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OUTPUT FEEDBACK MIN-MAX CONTROL PROBLEM FOR A CLASS OF UNCERTAIN LINEAR STOCHASTIC SYSTEMS ON UMD BANACH SPACE

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Abstract. In this paper we consider the problem of quadratic linear min-max regulator problem on a UMD (Unconditional Martingale Differences) Banach space where the system is to be regulated by output feedback subject to measurement uncertainty. The problem is to find an optimal feedback policy (an operator valued function) that minimizes the maximum risk (or loss). We prove existence of an optimal policy, as an output feedback operator valued function, and present the necessary conditions of optimality. Also we present a convergence theorem based on the necessary conditions of optimality. The results presented here are new even in the Hilbert space setting.

1. Introduction

In this paper we consider a class of linear stochastic systems on Banach spaces subject to measurement uncertainty (noise) and regulated by output feedback control. This is very different from the classical regulator problem on two different fronts. First, the optimal policy is based on output feedback rather than state feedback. Second, the output measurement is noisy or uncertain but bounded with no probabilistic structure assumed. The feedback operator must be chosen as a strongly measurable operator valued function taking values from a bounded subset of the space of Linear operators endowed

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with the strong operator topology. The objective is to find the optimal feedback control law that minimizes the maximum risk or, equivalently, minimizes the maximum loss. In a recent paper [2] we studied similar problems for infinite dimensional deterministic uncertain systems. The question of existence of optimal feedback controls have been studied in [5] and more recently in [1]. Here we consider an infinite dimensional uncertain stochastic system and develop necessary conditions for min-max problems. The min-max problems involving uncertain stochastic dynamic systems considered here are rarely studied in the literature though such problems are more realistic and natural. The probable reason is the difficulty encountered in constructing optimal strategies for mini-max problems with stochastic dynamic constraints.

The rest of the paper is organized as follows. In section 2, we introduce the system model and present the problem considered in the paper. In section 3, we present the basic assumptions including a brief discussion on the question of existence and uniqueness of solutions. In section 4, we present some preparatory results in terms of necessary conditions for potentially extreme uncertainty in the system and their characterization. In section 5, we present the major results of this paper proving existence of optimal policies (optimal output feedback operator valued functions) and present the necessary conditions of optimality. Also we present some results extending the uncertainty set from the closed unit ball $B_1(Y)$ to a ball $B_r(Y), r \geq 0$. In section 6, we present a result on the convergence of an algorithm based on the necessary conditions of optimality developed in section 5.

2. System model and formulation of control problem

The system is governed by the following set of equations:

$$dx = Axdt + B(t)udt + \sigma(t)dW_H(t),\ x(0) = x_0, \tag{2.1}$$

$$y(t) = L(t)x + \xi(t) \tag{2.2}$$

$$u(t) = K(t)y(t), t \in I, \tag{2.3}$$

on a Banach space E. The first equation represents the dynamic system (to be controlled), a linear stochastic evolution equation on a Banach space E, the second equation represents the sensor observing the sate x in an uncertain or noisy environment ξ, and the third equation represents the controller that uses the output data y to deliver the control signal. The operator valued function B represents the actuator (controller), σ represents the diffusion operator, W_H represents the H-Brownian motion where H is a separable Hilbert space; L represents the observer (or the measurement operator) of the state x , ξ represents uncertainty in the observed (or measured) output y, and K represents the output feedback controller. The objective (cost) functional is given by

$$J(K,\xi) \equiv (1/2)\mathbf{E}\left\{\int_0^T < Q(t)x, x > dt + < Mx(T), x(T) >\right\} \tag{2.4}$$

where $\mathbf{E}(\cdot)$ denotes the expectation operation, Q is a positive and symmetric operator valued function taking values from $B_0(I, \mathcal{L}(E, E^*))$ and the operator $M \in \mathcal{L}(E, E^*)$ is also positive and symmetric. The objective is to find a controller K that minimizes the maximum loss (or penalty). This is equivalent to minimizing the maximum risk and hence we are faced with the min-max problem:

$$\inf_{K\in\mathcal{B}_{ad}} \sup_{\xi\in\mathcal{D}} J(K,\xi) \equiv \inf\{\sup\{J(K,\xi), \xi \in \mathcal{D}\}, K \in \mathcal{B}_{ad}\}, \tag{2.5}$$

where $\mathcal{D}$ denotes the set (class) of measurement uncertainties and $\mathcal{B}_{ad}$ denotes the class of admissible feedback operators to be defined shortly.

Stochastic convolutions are well defined in the class of general UMD (unconditional martingale difference sequences) Banach spaces as discussed in Neerven, Veraar, Weis,[9]. For simplicity we consider UMD-type-2 Banach spaces only. However we believe that the results presented here also hold for general UMD spaces. For details on UMD spaces see also [Burkholder, 13]. Since UMD spaces are reflexive, it is clear that both $Q(t), t \in I$, and M, along with their duals, take values in $\mathcal{L}(E, E^*)$.

3. Basic Assumptions and Existence of Solutions

Let H be a separable real Hilbert space and W_H a H-cylindrical Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ where $\mathcal{F}_{t\geq 0}$ is a nondecreasing family of subsigma algebras of the sigma algebra $\mathcal{F}$. We introduce the following assumptions:

- (A1) : The operator A is the infinitesimal generator of a C_0 semigroup $S(t), t \geq 0$, on a Banach space E where E is a UMD-type-2 Banach space.
- (A2) : The operator valued function B is measurable in the strong operator topology and belongs to the space $B_0(I, \mathcal{L}(U, E))$ where U is another real Hilbert space.
- (A3) : The operator valued function $\sigma \in B_0(I, \gamma(H, E))$ the space of bounded strongly measurable operator valued functions taking values in the space of γ-Radonifying operators $\gamma(H, E)$).
- (A4) : The output space Y is a reflexive Banach space and L is a strongly measurable operator valued function with values in $B_0(I, \mathcal{L}(E, Y))$ and ξ is any strongly measurable function with values in the closed unit

ball $B_1(Y)$. We denote this class of (measurement) uncertainty by $\mathcal{D} \equiv B_0(I, B_1(Y))$.

(A5) : The output feedback operators $\{K\}$ are strongly measurable functions defined on I and taking values in $\Lambda \subset \mathcal{L}(Y, U)$, a closed bounded convex set, which is compact with respect to the strong operator topology τ_{so}. Let $\mathcal{B}_{ad} \equiv B(I, \Lambda)$ denote the admissible class of feedback operators equipped with the Tychnoff product topology denoted by τ_{tp}.

Combining the equations (2.1)-(2.3) we obtain the following linear stochastic system:

$$dx = Axdt + BKLxdt + BK\xi dt + \sigma dW_H, x(0) = x_0, t \in I \equiv [0, T], \quad (3.1)$$

where all the operators are functions of time except the unbounded operator A which is assumed to be the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, on the Banach space E. Using the semigroup we can reformulate this as an integral equation as follows:

$$\begin{aligned} x(t) = S(t)x_0 &+ \int_0^t S(t-s)B(s)K(s)L(s)x(s)ds \\ &+ \int_0^t S(t-s)B(s)K(s)\xi(s)ds \\ &+ \int_0^t S(t-s)\sigma(s)dW_H(s), t \in I. \end{aligned} \quad (3.2)$$

Clearly, this is a linear stochastic integral equation. Since all the operators $\{B, K, L, \sigma\}$ are assumed to be bounded and measurable in the strong operator topology, and ξ is absolutely norm bounded, and $\sigma \in B_0(I, \gamma(H, E))$, using Banach fixed point theorem it is easy to verify that, for each $\mathcal{F}_0$-measurable initial state $x_0 \in L_2(\Omega, E)$ and $\xi \in \mathcal{D}$ and $K \in \mathcal{B}_{ad}$, this equation has a unique $\mathcal{F}_t$-adapted solution $x \in C(I, E)$ P-a.s. and further we have $\mathbf{E} \parallel x \parallel^2_{C(I,E)} < \infty$. Thus $x \in L_2^a(\Omega, C(I, E))$ where $L_2^a(\Omega, C(I, E))$ denotes the space of $\mathcal{F}_t$-adapted stochastic processes with values in the Banach space E having continuous (version) sample paths and possessing finite second moments. The fact that, pathwise, the process $x(\cdot) \equiv x(\cdot, \omega) \in C(I, E)$ follows from the regularity of the stochastic convolution:

$$v(t) \equiv \int_0^t S(t-s)\sigma(s)dW_H(s), t \in I.$$

In Hilbert space setting, the most well known regularity result is due to Prato, Kwapien and Zabczyk [6] and this is based on factorization technique where the Ito integral is written as

$$v(t) = \frac{sin(\alpha\pi)}{\pi}\int (t-s)^{\alpha-1}S(t-s)\varpi(s)ds, t \in I,$$

where,

$$\varpi(t) = \int_0^t (t-s)^{-\alpha}S(t-s)\sigma(s)dW_H(s), t \in I.$$

4. Necessary conditions for extremality of uncertainty

Notice that in order to solve the problem (2.5) we must first solve the extremality problem:

$$J_0(K) \equiv \sup\{J(K,\xi), \xi \in \mathcal{D}\}. \tag{4.1}$$

This is because, in general, inf sup $\neq$ sup inf . Generally the following inequality holds,

$$\inf_{K\in\mathcal{B}_{ad}}\sup_{\xi\in\mathcal{D}} J(K,\xi) \geq \sup_{\xi\in\mathcal{D}}\inf_{K\in\mathcal{B}_{ad}} J(K,\xi).$$

In case the reverse inequality holds, we have a saddle point. For the existence of saddle point we need much stronger conditions on the functional J which we usually don't have. So first we concentrate on the extremality problem (4.1). Since Y is a reflexive Banach space the closed unit ball $B_1(Y)$ is weakly compact. Thus $\mathcal{D} \equiv B_0(I, B_1(Y))$, equipped with the Tychonoff product topology $\tau_{\tau\pi}$ is compact (Hausdorff). Using the integral equation (3.2) and Gronwall inequality one can easily verify that for each $K \in \mathcal{B}_{ad}$, $\xi \longrightarrow J(K,\xi)$ is continuous on $\mathcal{D}$ with respect to the Tychonoff product topology $\tau_{\tau\pi}$. Hence, for each $K \in \mathcal{B}_{ad}$, there exists a $\xi^o \equiv \xi^o(K) \in \mathcal{D}$ dependent on K such that

$$J(K,\xi^o) \geq J(K,\xi)\ \forall\xi \in \mathcal{D}.$$

Thus $J_0(K)$ is well defined. Here we are interested in the necessary conditions of extremality of ξ^o.

