{\infty} i^l \frac{t^l}{l!} \sum_{m=0}^{\infty} M_m(x) \frac{t^m}{m!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} i^l b_i(N, x) M_{k-l}(x) \right) \frac{t^k}{k!}. \end{aligned} \tag{3.40}$$

Comparing (3.39) with (3.40) gives Theorem 3.5. □

4. Zeros of the Sheffer-Mahler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the Sheffer-Mahler polynomials $M_n(x)$. By using computer, the Sheffer-Mahler polynomials $M_n(x)$ can be determined explicitly. A few of them are

$$\begin{aligned} M_0(x) &= 1, \\ M_1(x) &= -2x, \\ M_2(x) &= -x + 4x^2, \\ M_3(x) &= -x + 6x^2 - 8x^3, \\ M_4(x) &= -x + 11x^2 - 24x^3 + 16x^4, \end{aligned}$$

$$M_5(x) = -x + 20x^2 - 70x^3 + 80x^4 - 32x^5,$$
$$M_6(x) = -x + 37x^2 - 195x^3 + 340x^4 - 240x^5 + 64x^6,$$
$$M_7(x) = -x + 70x^2 - 539x^3 + 1330x^4 - 1400x^5 + 672x^6 - 128x^7,$$
$$M_8(x) = -x + 135x^2 - 1498x^3 + 5033x^4 - 7280x^5 + 5152x^6 - 1792x^7 + 256x^8.$$

We display the shapes of the Sheffer-Mahler polynomials $M_n(x)$ and investigate the zeros of the Sheffer-Mahler polynomials $M_n(x)$. For $n = 1, \cdots, 10$, we can draw a plot of the Sheffer-Mahler polynomials $M_n(x)$, respectively. This shows the ten plots combined into one. We display the shape of $M_n(x)$, $-1 \leq x \leq 1$. (Figure1). We investigate the beautiful zeros of the Sheffer-

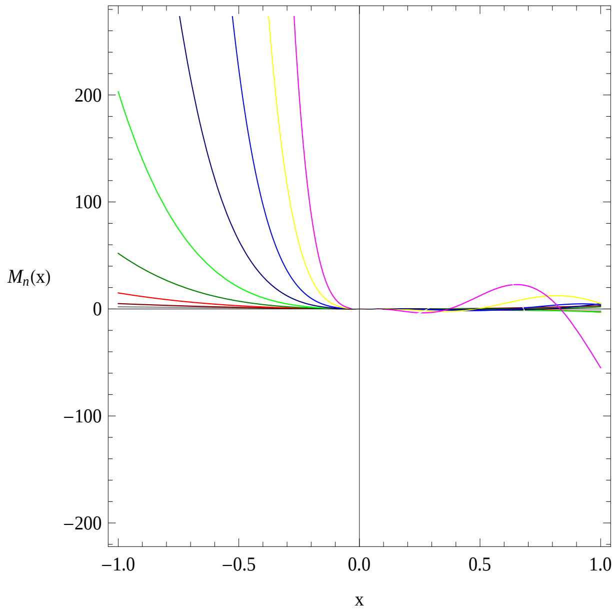

FIGURE 1. Curve of the Sheffer-Mahler polynomials $M_n(x)$

Mahler polynomials $M_n(x)$ by using a computer. We plot the zeros of the $M_n(x)$ for $n = 5, 10, 15, 20$ and $x \in \mathbb{C}$(Figure 2). In Figure 2(top-left), we choose $n = 5$. In Figure 2(top-right), we choose $n = 10$. In Figure 2(bottom-left), we choose $n = 15$. In Figure 2(bottom-right), we choose $n = 20$. Prove that $M_n(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions(see Figure 2). Stacks of zeros of the Sheffer-Mahler polynomials $M_n(x)$ for $1 \leq n \leq 20$ from a 3-D structure are presented(Figure 3). Our numerical results for approximate solutions of real zeros of the Sheffer-Mahler polynomials $M_n(x)$ are displayed(Tables 1, 2).

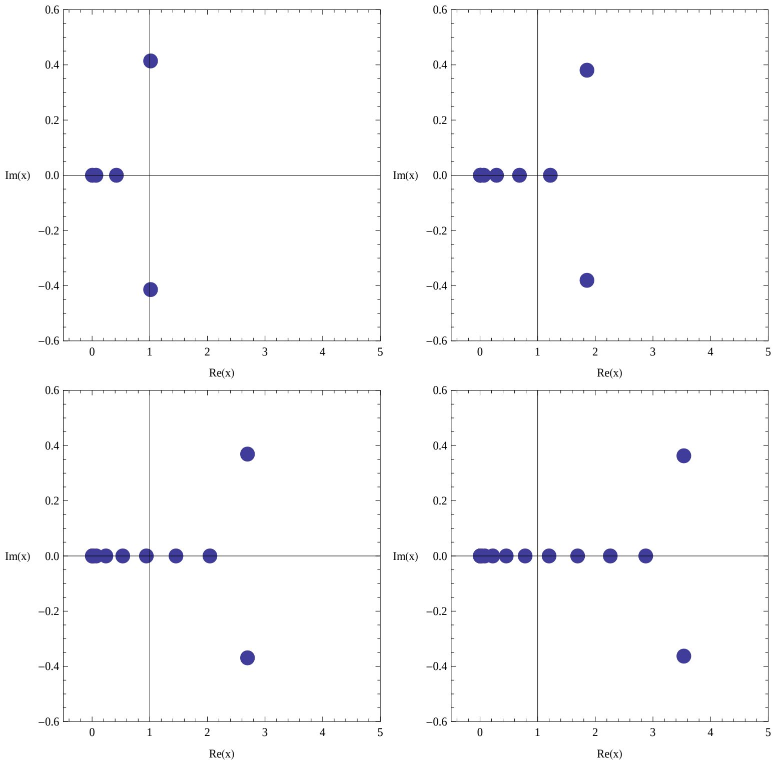

FIGURE 2. Zeros of $M_n(x)$

Table 1. Numbers of real and complex zeros of $M_n(x)$

degree n	real zeros	complex zeros
1	1	0
2	2	0
3	3	0
4	2	2
5	3	2
6	4	2
7	5	2
8	6	2
9	5	4
10	6	4
11	7	4
12	8	4
13	9	4
14	8	6