Lemma 4.1. *Consider the system* (2.1)-(2.3) *or equivalently the system* (3.1) *and suppose the assumptions* (*A*1)-(*A*5) *hold. Then in order that* ξ^o *be the extremal (optimal), it is necessary that there exists a* $\varphi \in L_2^a(\Omega, C(I,E^*))$ *and* $x^o \in L_2^a(\Omega, C(I,E))$ *such that the triple* $\{\xi^o, x^o, \varphi\}$ *satisfies the following inequality and the evolution equations:*

$$\begin{aligned}&\mathbf{E}\int_0^T < B(t)K(t)\xi^o(t), \varphi(t) >_{E,E^*} dt\\ \geq\ &\mathbf{E}\int_0^T < B(t)K(t)\xi(t), \varphi(t) >_{E,E^*} dt,\ \forall\ \xi \in \mathcal{D},\end{aligned} \tag{4.2}$$

$$dx^o = Ax^o dt + BKLx^o dt + BK\xi^o dt + \sigma dW_H, x^o(0) = x_0, \quad (4.3)$$

$$-d\varphi = A^*\varphi dt + L^*K^*B^*\varphi dt + Qx^o dt, \varphi(T) = Mx^o(T). \quad (4.4)$$

Proof. Fix $K \in \mathcal{B}_{ad}$ and let $\xi^o \in \mathcal{D}$ be the maximizer of the functional $\xi \longrightarrow J(K, \xi)$ given by the expression (2.4). Since $\mathcal{D}$ is convex, it is clear that $\xi^\varepsilon \equiv \xi^o + \varepsilon(\xi - \xi^o) \in \mathcal{D}$ for all $\varepsilon \in [0, 1]$ and all $\xi \in \mathcal{D}$. Thus we have the following inequality:

$$J_o(K) \equiv J(K, \xi^o) \geq J(K, \xi^o + \varepsilon(\xi - \xi^o)), \ \forall \varepsilon \in [0, 1] \text{ and } \xi \in \mathcal{D}. \quad (4.5)$$

Let $\{x^\varepsilon, x^o\}$ denote the (mild) solutions of the evolution equation (3.1) corresponding to $\{\xi^\varepsilon, \xi^o\}$ respectively. Subtracting equation (3.2) corresponding to ξ^o, from the same equation corresponding to ξ^ε, and taking the limit $\varepsilon \to 0$, and defining the process z given by $z(t) \equiv \lim(1/\varepsilon)(x^\varepsilon(t) - x^o(t)), t \in I$, we obtain the following integral equation:

$$\begin{aligned} z(t) &= \int_0^t S(t-s)B(s)K(s)L(s)z(s)ds \\ &\quad + \int_0^t S(t-s)B(s)K(s)(\xi(s) - \xi^o(s))ds, t \in I. \end{aligned} \quad (4.6)$$

In other words, z is the mild solution of the variational equation:

$$dz = Azdt + BKLzdt + BK(\xi - \xi^o)dt, z(0) = 0. \quad (4.7)$$

Note that the above equation is an ordinary differential equation (not an Itô differential equation) with random input $BK(\xi - \xi^o)$. Since the operators $\{B, K, L\}$ satisfy the assumptions (A2)-(A5), it follows from the above equation that the map $BK(\xi - \xi^o) \longrightarrow z$ is a bounded linear hence continuous map from $B_0(I, E)$ to itself P-a.s. Now, for the fixed $K \in \mathcal{B}_{ad}$, using the cost functional J given by (2.4) and carrying out some elementary computations, we obtain the Gâteaux differential $dJ(K, \xi^o, \xi - \xi^o)$ of J at $\xi^o \in \mathcal{D}$ in the direction $(\xi - \xi^o)$. Further, it follows from (4.5) that it satisfies the following inequality,

$$\begin{aligned} dJ(K, \xi^o; \xi - \xi^o) &= \mathbf{E}\bigg\{\int_0^T < Q(s)x^o(s), z(s) >_{E^*,E} ds \\ &\quad + < Mx^o(T), z(T) >_{E^*,E}\bigg\} \leq 0 \end{aligned} \quad (4.8)$$

for all $\xi \in \mathcal{D}$. Define the functional $\ell(z)$ as follows:

$$\ell(z) \equiv \mathbf{E}\bigg\{\int_0^T < Q(s)x^o(s), z(s) >_{E^*,E} ds + < Mx^o(T), z(T) >_{E^*,E}\bigg\}. \quad (4.9)$$

Since by assumption $Q \in B_0(I, \mathcal{L}(E, E^*))$ and $M \in \mathcal{L}(E, E^*)$, it is easy to verify that $z \longrightarrow \ell(z)$ is a continuous linear functional on $L_2^a(\Omega, C(I, E)) \subset$

$L_2^a(\Omega, B_0(I,E))$. As the operators $\{B, K\}$ are bounded and strongly measurable (strong operator topology) and ξ is also a strongly measurable function on I with values in $B_1(Y)$, it is clear that $BK(\xi - \xi^o) \in L_2^a(\Omega, B_0(I,E))$. Thus the composition map $BK(\xi - \xi^o) \longrightarrow z \longrightarrow \ell(z)$ is a continuous linear functional on $L_2^a(\Omega, B_0(I,E))$. Hence by duality, there exists a $\mu \in L_2^a(\Omega, M_{fa}(I,E^*))$ so that

$$\ell(z) \equiv \tilde{\ell}(BK(\xi - \xi^o)) \equiv \mathbf{E}\int_0^T < BK(\xi - \xi^o), \mu(dt) > \tag{4.10}$$

where $M_{fa}(I,E^*) \equiv M_{fa}(\Sigma(I), E^*)$ denotes the space of finitely additive vector measures defined on the sigma algebra $\Sigma(I)$ (of all Lebesgue measurable subsets of the set I) with values in the dual E^* of the space E. For details on vector measures see Dunford and Schwartz [7] and Diestel and Uhl Jr [18]. In a recent paper [14, Theorem 4.5, p481] Bongiornio, Piazza and Musial have proved that, under some mild conditions, finitely additive measures with values in Banach spaces satisfying weak Radon-Nikodym property (WRNP) posses densities which are Henstock-Kurzweil-Pettis integrable. We denote the space of Henstock-Kurzweil-Pettis integrable functions with values in a Banach space X by $L_{HKP}(I,X)$. Since, in our case, E is a reflexive Banach space, its dual is also a reflexive Banach space and therefore E satisfies RNP (Radon Nikodym Property). Thus evidently E also satisfies the weak Radon-Nikodym property (WRNP). Hence the measure μ with values in E^* has a density and there exists a $\varphi \in L_{HKP}(I,E^*)$ $P-$a.s so that $\mu(dt) = \varphi(t)dt$. Thus the expression (4.10) can be rewritten as

$$\ell(z) = \mathbf{E}\int_0^T < BK(\xi - \xi^o), \varphi(t) > dt. \tag{4.11}$$

Now consider the Itô differential of the duality product $(z(t), \varphi(t))_{E,E^*}$. Since equation (4.7) is actually an ordinary differential equation on the Banach space E, the Itô derivative of $(z(t), \varphi(t))$ is given by

$$d(z,\varphi) = (dz, \varphi) + (z, d\varphi) + (<< dz, d\varphi >>= 0). \tag{4.12}$$

In other words, the quadratic variation term is identically zero. Since $z(0) = 0$, integrating the above equation we obtain

$$\begin{aligned}\mathbf{E}(z(T), \varphi(T)) \quad &= \quad \mathbf{E}\int_0^T (Az + BKLz + BK(\xi - \xi^o), \varphi)dt \\ &\quad +\mathbf{E}\int_0^T (z, d\varphi)\end{aligned} \tag{4.13}$$

and hence integrating by parts formally (as justified below) we arrive at the following expression,

$$\begin{aligned}\mathbf{E}(z(T),\varphi(T)) &= \mathbf{E}\int_0^T (z, d\varphi + A^*\varphi dt + L^*K^*B^*\varphi dt) \\ &\quad +\mathbf{E}\int_0^T (BK(\xi-\xi^o),\varphi)_{E,E^*}dt. \qquad (4.14)\end{aligned}$$

Since we are interested in the mild solutions, this can be justified by use of Yosida approximation of the identity $I_n \equiv nR(n,A)$ where $R(n,A)$ is the resolvent of the unbounded operator A for $n \in \rho(A)$ (the resolvent set of A). We rewrite the variational equation (4.7) by its Yosida approximation as follows:

$$dz_n = Az_n dt + I_n BKLz_n dt + I_n BK(\xi-\xi^o)dt, z_n(0) = 0.$$

It is easy to verify that the mild solution z_n of the above equation satisfies the following properties: $z_n(t) \in D(A)$ for all $t \in I$, and that $z_n(t) \xrightarrow{s} z(t)$ in E uniformly in $t \in I$, P-a.s. Letting I_n^* denote the adjoint of the operator I_n, and by replacing z by z_n and φ by $I_n^*\varphi$ in equation (4.12) and carrying out the integration by parts we obtain

$$\begin{aligned}(z_n(T), I_n^*\varphi(T)) &= \int_0^T (z_n(t), d(I_n^*\varphi) + A^*(I_n^*\varphi)dt + (I_n(BKL))^* I_n^*\varphi dt) \\ &\quad + \int_0^T (I_n BK(\xi-\xi^o), I_n^*\varphi)dt.\end{aligned}$$

Since both I_n and I_n^* converge in the strong operator topology to the identity in E and E^* respectively, by letting $n \to \infty$ in the above expression, we obtain equation (4.14). Now setting

$$-d\varphi = A^*\varphi dt + L^*K^*B^*\varphi dt + Qx^o dt, t \in I, \qquad (4.15)$$

$$\varphi(T) = Mx^o(T) \qquad (4.16)$$

we find that the identity (4.14) is equivalent to

$$\begin{aligned}&\mathbf{E}(z(T), Mx^o(T))_{E,E^*} + \mathbf{E}\int_0^T (z(t), Q(t)x^o(t))_{E,E^*}dt \\ &= \mathbf{E}\int_0^T (BK(\xi-\xi^o),\varphi)_{E,E^*}dt. \qquad (4.17)\end{aligned}$$

It follows from the expression (4.9) that the left hand side of the above expression coincides with the functional $\ell(z)$. Then, by virtue of the expression (4.8) we arrive at the following inequality

$$\mathbf{E}\int_0^T (BK(\xi-\xi^o),\varphi)_{E,E^*}dt \le 0, \text{ for all } \xi \in \mathcal{D}. \qquad (4.18)$$

This proves the necessary condition (4.2). The pair of equations (4.15)-(4.16) gives the necessary condition (4.4). The necessary condition (4.3) represents the dynamic system corresponding to the extreme uncertainty $\xi^o \in \mathcal{D}$ and x^o is the corresponding state trajectory. This proves all the necessary conditions as stated. □

This only solves half of the problem (2.5). Now we must solve the problem

$$\inf\{J_0(K) : K \in \mathcal{B}_{ad}\} \tag{4.19}$$

where the functional $J_0(K)$ is given by the expression (4.1). Here we will need the notion of duality map. Let Y be a real Banach space with the dual Y^* and let 2^{Y^*} denote the power set of Y^*. In general, a mapping $\Delta : Y \setminus \emptyset \longrightarrow 2^{Y^*}$ given by

$$\Delta(y) \equiv \{y^* \in Y^* : y^*(y) = \| y^* \| \| y \| = \| y \|^2 = \| y^* \|^2\}$$

is called the duality map. Generally this is a multivalued map and it follows from Hahn-Banach theorem that it is nonempty. The normalized duality map is given by a similar expression

$$\Delta(y) \equiv \{y^* \in B_1(Y^*) : y^*(y) = \| y \|\} \tag{4.20}$$

where $B_1(Y^*)$ is the closed unit ball of Y^*. Similarly one can define a duality map from Y^* to Y as $\nu : Y^* \setminus \emptyset \longrightarrow 2^Y$ and it is given by

$$\nu(y^*) \equiv \{y \in B_1(Y) : (y^*, y) = \| y^* \|\}. \tag{4.21}$$

This later duality map demands that y^* attains its norm on the closed unit ball $B_1(Y)$. This is not true for general Banach spaces. However, if Y is a reflexive Banach space, $B_1(Y)$ is weakly compact and hence y^* attains its norm on it and hence $\nu(y^*)$ is well defined. In fact the norm is attained on the unit sphere $S_1(Y) = \partial B_1(Y)$.

Now we return to our problem and present the following intermediate result.

Lemma 4.2. *Suppose the assumptions of Lemma* 4.1 *hold. Further, let the Banach space Y, representing the output space, be a strictly convex reflexive Banach space. Then there exists a unique extremal ξ^o in $\mathcal{D}$, as stated in Lemma* 4.1*, and it is given by*

$$\xi^o(t) = \nu(K^*(t)B^*(t)\varphi(t)) \text{ for all } t \in I. \tag{4.22}$$

Proof. It follows from Lemma 4.1 that the extremal ξ^o must satisfy the inequality (4.2). By our assumptions (A2) and (A5), it is clear that for all $t \in I$, $K^*(t)B^*(t) : E^* \longrightarrow Y^*$ and hence the inequality (4.2) is equivalent to the following inequality

$$\mathbf{E}\int_0^T (\xi^o, K^*B^*\varphi)_{Y,Y^*} dt \geq \mathbf{E}\int_0^T (\xi, K^*B^*\varphi)_{Y,Y^*} dt, \quad \forall\ \xi \in \mathcal{D}. \tag{4.23}$$

By assumptions, the operator valued functions B and K are bounded and strongly measurable, and we have seen that φ is E^* valued strongly measurable random process. Hence the process $K^*B^*\varphi$ is strongly measurable random process and Bochner integrable. Thus the map

$$\xi \longrightarrow \Upsilon(\xi) \equiv \mathbf{E}\int_0^T (\xi, K^*B^*\varphi)_{Y,Y^*}dt$$

is well defined and it is a continuous and bounded linear functional on $\mathcal{D}$ with respect to the Tychonoff product topology. Since $\mathcal{D}$ is compact in this topology, Υ attains its maximum at some point $\xi^o \in \mathcal{D}$. Hence by Lebesgue density argument $\xi^o(t) \in \partial B_1(Y)$ for each $t \in I$, P-a.s. Since $B_1(Y)$ is strictly convex, the extremal ξ^o is unique. Strict convexity of Y also implies that the duality map ν is single valued. Hence, it follows from the definition of the duality map ν, that $\xi^o(t) = \nu(K^*(t)B^*(t)\varphi(t))$ for all $t \in I$, P a.s. This completes the proof as stated. □

Remark 4.3 It is interesting to note that the mild solution of the adjoint equation (4.15) with the terminal condition (4.16) is actually path wise continuous though it is the weak Radon-Nikodym derivative of a finitely additive E^* valued vector measure μ and it belongs to $L_2^a(\Omega, L_{HKP}(I, E^*))$. So it is more smooth than that implied by the WRNP.

Remark 4.4 Recall that the uncertainty set was given by $\mathcal{D} \equiv B_0(I, B_1(Y))$ equipped with the Tychonoff product topology. The range of uncertainty $B_1(Y)$ can be easily relaxed by replacing it by any weakly compact convex subset $\mathcal{C} \subset Y$.

5. Optimal output feedback control policy

In this section we prove the existence of optimal output feedback control policies and present the necessary conditions of optimality. Now using Lemma 4.2, we can rewrite the necessary conditions (4.3)-(4.4) in the closed form as a coupled system of differential equations on the product space $E \times E^*$ as follows:

$$dx = Axdt + BKLxdt + BK\nu(K^*B^*\varphi)dt + \sigma dW_H, x(0) = x_0. \quad (5.1)$$

$$-d\varphi = A^*\varphi dt + L^*K^*B^*\varphi dt + Qxdt, \varphi(T) = Mx(T). \quad (5.2)$$

This pair of equations represents the system in the state of extreme uncertainty (due to imperfect measurement) and our objective is to find an output feedback operator K that minimizes the maximum risk as determined by the functional

$J_o(K)$ given by,

$$J_0(K) \equiv (1/2)\mathbf{E}\left\{\int_0^T < Q(t)x, x > dt + < Mx(T), x(T) >\right\}, \tag{5.3}$$

where now x is the first component of the mild solution $\{x, \varphi\}$ of the two point boundary value problem (5.1)-(5.2). Note that the system is no more linear even though we started with linear system. Here the first question that arises naturally is the question of existence of solution of the two point boundary value problem (5.1)-(5.2). This raises the question of regularity of the duality map ν. The question of regularity of the duality map is intimately related to smoothness of the unit ball of the Banach space which is again related to differentiability of the norm. There are well known sufficient conditions [Zemek, 10] for Lipschitz continuity of the duality map ν defined on the dual space Y^*. It is known that if Y^* satisfies the differentiability condition, that is, if there exists a constant $c > 0$ such that for all $x^* \in \partial B_1(Y^*)$ and for all $y^* \in Y^*$, and all $z \in \nu(x^*)$

$$\| x^* + y^* \| - \| x^* \| - y^*(z) \le c \| y^* \|^2,$$

then the duality map ν as defined above is Lipschitz. This follows from the main theorem in Zemek [10]. For more details on this topic see also [11]-[12]. This result is useful in the proof of the following existence theorem.

Theorem 5.1. *Suppose the dual Y^* of the Banach space Y has differentiable norm implying Lipschitz continuity of the duality map ν and that the operators $\{A, M\}$ and the operator valued functions $\{B, L, \sigma, Q\}$ satisfy the basic assumptions* $(\mathbf{A1}) - (\mathbf{A5})$. *Then for any $\mathcal{F}_0$ measurable $x_0 \in L_2(\Omega, E)$ and every $K \in \mathcal{B}_{ad}$, the two point boundary value problem given by the pair of equations* (5.1)-(5.2) *has a unique mild solution* $(x^o, \varphi^o) \in L_2^a(\Omega, C(I, E)) \times L_2^a(\Omega, C(I, E^*))$.

Proof. We use successive approximation and Banach fixed point theorem to prove the existence and uniqueness of solutions of the following system of coupled forward backward stochastic integral equations,

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s)(BKL)x(s)ds \\ &\quad + \int_0^t S(t-s)(BK)\nu(K^*B^*\varphi)ds + \int_0^t S(t-s)\sigma(s)dW_H,\ t \in I, \end{aligned} \tag{5.4}$$

$$\begin{aligned} \varphi(t) &= S^*(T-t)Mx(T) + \int_t^T S^*(s-t)(L^*K^*B^*)\varphi(s)ds \\ &\quad + \int_t^T S^*(s-t)Q(s)x(s)ds, \quad t \in I. \end{aligned} \tag{5.5}$$

Since the proof is classical we present only an outline. For any given $x \in L_2^a(\Omega, C(I,E))$, it follows from Banach fixed point theorem that equation (5.5) has a unique solution say, φ that belongs to $L_2^a(\Omega, C(I,E^*))$. Hence there exists a bounded linear operator $\mathbf{\Gamma}$ that maps $L_2^a(\Omega, C(I,E))$ to $L_2^a(\Omega, C(I,E^*))$ giving $\varphi = \mathbf{\Gamma} x$. Substituting this in equation (5.5) we obtain the following integral equation for x,

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s)(BKL)x(s)\, ds \\ &\quad + \int_0^t S(t-s)(BK)\nu(K^*B^*\mathbf{\Gamma}x)\, ds \\ &\quad + \int_0^t S(t-s)\sigma(s)dW_H, t \in I. \end{aligned} \tag{5.6}$$

Since Y^* has differentiable norm, it follows from [10, Main Theorem, p62] that the duality map $\nu : Y^* \longrightarrow Y$ is Lipschitz. Hence by use of successive (Piccard) approximation technique as seen in Hu and Peng [8] one can prove that the integral equation (5.6) has a unique solution $x^o \in L_2^a(\Omega, C(I,E))$. This implies that equation (5.5) has a unique solution $\varphi^o = \mathbf{\Gamma} x^o$. This completes the outline of our proof. □

As stated in this section, the control problem here is to find an operator valued function $K \in \mathcal{B}_{ad}$ that minimizes the functional (5.3) subject to the 2-point boundary value problem (5.1)-(5.2). The first question that arises naturally is the question of existence of an optimal policy. If no such policy exists, it does not make much sense to try and characterize something that does not even exist. So, first we consider this problem and prove the following existence result.