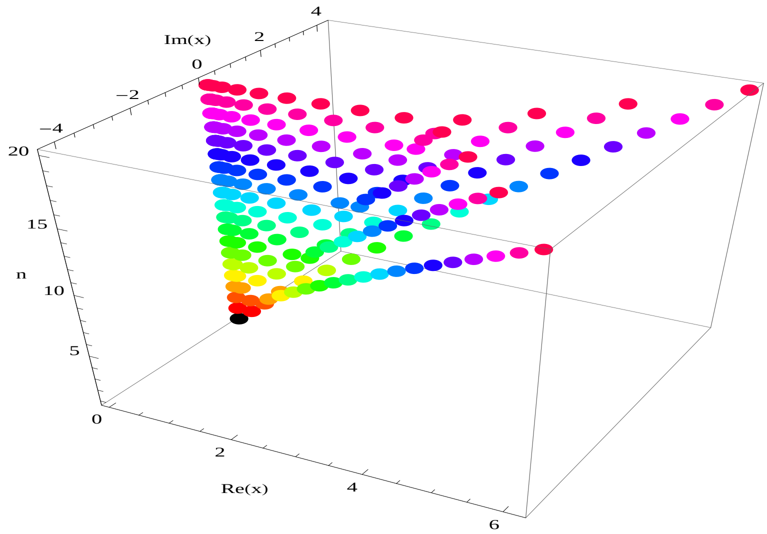

FIGURE 3. Stacks of zeros of $M_n(x), 1 \leq n \leq 20$

Plot of real zeros of $M_n(x)$ for $1 \leq n \leq 30$ structure are presented(Figure 4). We observe a remarkable regular structure of the complex roots of the

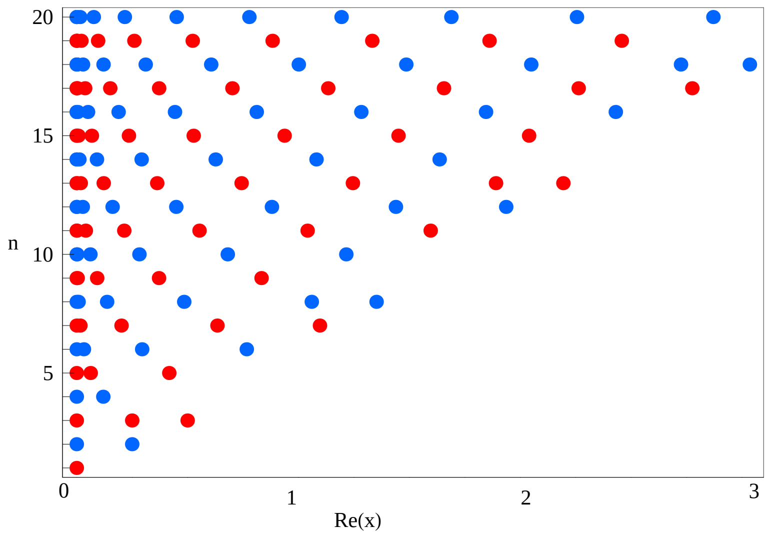

FIGURE 4. Real zeros of $M_n(x)$ for $1 \leq n \leq 20$

Sheffer-Mahler polynomials $M_n(x)$. We hope to verify a remarkable regular structure of the complex roots of the Sheffer-Mahler polynomials $M_n(x)$ (Table 1). Next, we calculated an approximate solution satisfying $M_n(x) = 0, x \in \mathbb{C}$. The results are given in Table 2.

Table 2. Approximate solutions of $M_n(x) = 0, x \in \mathbb{C}$

degree n	x
1	0
2	0, 0.25000
3	0, 0.25000, 0.50000
4	0, 0.11966, $0.69017 - 0.21447i$, $0.69017 + 0.21447i$
5	0, 0.062862, 0.41719, $1.00998 - 0.41418i$, $1.00998 + 0.41418i$
6	0, 0.032188, 0.29492, 0.76667, $1.3281 - 0.6189i$, $1.3281 + 0.6189i$
7	0, 0.016235, 0.20182, 0.63436, 1.0966, $1.6505 - 0.8386i$, $1.6505 + 0.8386i$
8	0, 0.0081192, 0.13688, 0.48452, 1.0598, 1.3520, $1.9794 - 1.0701i$, $1.9794 + 1.0701i$

Remark 4.1. We can consider the more general problems based on numerical experiments. How many zeros does $M_n(x)$ have? We are not able to decide if $M_n(x) = 0$ has n distinct solutions(see Table 2). We would also like to know the number of complex zeros $C_{M_n(x)}$ of $M_n(x), Im(x) \neq 0$. Since n is the degree of the polynomial $M_n(x)$, the number of real zeros $R_{M_n(x)}$ lying on the real line $Im(x) = 0$ is then $R_{M_n(x)} = n - C_{M_n(x)}$, where $C_{M_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{M_n(x)}$ and $C_{M_n(x)}$. The authors have no doubt that investigations along these line will lead to a new approach employing numerical method in the research field of the Sheffer-Mahler polynomials $M_n(x)$ which appear in applied mathematics and mathematical physics(see [6], [7]).

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