Theorem 5.2. *Consider the system* (5.1)-(5.2) *with the cost functional given by the expression* (5.3) *and suppose the assumptions of Theorem* 5.1 *hold. Then there exists an output feedback operator valued function $K_o \in \mathcal{B}_{ad}$ that minimizes the cost functional* (5.3).

Proof. The proof is long. We give a broad outline of the proof. First we prove that the map $K \longrightarrow (x, \varphi)$ from $\mathcal{B}_{ad}$ to $L_2^a(\Omega, C(I, E \times E^*))$ is continuous with respect to the Tychonoff product topology on $\mathcal{B}_{ad}$ and the norm topology on $L_2^a(\Omega, C(I, E \times E^*))$ respectively. Then we prove that J_0 is lower semicontinuous on $\mathcal{B}_{ad}$ with respect to the Tychonoff product topology τ_{ty}. For the first part we proceed as follows. For economy of notation let V denote the product space $E \times E^*$. Let $K_n \xrightarrow{\tau_{tp}} K_o$ in $\mathcal{B}_{ad}$ and let $(x_n, \varphi_n) \in L_2^a(\Omega, C(I,V))$ and $(x_o, \varphi_o) \in L_2^a(\Omega, C(I,V))$ denote the mild solutions of the system (5.1)-(5.2) (equivalently, the solutions of the integral equations (5.5)-(5.6) corresponding to K_n and K_o respectively. Subtracting (x_o, φ_o) from (x_n, φ_n) using the expressions (5.5)-(5.6) and defining, for each $t \in I$, $\xi_n(t) \equiv \| x_n(t) - x_o(t) \|_E$ and $\eta_n(t) \equiv \| \varphi_n(t) - \varphi_o(t) \|_{E^*}$ we arrive at the following inequalities:

$$\begin{aligned} \xi_n(t) \quad \leq \quad & C_1 \int_0^t \xi_n(s)ds + C_2 \int_0^t \eta_n(s)ds \\ & + M_o b\kappa\beta \int_0^t \| (K_n^* - K_o^*)B^*\varphi_o \|_{Y^*} \, ds \\ & + M_0 b \int_0^t \| (K_n - K_o)[Lx_o + \nu(K_o^* B^* \varphi_o)] \|_U \, ds, t \in I, P - a.s. \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \eta_n(t) \quad \leq \quad & C_3 \| x_n(T) - x_o(T) \|_E + C_4 \int_t^T \eta_n(s)ds + C_5 \int_t^T \xi_n(s)ds \\ & + M_o \ell \int_t^T \| (K_n^* - K_o^*)B^*\varphi_o \|_{Y^*} \, ds, t \in I, P - a.s. \end{aligned} \tag{5.8}$$

where the constants are given by $C_1 = M_o b\kappa\ell$, $C_2 = M_o\beta(\kappa b)^2$, $C_3 = M_o m$, $C_4 = M_o \ell b \kappa$, $C_5 = M_o q$ and they are determined from the following bounds:

$$\begin{aligned} & \sup\{\| S(t) \|_{\mathcal{L}(E)}, t \in I\} = M_o, \ \sup\{\| L(t) \|_{\mathcal{L}(E,Y)}, t \in I\} = \ell, \\ & \sup\{\| B(t) \|_{\mathcal{L}(U,E)}, t \in I\} = b, \ \sup\{\| K \|_{\mathcal{L}(Y,U)}, K \in \Lambda\} = \kappa, \\ & \| M \|_{\mathcal{L}(E,E^*)} = m, \ \sup\{\| Q(t) \|_{\mathcal{L}(E,E^*)}, t \in I\} = q, \\ & \| \nu(y_1^*) - \nu(y_2^*) \|_E \leq \beta \| y_1^* - y_2^* \|_{Y^*}, \ \forall \, y_1^*, y_2^* \in Y^*, \end{aligned}$$

which follow from the basic assumptions (A1)-(A5). Defining

$$\begin{aligned} e_n(t) \equiv M_o b\kappa\beta & \int_0^t \| (K_n^* - K_o^*)B^*\varphi_o \|_{Y^*} \, ds \\ & + M_0 b \int_0^t \| (K_n - K_o)[Lx_o + \nu(K_o^* B^* \varphi_o)] \|_U \, ds, t \in I, \end{aligned} \tag{5.9}$$

and

$$f_n(t) \equiv M_o \ell \int_t^T \| (K_n^* - K_o^*)B^*\varphi_o \|_{Y^*} \, ds, \ \ t \in I, \tag{5.10}$$

we can rewrite the inequalities (5.7) and (5.8) in the following compact form,

$$\xi_n(t) \leq C_1 \int_0^t \xi_n(s)ds + C_2 \int_0^t \eta_n(s)ds + e_n(t), \ \ t \in I, \ \ P - a.s. \tag{5.11}$$

and

$$\eta_n(t) \leq C_3 \xi_n(T) + C_4 \int_t^T \eta_n(s)ds + C_5 \int_t^T \xi_n(s)ds + f_n(t) \tag{5.12}$$

for all $t \in I$.

Note that

$$e_n(t) \leq e_n(T),\ \forall\ t \in I; f_n(t) \leq f_n(0),\ \forall\ t \in I, P - a.s.$$

Hence the inequalities (5.11) and (5.12) can be simplified further as follows:

$$\xi_n(t) \leq C_1 \int_0^t \xi_n(s)ds + C_2 \int_0^t \eta_n(s)ds + e_n(T), t \in I, P - a.s, \qquad (5.13)$$

and

$$\begin{aligned} \eta_n(t) \quad \leq \quad & C_3 \xi_n(T) + C_4 \int_t^T \eta_n(s)ds \\ & + C_5 \int_t^T \xi_n(s)ds + f_n(0), t \in I, \quad P - a.s. \end{aligned} \qquad (5.14)$$

Since $K_n \xrightarrow{\tau_{tp}} K_o$ (in the Tychonoff product topology) on $\mathcal{B}_{ad}$, it is clear that

$$\lim_{n\to\infty} e_n(T) = 0, \text{ and } \lim_{n\to\infty} f_n(0) = 0, \quad P - a.s.$$

Using this fact and Gronwall lemma and carrying out somewhat laborious computation using the above inequalities, we can verify that

$$\lim_{n\to\infty} \xi_n(t) = 0 \text{ uniformly in } t \in I, \quad P - a.s;$$

and

$$\lim_{n\to\infty} \eta_n(t) = 0 \text{ uniformly in } t \in I, \quad P - a.s.$$

Hence, we conclude that as $K_n \xrightarrow{\tau_{tp}} K_o$, we have $x_n \xrightarrow{s} x_o$ $P - a.s$ in the norm topology of the Banach space E uniformly on I and $\varphi_n \xrightarrow{s} \varphi_o$ P-a.s in the norm topology of the dual space E^* uniformly on I. Furthermore, by assumption (A3), $\sigma \in B_0(I, \gamma(H, E))$, and by assumption (A5), Λ is a closed bounded convex subset of $\mathcal{L}(Y, U)$. Using these facts and the integral equations (5.5) and (5.6) and Gronwall inequality, one can verify that the sequence $\{(x_n, \varphi_n)\}$ is dominated by an integrable process belonging to $L_2^a(\Omega, C(I, V))$ $(V \equiv E \times E^*)$. Thus it follows from dominated convergence theorem that

$$(x_n, \varphi_n) \longrightarrow (x_o, \varphi_o) \ \text{ strongly in } L_2^a(\Omega, C(I, V)).$$

Since both M and $Q(t), t \in I$, are symmetric and positive, it follows from elementary algebraic computation using the cost functional (5.3) that

$$\begin{aligned} J_0(K_n) \quad \geq \quad & J_0(K_o) + \mathbf{E}\Big\{ \int_0^T < Qx_o, x_n - x_o >_{E^*,E} dt \\ & + < Mx_o(T), x_n(T) - x_o(T) >_{E^*,E} \Big\}. \end{aligned} \qquad (5.15)$$

Thus it follows from strong convergence of x_n to x_o that $\underline{\lim}\, J_0(K_n) \geq J_0(K_o)$ proving lower semicontinuity of J_0 on $\mathcal{B}_{ad}$ with respect to the Tychonoff product topology. Since the set $\mathcal{B}_{ad}$ is compact in this topology, it is clear that J_0 attains its minimum on it. This proves the existence of an optimal policy in $\mathcal{B}_{ad}$ thereby completing the proof. □

Remark 5.3. We have proved that J_0 is lower semicontinuous on $\mathcal{B}_{ad}$. In fact, one can verify that $K \longrightarrow J_0(K)$ is also upper semicontinuous and hence J_0 is continuous with respect to the Tychonoff product topology.

Now we are prepared to construct the necessary conditions of optimality for our original min-max problem. As seen above, the original problem is reduced to the optimization problem: Find $K \in \mathcal{B}_{ad}$ that minimizes the functional $J_0(K)$ given by the expression (5.3) subject to the evolution equations (5.1)-(5.2) (a 2-point boundary value problem). We are now prepared to prove the following necessary conditions of optimality.

Note that we continue to use the standard notation for stochastic differential equations even though the underlying equation is a true differential equation without the martingale term. For example, equation (4.4) with random x^o but without a martingale term, like in equation (4.3). Similarly, we have random evolution equations like (5.17)-(5.18) as seen below.

Theorem 5.4. *Consider the (extremal) system* (5.1)-(5.2) *along with the cost functional J_0 given by* (5.3) *and the set of admissible (control) operators $\mathcal{B}_{ad}$. Suppose the assumptions of Theorem* 5.2 *hold and the duality map ν is once continuously Fréchet differentiable. Then, in order that $K_o \in \mathcal{B}_{ad}$ be optimal, it is necessary that there exists a pair $\{\psi_1, \psi_2\} \in L_2^a(\Omega, C(I, E^*)) \times L_2^a(\Omega, C(I, E))$ and the (mild) solutions*

$$(x_o, \varphi_o) \in L_2^a(\Omega, C(I, E)) \times L_2^a(\Omega, C(I, E^*))$$

of the system (5.1)-(5.2) *corresponding to K_o such that the set of quintuple $\{K_o\psi_1, \psi_2, x_o, \varphi_o\}$ satisfies the following inequality,*

$$\begin{aligned} &dJ_0(K_o, K - K_o) \\ &\equiv \mathbf{E} \int_0^T \Big\{ < (K - K_o)[Lx_o + \nu(K_o^* B^* \varphi_o)], B^* \psi_1 >_U \\ &\quad + < (K - K_o)[\Gamma_o^* K_o^* B^* \psi_1 - L\psi_2], B^* \varphi_o >_U \Big\} dt \ \geq 0 \end{aligned} \tag{5.16}$$

for all $K \in \mathcal{B}_{ad}$, where $\Gamma_o \equiv D\nu(K_o^* B^* \varphi_o) \in B_0(I, \mathcal{L}(Y^*, Y))$ is a bounded strongly measurable operator valued function, and the pair $\{\psi_1, \psi_2\}$ satisfies *the following two point boundary value problem:*

$$-d\psi_1 = A^*\psi_1 dt + L^*K_o^*B^*\psi_1 dt - Q\psi_2 dt + Qx_o dt, t \in I, \quad (5.17)$$

$$d\psi_2 = A\psi_2 dt + BK_oL\psi_2 dt - BK_o\Gamma_o^*K_o^*B^*\psi_1 dt, \quad (5.18)$$

$$\psi_1(T) + M\psi_2(T) = Mx_o(T), \psi_2(0) = 0.$$

Proof. Let $K_o \in \mathcal{B}_{ad}$ denote the optimal policy and $K \in \mathcal{B}_{ad}$ be any other element. Define $K_\varepsilon \equiv K_o + \varepsilon(K - K_o)$. Since $\mathcal{B}_{ad}$ is a closed convex set, $K_\varepsilon \in \mathcal{B}_{ad}$ for all $\varepsilon \in [0, 1]$. By optimality of K_o it is clear that the Gâteaux derivative of J_0 at K_o in the direction $(K - K_o)$ satisfies the following inequality,

$$dJ_0(K_o, K - K_o) = \lim_{\varepsilon \to 0}(1/\varepsilon)\big(J_0(K_\varepsilon) - J_0(K_o)\big) \geq 0, \ \forall \ K \in \mathcal{B}_{ad}. \quad (5.19)$$

Let $\{x_\varepsilon, x_o\}$ denote the mild solutions of equation (5.1) and $\{\varphi_\varepsilon, \varphi_o\}$ the mild solutions of equation (5.2) corresponding to the control operators $\{K_\varepsilon, K_o\}$ respectively. One can verify that as $\varepsilon \downarrow 0$, $K_\varepsilon \to K_o$ in the uniform operator topology, $x_\varepsilon \xrightarrow{s} x_o$ in $C(I, E)$ P-a.s and $\varphi_\varepsilon \xrightarrow{s} \varphi_o$ in $C(I, E^*)$ P-a.s. Define

$$z_1(t) \equiv (1/\varepsilon)(x_\varepsilon(t) - x_o(t)), t \in I, \quad (5.20)$$

$$z_2(t) \equiv (1/\varepsilon)(\varphi_\varepsilon(t) - \varphi_o(t)), t \in I. \quad (5.21)$$

By direct (but tedious) computation one can verify that z_1 and z_2 satisfy the following pair of evolution equations (with stochastic coefficients) in the mild sense

$$\begin{aligned} dz_1 \ &= \ Az_1 dt + BK_oLz_1 dt + B(K - K_o)Lx_o dt \\ &\quad +BK_oD\nu(K_o^*B^*\varphi_o)K_o^*B^*z_2 dt \\ &\quad +BK_oD\nu(K_o^*B^*\varphi_o)(K^* - K_o^*)B^*\varphi_o dt \\ &\quad +B(K - K_o)\nu(K_o^*B^*\varphi_o)dt, \ \ t \in I \end{aligned} \quad (5.22)$$

and

$$\begin{aligned} -dz_2 \ &= \ A^*z_2 dt + L^*K_o^*B^*z_2 dt + L^*(K^* - K_o^*)B^*\varphi_o dt \\ &\quad +Qz_1 dt, \ \ t \in I, \end{aligned} \quad (5.23)$$

with initial boundary conditions given by

$$z_1(0) = 0, \ z_2(T) = Mz_1(T). \quad (5.24)$$

Using the semigroup along with its adjoint $\{S(t), S^*(t), t \geq 0\}$ and Duhamel's formula one can rewrite these equations as integral equations and using Banach fixed point theorem one can verify that these equations have unique mild solutions $z_1 \in L_2^a(\Omega, C(I, E))$ and $z_2 \in L_2^a(\Omega, C(I, E^*))$ respectively. It is convenient to write this as a system on the product space $V = E \times E^*$ as follows.

Define the vector $z \equiv (z_1, z_2)'$ in V and note that z satisfies the following system:

$$\begin{aligned} dz &= \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} zdt + \begin{bmatrix} BK_oL & BK_o\Gamma_o K_o^* B^* \\ -Q & -L^* K_o^* B^* \end{bmatrix} zdt \\ &\quad + \begin{bmatrix} B(K-K_o)\big(Lx_o + \nu(K_o^* B^* \varphi_o)\big) \\ +BK_o\Gamma_o(K^* - K_o^*)B^*\varphi_o \\ -L^*(K^* - K_o^*)B^*\varphi_o. \end{bmatrix} dt \end{aligned} \tag{5.25}$$

subject to the boundary conditions (5.24). For notational convenience we write the above equation in compact form as follows:

$$\begin{aligned} & dz = \mathcal{A}zdt + \mathcal{B}(t)zdt + F_o dt, \quad t \in I, \\ & z_1(0) = 0, \quad z_2(T) = Mz_1(T), \end{aligned} \tag{5.26}$$

where one can easily identify the operators $\{\mathcal{A}, \mathcal{B}(t), t \in I\}$. The nonhomogeneous term F_o is given by

$$F_o \equiv \begin{bmatrix} F_o^{(1)} \\ F_o^{(2)} \end{bmatrix} \equiv \begin{bmatrix} B(K-K_o)\big(Lx_o + \nu(K_o^* B^* \varphi_o)\big) + BK_o\Gamma_o(K^* - K_o^*)B^*\varphi_o \\ -L^*(K^* - K_o^*)B^*\varphi_o \end{bmatrix}.$$

The evolution equation (5.26) is defined on the product space $V \equiv E \times E^*$ with the nonhomogeneous term F_o given by the above expression. Since the operators appearing in the expression for F_o are all bounded and strongly measurable and $x_o \in L_2^a(\Omega, C(I, E))$ and $\varphi_o \in L_2^a(\Omega, C(I, E^*))$ it is easy to verify that $F_o \in L_2^a(\Omega, B_0(I, V))$. Note that the operator valued function $\mathcal{B}$ is a bounded strongly measurable function on I with values in $\mathcal{L}(V)$ P-a.s. Thus it is clear from the preceding analysis that the map $F_o \longrightarrow z$ is continuous and linear from $L_2^a(\Omega, B_0(I, V))$ to $L_2^a(\Omega, C(I, V))$. By direct computation, it follows from the expression (5.3) that the Gâteaux derivative of J_0 at K_o in the direction $K - K_o$ satisfies, for all $K \in \mathcal{B}_{ad}$, the following inequality,

$$\begin{aligned} & dJ_0(K_o, K - K_o) \\ & = \mathbf{E}\bigg\{ \int_0^T < Qx_o, z_1 >_{E^*,E} dt + < Mx_o(T), z_1(T) >_{E^*,E} \bigg\} \geq 0. \end{aligned} \tag{5.27}$$

Define the functional

$$\ell(z_1) \equiv \mathbf{E}\bigg\{ \int_0^T < Qx_o, z_1 >_{E^*,E} dt + < Mx_o(T), z_1(T) >_{E^*,E} \bigg\}.$$

Clearly, $z_1 \longrightarrow \ell(z_1)$ is a continuous linear functional on $C(I, E)$ P-a.s. Letting P_1 denote the projection of V to the first component E of V, it is clear that $z \longrightarrow \ell(P_1 z)$ is a continuous linear functional of z. Thus the composition map

$$F_o \longrightarrow z \longrightarrow \ell(P_1 z) \equiv \tilde{\ell}(F_o) \tag{5.28}$$

is a continuous linear functional of $F_o \in L_2^a(\Omega, B_0(I,V))$. Hence there exists a vector measure

$$\Xi \in L_2^a(\Omega, B_0(I,V))^* \cong L_2^a(\Omega, M_{fa}(\Sigma(I), V^*))$$

such that

$$\tilde{\ell}(F_o) = \mathbf{E}\int_0^T < F_o, \Xi(dt) >_{V,V^*} .$$

Again, we have used $M_{fa}(I,V^*)$ to denote the space of finitely additive V^* valued vector measures defined on the sigma algebra $\Sigma(I)$ of Lebesgue measurable subsets of the interval I. Since V and, hence, V^* are reflexive Banach spaces they satisfy RNP (Radon Nikodym Property) and hence they satisfy also weak RNP (WRNP). Thus it follows from [14, Theorem 4.5, p481] that there exists a $\Psi \in L_2^a(\Omega, L_{HKP}(I,V^*))$ such that

$$\begin{aligned} \tilde{\ell}(F_o) &= \mathbf{E}\int_0^T < F_o, \Xi(dt) >_{V,V^*} \\ &= \mathbf{E}\int_0^T < F_o, \Psi(t) >_{V,V^*} dt. \end{aligned}$$

Note that the function Ψ has two components: $\Psi \equiv (\psi_1, \psi_2)'$ with $\psi_1 \in L_2^a(\Omega, L_{HKP}(I,E^*))$ and $\psi_2 \in L_2^a(\Omega, L_{HKP}(I,E))$. Using these we obtain

$$\begin{aligned} \tilde{\ell}(F_o) &= \mathbf{E}\int_0^T < F_o, \Psi(t) >_{V,V^*} dt \\ &= \mathbf{E}\int_0^T \Big\{ < B(K-K_o)[Lx_o + \nu(K_o^* B^* \varphi_o)], \psi_1(t) >_{E,E^*} \\ &\quad + < BK_o\Gamma_o(K^* - K_o^*)B^*\varphi_o, \psi_1(t) >_{E,E^*} \Big\} dt \qquad (5.29) \\ &\quad -\mathbf{E}\int_0^T \Big\{ < L^*(K^* - K_o^*)B^*\varphi_o, \psi_2(t) >_{E^*,E} \Big\} dt \geq 0 \ \forall\ K \in \mathcal{B}_{ad}. \end{aligned}$$

Rearranging terms we can rewrite the above inequality in the following convenient form,

$$\begin{aligned} \tilde{\ell}(F_o) &= \mathbf{E}\int_0^T \Big\{ < (K-K_o)[Lx_o + \nu(K_o^* B^* \varphi_o)], B^*\psi_1 >_U \qquad (5.30) \\ &\quad + < (K-K_o)[\Gamma_o^* K_o^* B^* \psi_1 - L\psi_2], B^*\varphi_o >_U \Big\} dt, \forall\ K \in \mathcal{B}_{ad}. \end{aligned}$$

This proves the necessary condition (5.16). Now applying the Itô differential rule to the duality product $(z(t), \Psi(t))_{V,V^*}$ and noting that the system (5.26) is free of any martingale component, we obtain

$$d(z(t),\Psi(t))_{V,V^*} =< dz,\Psi > + < z, d\Psi > . \tag{5.31}$$

Integrating the expression on the left we have

$$\mathbf{E}\int_0^T d(z,\Psi) = \mathbf{E}\{< z(T),\Psi(T) > - < z(0),\Psi(0) >\} \tag{5.32}$$

and then using the boundary conditions (5.24) we obtain

$$\begin{aligned}\mathbf{E}\int_0^T d(z,\Psi) &= \mathbf{E}\{< z(T),\Psi(T) > - < z(0),\Psi(0) >\} \qquad (5.33)\\ &= \mathbf{E}\big\{< z_1(T),\psi_1(T)+M\psi_2(T) >_{E,E^*} - < z_2(0),\psi_2(0) >_{E^*,E}\big\}.\end{aligned}$$

Next, considering the first term on the right hand side of the expression (5.31) and formally integrating by parts we find that

$$\begin{aligned}&\mathbf{E}\int_0^T < dz,\Psi >\\ &= \mathbf{E}\int_0^T < z_1, A^*\psi_1 + L^*K_o^*B^*\psi_1 - Q\psi_2 >_{E,E^*} dt\\ &\quad +\mathbf{E}\int_0^T < z_2, BK_o\Gamma_o^*K_o^*B^*\psi_1 - BK_oL\psi_2 - A\psi_2 >_{E^*,E} dt\\ &\quad +\mathbf{E}\int_0^T \{< F_o^{(1)},\psi_1 >_{E,E^*} + < F_o^{(2)},\psi_2 >_{E^*,E}\}dt, \qquad (5.34)\end{aligned}$$

where $\{F_o^{(1)}, F_o^{(2)}\}$ denote the first and the second components of F_o respectively. Again the formal derivation can be justified by use of Yosida approximation as we did in case of Lemma 5.1. Now considering the last term of equation (5.31) we obtain

$$\mathbf{E}\int_0^T < z, d\Psi >_{V,V^*} = \mathbf{E}\int_0^T\{< z_1, d\psi_1 >_{E,E^*} + < z_2, d\psi_2 >_{E^*,E}\}. \tag{5.35}$$

Adding (5.34) and (5.35) we arrive at the coordinate wise expression for the integral of the sum on the righthand side of equation (5.31) as follows:

$$
\begin{aligned}
&\mathbf{E}\int_0^T \{< dz, \Psi > + < z, d\Psi >\}\\
&= \mathbf{E}\int_0^T < z_1, d\psi_1 + A^*\psi_1 dt + L^*K_o^*B^*\psi_1 dt - Q\psi_2 dt >_{E,E^*}\\
&\quad + \mathbf{E}\int_0^T < z_2, d\psi_2 - A\psi_2 dt - BK_oL\psi_2 dt + BK_o\Gamma_o^*K_o^*B^*\psi_1 dt >_{E^*,E} dt\\
&\quad + \mathbf{E}\int_0^T \{< F_o^{(1)}, \psi_1 >_{E,E^*} + < F_o^{(2)}, \psi_2 >_{E^*,E}\} dt.
\end{aligned} \tag{5.36}
$$

Then setting

$$-d\psi_1 = A^*\psi_1 dt + L^*K_o^*B^*\psi_1 dt + Q(x_0 - \psi_2)dt \tag{5.37}$$

$$d\psi_2 = A\psi_2 dt + BK_oL\psi_2 dt - BK_o\Gamma_o^*K_o^*B^*\psi_1 dt, \tag{5.38}$$

it follows from equation (5.36) that

$$
\begin{aligned}
&\mathbf{E}\int_0^T \{< dz, \Psi > + < z, d\Psi >\}\\
&= \mathbf{E}\int_0^T < z_1, -Qx_o >_{E,E^*} dt\\
&\quad + \mathbf{E}\int_0^T \{< F_o^{(1)}, \psi_1 >_{E,E^*} + < F_o^{(2)}, \psi_2 >_{E^*,E}\} dt.
\end{aligned} \tag{5.39}
$$

By virtue of the identity (5.31) it follows from (5.34) and (5.39) that

$$
\begin{aligned}
&\mathbf{E} < z_1(T), \psi_1(T) + M\psi_2(T) >_{E,E^*} - \mathbf{E} < z_2(0), \psi_2(0) >_{E^*,E}\\
&+ \mathbf{E}\int_0^T < z_1(t), Qx_o(t) >_{E,E^*} dt\\
&= \mathbf{E}\int_0^T \{< F_o^{(1)}, \psi_1 >_{E,E^*} + < F_o^{(2)}, \psi_2 >_{E^*,E}\} dt.
\end{aligned} \tag{5.40}
$$

Now setting $\psi_1(T) + M\psi_2(T) = Mx_o(T)$ and $\psi_2(0) = 0$, equation (5.40) reduces to

$$
\begin{aligned}
&\mathbf{E} < z_1(T), Mx_o(T)) >_{E,E^*} + \mathbf{E}\int_0^T < z_1(t), Qx_o(t) >_{E,E^*} dt\\
&= \mathbf{E}\int_0^T \{< F_o^{(1)}, \psi_1 >_{E,E^*} + < F_o^{(2)}, \psi_2 >_{E^*,E}\} dt.
\end{aligned} \tag{5.41}
$$

Clearly, the expression on the left of the above identity gives the functional $\ell(z_1)$ as seen in the inequality (5.28). Hence it follows from (5.28) and the above identity that

$$dJ_0(K_o, K - K_o) = \mathbf{E}\int_0^T \{< F_o^{(1)}, \psi_1 >_{E,E^*} + < F_o^{(2)}, \psi_2 >_{E^*,E}\}dt \\ \geq 0, \tag{5.42}$$

for all $K \in \mathcal{B}_{ad}$. This is consistent with the necessary condition (5.16). Further, we have seen above (see (5.37)-(5.38)) that the pair (ψ_1, ψ_2) must satisfy the following (random) differential equations (in the mild sense) with the given boundary conditions,

$$-d\psi_1 = A^*\psi_1 dt + L^*K_o^*B^*\psi_1 dt + Q(x_0 - \psi_2)dt, \tag{5.43}$$

$$d\psi_2 = A\psi_2 dt + BK_oL\psi_2 dt - BK_o\Gamma_o^*K_o^*B^*\psi_1 dt, \tag{5.44}$$

$$\psi_1(T) + M\psi_2(T) = Mx_o(T),\ \ \psi_2(T) = 0. \tag{5.45}$$

This proves all the necessary conditions as stated. □

Remark 5.5. Note that the necessary conditions of optimality consist of the inequality (5.16), the (compound) state equations (5.1)-(5.2) and the pair of adjoint equations (5.17)-(5.18) with initial boundary conditions as indicated. Again, as seen in Remark 5.3, the solutions of the pair of adjoint equations (5.17)-(5.18) are considered in the mild sense and they are path wise continuous and belong to $L^2(\Omega, C(I, V^*))$. Thus they are more regular than that predicted by WRNP.

Remark 5.6. For the measurement error we have used the closed unit ball $B_1(Y)$ as the domain of uncertainty. If we wish to increase or decrease the size of uncertainty, we only have to multiply the unit ball by $r > 0$ giving $B_r(Y) = rB_1(Y)$. In this case the compound system (5.1)-(5.2) is given by

$$dx = Axdt + BKLxdt + rBK\nu(K^*B^*\varphi)dt + \sigma dW_H, x(0) = x_0. \tag{5.46}$$

$$-d\varphi = A^*\varphi dt + L^*K^*B^*\varphi dt + Qxdt, \varphi(T) = Mx(T). \tag{5.47}$$

The necessary conditions given by Theorem 5.4 remain intact modulo a multiplier r as shown below:

$$dJ_0(K_o, K - K_o) \\ \equiv \mathbf{E}\int_0^T \Big\{< (K - K_o)[Lx_o + r\nu(K_o^*B^*\varphi_o)], B^*\psi_1 >_U \tag{5.48} \\ + < (K - K_o)[r\Gamma_o^*K_o^*B^*\psi_1 - L\psi_2], B^*\varphi_o >_U\Big\}dt \ \ \geq 0, \ \ \forall\ K \in \mathcal{B}_{ad}$$

where $\Gamma_o \equiv D\nu(K_o^*B^*\varphi_o) \in B_0(I, \mathcal{L}(Y^*, Y))$ is a bounded strongly measurable operator valued function and the pair $\{\psi_1, \psi_2\}$ satisfies the following two point (adjoint) boundary value problem:

$$-d\psi_1 = A^*\psi_1 dt + L^*K_o^*B^*\psi_1 dt - Q\psi_2 dt + Qx_o dt, t \in I, \quad (5.49)$$
$$d\psi_2 = A\psi_2 dt + BK_oL\psi_2 dt - rBK_o\Gamma_o^*K_o^*B^*\psi_1 dt, \quad (5.50)$$
$$\psi_1(T) + M\psi_2(T) = Mx_o(T), \psi_2(0) = 0.$$

Remark 5.7. From the above result we can recover the necessary conditions of optimality for systems without measurement uncertainty. This is stated in the following corollary.

Corollary 5.8. *Consider the system* (2.1)-(2.3) *and the cost functional* (2.4) *with measurement uncertainty* $\xi \equiv 0$*. In order that an element* $K_o \in \mathcal{B}_{ad}$ *be optimal it is necessary that there exists a pair* $\{x_o, \psi\} \in L_2^a(\Omega, C(I, V)) \equiv L_2^a(\Omega, C(I, E)) \times L_2^a(\Omega, C(I, E^*))$ *such that it satisfies the inequality*

$$dJ(K_o, K - K_o) \equiv \mathbf{E}\int_0^T \{< (K - K_o)Lx_o, B^*\psi >_U\} dt \geq 0 \quad (5.51)$$

for all $K \in \mathcal{B}_{ad}$*, and the following pair of evolution equations,*

$$-d\psi = A^*\psi dt + L^*K_o^*B^*\psi dt + Qx_o dt, \psi(T) = Mx_o(T),\ \ t \in I, \quad (5.52)$$
$$dx_o = Ax_o dt + BK_oLx_o dt + \sigma(t)dW_H, x_o(0) = x_0. \quad (5.53)$$

Proof. Set $r = 0$ in all the equations (5.46)-(5.50). This reduces (5.46) to a system with perfect measurement. Inequality (5.49) reduces to

$$dJ_0(K_o, K - K_o) \equiv \mathbf{E}\int_0^T \Big\{< (K - K_o)Lx_o, B^*\psi_1 >_U + < -(K - K_o)L\psi_2, B^*\varphi_o >_U \Big\} dt \ \geq 0, \quad (5.54)$$

for all $K \in \mathcal{B}_{ad}$. The adjoint equations (5.49)-(5.50) reduce to

$$-d\psi_1 = A^*\psi_1 dt + L^*K_o^*B^*\psi_1 dt - Q\psi_2 dt + Qx_o dt,\ \ t \in I, \quad (5.55)$$
$$d\psi_2 = A\psi_2 dt + BK_oL\psi_2 dt, \quad (5.56)$$
$$\psi_1(T) + M\psi_2(T) = Mx_o(T),\ \ \psi_2(0) = 0. \quad (5.57)$$

Note that equation (5.56) is a linear homogeneous differential equation with zero initial state. Thus $\psi_2(t) \equiv 0, t \in I$. Hence equation (5.55) reduces to

$$-d\psi_1 = A^*\psi_1 dt + L^*K_o^*B^*\psi_1 dt + Qx_o dt,\ \ t \in I, \quad (5.58)$$
$$\psi_1(T) = Mx_o(T),$$

and the inequality (5.54) reduces to

$$dJ_0(K_o, K - K_o) \equiv \mathbf{E}\int_0^T \big\{< (K - K_o)Lx_o, B^*\psi_1 >_U\big\} dt \ \geq 0, \quad (5.59)$$

for all $K \in \mathcal{B}_{ad}$. Clearly equation (5.58) coincides with (5.52) and the inequality (5.59) coincides with (5.51). Thus we have recovered all the necessary conditions of optimality for the system without uncertainty (perfect sensor), from the case with uncertainty, by simply letting $r \to 0$. This completes the proof. □

Remark 5.9. If A is assumed to be the infinitesimal generator of an analytic semigroup, we can admit more relaxed assumptions for the operators $\{B, \sigma\}$ appearing in equation (2.1). So let A be the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, on the UMD-type-2 Banach space E. Without loss of generality we may assume that $0 \in \rho(A)$, the resolvent set of A. For any $\eta \in (0,1)$, we can define the linear space

$$E_\eta \equiv \; \{x \in E : \parallel x \parallel_{E_\eta} \equiv \parallel (-A)^\eta x \parallel < \infty\}$$

which is a normed space generated by the fractional power of $(-A)$. With respect to the above norm topology these spaces are Banach spaces. For details on fractional powers of generators of analytic semigroups see Ahmed [3]. In fact we have the continuous embeddings

$$E_\eta \hookrightarrow E \hookrightarrow E_{-\eta}$$

where the space $E_{-\eta}$ is given by the completion of E with respect to the norm topology $\parallel x \parallel_{E_{-\eta}} \equiv \parallel (-A)^{-\eta} x \parallel_E$. In this case we can admit $B \in B_0(I, \mathcal{L}(U, E_{-\eta}))$ and $\sigma \in L_2^a(\Omega, \gamma(H, E_{-\eta}))$ for $0 \leq \eta < 1/2$. Under these relaxed assumptions, all the results of this paper remain valid.

6. Convergence of algorithm

In order to construct the optimal feedback operator valued function K_o we use the necessary conditions of optimality stated in Theorem 5.4.

For this we need some basic results on duality pairings involving the space of bounded linear operators. It is known that the topological dual of the Banach space of nuclear operators is the (Banach) space of bounded linear operators. For any pair of real Banach spaces $\{E, F\}$, the dual of the Banach space of nuclear operators $\mathcal{L}_1(E, F)$ is given by $(\mathcal{L}_1(E, F))^* = \mathcal{L}(E^*, F^*)$. By examining the pairing under the integral sign in the inequality (5.16), we note that it is of the form $< K, y \otimes u >$ with $K \in \mathcal{L}(Y, U)$ where Y is a reflexive Banach space and U is a Hilbert space. Clearly, $y \otimes u$ is an elementary nuclear operator and it belongs to $\mathcal{L}_1(Y^*, U)$ whose dual is given by $(\mathcal{L}_1(Y^*, U))^* \cong \mathcal{L}(Y, U)$. In general, an element $L \in \mathcal{L}_1(Y^*, U)$ has the form $L(y^*) = \sum y^*(y_i) u_i$ for all $y^* \in Y^*$ with $\{y_i \in Y, u_i \in U\}$ satisfying $\sum \parallel y_i \parallel_Y \parallel u_i \parallel_U < \infty$. For $\Lambda \in \mathcal{L}(Y, U)$ and $L \in \mathcal{L}_1(Y^*, U)$, the duality pairing

$$< \Lambda, L >= \sum (\Lambda(y_i), u_i)_U,$$

is well defined and it is easy to verify that

$$|< \Lambda, L >| \leq \| \Lambda \|_{\mathcal{L}(Y,U)} \| L \|_{\mathcal{L}_1(Y^*,U)} .$$

Using this notation, we can rewrite the inequality (5.16) in the following form:

$$\begin{aligned} & dJ_0(K_o, K - K_o) \\ & \equiv \mathbf{E} \int_0^T < (K - K_o), N_o >_{\mathcal{L}(Y,U),\mathcal{L}_1(Y^*,U)} \; dt \; \geq 0 \end{aligned} \tag{6.1}$$

for all $K \in \mathcal{B}_{ad}$, where

$$N_o \equiv [Lx_o + \nu(K_o^* B^* \varphi_o)] \otimes B^* \psi_1 + [\Gamma_o^* K_o^* B^* \psi_1 - L\psi_2] \otimes B^* \varphi_o. \tag{6.2}$$

By examining the elements defining N_o, one can easily check that it is an element of the tensor product space $Y \hat{\otimes} U$ for almost all $(t, \omega) \in I \times \Omega$. In fact we can consider N_o to be a function with values in the space of nuclear operators $\mathcal{L}_1(Y^*, U^*) = \mathcal{L}_1(Y^*, U)$, (as U is a Hilbert space identified with its own dual). Being the sum of tensor products of strongly measurable functions, it is a strongly measurable function with values in $\mathcal{L}_1(Y^*, U)$. Thus this process (function) is Bochner integrable (with respect to P measure) in the sense that

$$\mathbf{E}\{N_o\}(t) \equiv \int_\Omega N_o(t, \omega) P(d\omega) \equiv \hat{N}_o(t), t \in I, \tag{6.3}$$

is well defined and we have $\hat{N}_o(t) \in \mathcal{L}_1(Y^*, U), t \in I$. Since we want our feedback operator to be deterministic, using Fubini's theorem we can rewrite the inequality (6.1) as

$$\begin{aligned} & dJ_0(K_o, K - K_o) \\ & \equiv \int_0^T < (K - K_o), \hat{N}_o >_{\mathcal{L}(Y,U),\mathcal{L}_1(Y^*,U)} \; dt \; \geq 0, \; \forall \; K \in \mathcal{B}_{ad}, \end{aligned} \tag{6.4}$$

where $< \cdot, \cdot >$ denotes the duality pairing as indicated above. In order to proceed further we need the duality map $\Delta : \mathcal{L}_1(Y^*, U) \longrightarrow \mathcal{L}(Y, U)$ given by

$$\begin{aligned} \Delta(N) \equiv \Big\{ & R \in \mathcal{L}(Y, U) :< R, N >= \| R \|_{\mathcal{L}(Y,U)} \| N \|_{\mathcal{L}_1(Y^*,U)} \\ & = \| R \|^2_{\mathcal{L}(Y,U)} = \| N \|^2_{\mathcal{L}_1(Y^*,U)} \Big\}. \end{aligned}$$

This is a multivalued map. It follows from Hahn-Banach theorem that for every $N \neq 0$, the set $\Delta(N) \neq \emptyset$. With the above preparation, using the necessary conditions of optimality given by Theorem 5.4, we prove the following convergence result.

Theorem 6.1. *Suppose the assumptions of Theorem 5.4 hold and that the Banach space Y is separable. Then there exists a sequence of feedback operator valued functions $\{K_n\} \in \mathcal{B}_{ad}$ along which the functional J_0 is monotone decreasing and it converges to a (possibly local) minimum, say, $m_0 \geq 0$.*

Proof. Choose an arbitrary $K_1 \in \mathcal{B}_{ad}$ and solve the pair of equations (5.1)-(5.2) in the mild sense bearing on Theorem 5.1 and denote these mild solutions by $\{x_1, \varphi_1\}$. Use these solutions in the adjoint pair of equations (5.17)-(5.18) replacing the triple $\{K_o, x_o, \varphi_o\}$ by $\{K_1, x_1, \varphi_1\}$ and solve for the pair $\{\psi_{1,1}, \psi_{2,1}\}$ yielding the quintuple $\{K_1, x_1, \varphi_1, \psi_{1,1}, \psi_{2,1}\}$. Using this quintuple and the expression (6.2), we construct N_1 replacing the quintuple $\{K_o, x_o, \varphi_o, \psi_1, \psi_2\}$ by $\{K_1, x_1, \varphi_1, \psi_{1,1}, \psi_{2,1}\}$ giving

$$N_1 \equiv [Lx_1 + \nu(K_1^* B^* \varphi_1)] \otimes B^* \psi_{1,1} + [\Gamma_1^* K_1^* B^* \psi_{1,1} - L\psi_{2,1}] \otimes B^* \varphi_1, \quad (6.5)$$

where Γ_1 is given by $\Gamma_1 \equiv D\nu(K_1^* B^* \psi_{1,1})$. From this we obtain it's Bochner integral $\hat{N}_1$ as defined by the expression (6.3). Being the sum of tensor products of strongly measurable functions, $\hat{N}_1$ is a strongly measurable function on I with values in the Banach space $\mathcal{L}_1(Y^*, U)$. Let $\mathcal{L}_{so}(Y, U)$ denote the space of bounded linear operators $\mathcal{L}(Y, U)$ endowed with the strong operator topology τ_{so}. This is a locally convex sequentially complete Hausdorff topological space. Since Y is assumed to be separable, this topology is metrizable with the metric

$$d(T, L) \equiv \sum_{i=1}^{\infty} (1/2^i) \min\{1, \| (T - L)y_i \|_U\},$$

where $\{y_i\}$ is a dense subset of the closed unit ball $B_1(Y)$. We denote this metric space by $\mathcal{L}_{sod}(Y, U)$. It is easy to verify that this is a complete metric space but not separable. This metric topology is equivalent to the strong operator topology. Using this metric topology one can verify that the graph

$$\mathcal{G}r(\Delta) \equiv \{(T, N) \in \mathcal{L}(Y, U) \times \mathcal{L}_1(Y^*, U) : T \in \Delta(N)\}$$

of the multifunction Δ is closed with respect the norm topology on $\mathcal{L}_1(Y^*, U)$ and the metric topology d on $\mathcal{L}(Y, U)$. Thus the duality map $\Delta : \mathcal{L}_1(Y^*, U) \longrightarrow \mathcal{L}_{sod}(Y, U) = (\mathcal{L}(Y, U), d)$ is an upper semi-continuous multi function. Since $t \longrightarrow \hat{N}_1(t)$ is strongly measurable, the composition map $I \ni t \longrightarrow \hat{\Delta}_1(t) \equiv (\Delta \circ \hat{N}_1)(t) \equiv \Delta(\hat{N}_1(t))$ is a weakly measurable multi function in the sense that, for every open set $\mathcal{O} \in \mathcal{L}_{sod}(Y, U)$, the set $\{t \in I : \hat{\Delta}_1(t) \cap \mathcal{O} \neq \emptyset\}$ is Lebesgue measurable. We have seen that the metric space $\mathcal{L}_{sod}(Y, U)$ is complete but not separable, and so, not a Souslin space (continuous image of a Polish space). Thus, unfortunately, a well known measurable selection theorem like that of Yankov-Von Neumann and Aumann (selection theorem) [15, Theorem 1, p26] is not applicable. Since $\mathcal{L}_{sod}(Y, U)$ with the metric topology is not σ-compact either, measurable selection theorems, requiring the target space to be σ-compact [16, Theorem 1, p25], are also not applicable. In an excellent survey paper [17, Theorem 3.5, p98], Graf presents, under some mild assumptions on additivity and reducibility (with respect to the class of Lebesgue measurable subsets of the interval I), a general result on selection theorem which does not require separability.

Based on this result [17, Theorem 3.5, p98], we can assert that the multifunction $\hat{\Delta}_1$ has measurable selections. So we can choose a measurable selection $R_1 \in \hat{\Delta}_1$, that is, $R_1(t) \in \Delta(\hat{N}_1(t)), t \in I$, and define

$$K_2 \equiv K_1 - \varepsilon R_1,$$

for $\varepsilon > 0$ sufficiently small, so that $K_2 \in \mathcal{B}_{ad}$.

Now we evaluate J_0 at K_2 giving

$$\begin{aligned} J_0(K_2) &= J_0(K_1) + dJ_0(K_1, K_2 - K_1) + o(\varepsilon) \\ &= J_0(K_1) + \int_I < K_2 - K_1, \hat{N}_1 >_{\mathcal{L}(Y,U),\mathcal{L}_1(Y^*,U)} dt + o(\varepsilon) \\ &\equiv J_0(K_1) + << K_2 - K_1, \hat{N}_1 >> + o(\varepsilon), \qquad (6.6) \\ &\equiv J_0(K_1) - \varepsilon \parallel R_1 \parallel^2 + o(\varepsilon) = J_0(K_1) - \varepsilon \parallel \hat{N}_1 \parallel^2 + o(\varepsilon), \end{aligned}$$

where

$$\parallel R_1 \parallel^2 = \int_I \parallel R_1(t) \parallel^2_{\mathcal{L}(Y,U)} dt, \text{ and } \parallel \hat{N}_1 \parallel^2 = \int_I \parallel \hat{N}_1(t) \parallel^2_{\mathcal{L}_1(Y^*,U)} dt.$$

It is known that, for $1 \leq p < \infty, ((1/p) + (1/q) = 1)$, the dual of the Banach space $L_p(I, X)$ is given by $L_q(I, X^*)$ if, and only if, X^* has the Radon-Nikodym property (RNP) with respect to Lebesgue measure (or any finite measure) [17, Theorem 1, p98]. Generally, $\mathcal{L}(Y, U)$ does not have the RNP. Thus the double angle bracket in the expression (6.7) stands for the duality pairing between $L_2(I, \mathcal{L}_1(Y^*, U))$ and its weak dual $L_2^w(I, \mathcal{L}(Y, U))$. It is clear from the expression (6.7) that, for $\varepsilon > 0$ sufficiently small, we have $J_0(K_2) < J_0(K_1)$. Next, we use K_2 as constructed above and again solve the pair of equations (5.1)-(5.2) in the mild sense and denote these mild solutions by $\{x_2, \varphi_2\}$. Use these solutions in the adjoint pair of equations (5.17)-(5.18) replacing the triple $\{K_o, x_o, \varphi_o\}$ by $\{K_2, x_2, \varphi_2\}$ and solve for the pair $\{\psi_{1,2}, \psi_{2,2}\}$ yielding the quintuple $\{K_2, x_2, \varphi_2, \psi_{1,2}, \psi_{2,2}\}$. Using this and the expression (6.2), we construct N_2 replacing the quintuple $\{K_o, x_o, \varphi_o, \psi_1, \psi_2\}$ by $\{K_2, x_2, \varphi_2, \psi_{1,2}, \psi_{2,2}\}$ giving

$$N_2 \equiv [Lx_2 + \nu(K_2^* B^* \varphi_2)] \otimes B^* \psi_{1,2} + [\Gamma_2^* K_2^* B^* \psi_{1,2} - L\psi_{2,2}] \otimes B^* \varphi_2, \quad (6.7)$$

where Γ_2 is given by $\Gamma_2 \equiv D\nu(K_2^* B^* \psi_{1,2})$. Denote the Bochner integral of N_2 by $\hat{N}_2$ and construct the corresponding multifunction $\hat{\Delta}_2$ as in the first step, that is, $\hat{\Delta}_2(t) \equiv \Delta(\hat{N}_2(t)), t \in I$. Choose a measurable selection $R_2 \in \hat{\Delta}_2$, and define $K_3 = K_2 - \varepsilon R_2$, for $\varepsilon > 0$ sufficiently small. Using this K_3 we obtain an expression similar to that of (6.7) giving

$$J_0(K_3) = J_0(K_2) - \varepsilon \parallel R_2 \parallel^2 + o(\varepsilon) = J_0(K_2) - \varepsilon \parallel \hat{N}_2 \parallel^2 + o(\varepsilon). \qquad (6.8)$$

It is clear from this expression that, for $\varepsilon > 0$ sufficiently small, $J_0(K_3) < J_0(K_2)$. Hence we have $J_0(K_3) < J_0(K_2) < J_0(K_1)$. Repeating this process, we can construct a sequence of operator valued functions $\{K_n\} \subset \mathcal{B}_{ad}$ which satisfies the following inequalities

$$\cdots < J_0(K_n) < J_0(K_{n-1}) < J_0(K_{n-2}) < \cdots < J_o(K_2) < J_0(K_1).$$

Since $J_0(K) \geq 0$, for all $K \in \mathcal{B}_{ad}$, this shows that there exists a sequence $\{K_n\}$ in $\mathcal{B}_{ad}$ and a finite positive number m_0 such that $\lim_{n\to\infty} J_0(K_n) = m_0 \geq 0$, where m_0 is possibly a local minimum. This completes the proof. □

Open Problem: It would be interesting to extend the results of this paper to nonlinear systems of the form

$$dx = Axdt + F(t,x)dt + Budt + \sigma(t,x)dW, x(0) = x_0, \qquad (6.9)$$

$$y(t) = L(t,x) + \xi(t), \qquad (6.10)$$

$$u(t) = K(t)y(t), t \in I. \qquad (6.11)$$

The objective is to find $K \in \mathcal{B}_{ad}$ that minimizes the functional $J_o(K) \equiv \sup\{J(K,\xi), \xi \in \mathcal{D}\}$, where

$$J(K,\xi) = \mathbf{E}\Big\{\int_I \ell(t,x)dt + \Psi(x(T))\Big\}. \qquad (6.12)$$

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