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ON APPLICATIONS OF GENERALIZED F-CONTRACTION TO DIFFERENTIAL EQUATIONS

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Abstract. In the present work, we introduce the new concept of a generalization of Geraghty type F-Berinde contraction mappings and establish certain existence results for such mappings. Some examples will embellish the results, for the same computer simulation is done. Our examples involve a series of complicated structured functions which cannot be treated by classical fixed point methods. Our findings extend, unify and enrich a multitude results in the existing literature. As an application, we apply our abstract results to establish

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the existence of solution of differential equations of first and second order to exhibit the potency and viability of our results. At the end, as an open problem, we suggest storekeeper's control problem in terms of Volterra integral equation whose solution may be procured from the established results.

1. Introduction and preliminaries

Banach contraction principle is one of the decisive results and fascinating emerging field in the nonlinear analysis especially in fixed point theory and it has been developed in many directions. Geraghty [7] established a fixed point result as a generalization of Banach contraction principle[2] by introducing family of functions Θ . In 2003, Berinde [3] adorned the concept of nonlinear type weak contraction operating a comparison function in a metric space. He enhanced that Kannan's, Banach's, Chatterjea's mappings are weak contractions (see [5], [6] also). Subsequently, a lot of generalizations of these results in some spaces arrived in the literature. Recently, Shukla [13] clubbed both the concepts of partial metric spaces and b-metric spaces and introduced partial b-metric spaces as a generalization of both of these spaces. This concept was further refined by Mustafa [9] to reveal that each partial b-metric p_b generates a b-metric d_{p_b}.

In recent investigations, Wardowski [15] described a new contraction called F-contraction and acquired a fixed point result as a generalization of Banach contraction principle. Recently, Piri et al. [11] purified the result of Wardowski [15] by launching some weaker conditions on the self mapping concerning a complete metric space and over the mapping F .

The importance of the systems of integral, differential and integro-differential equations in the study of problems emerging from the real world, have made them an extensive and influential research topic in science and engineering, and it is often significant develop numerical techniques to legitimize the existence problems of the solutions for such problems. These methods have some noteworthy advantages over others numerical techniques because these are very smooth to implement in a computer as per the need.

In the aforesaid context, the purpose of this paper is to establish the existence of solution of ordinary differential equations of first and second order by inaugurating a new class of generalized F-contraction called Geraghty Type F-Berinde contraction in the framework of partial b-metric spaces. Further our findings are authenticated with some non-trivial examples.

For the convenience of the readers and selfdependency of the article, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote the set of natural numbers, the set of all non-negative real numbers and the set real numbers respectively.

First we start with some notations and concepts that are useful tool in subsequent analysis.

Definition 1.1. ([13]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(p_{b1}) $x = y$ iff $p_b(x,x) = p_b(x,y) = p_b(y,y)$;
(p_{b2}) $p_b(x,x) \leq p_b(x,y)$;
(p_{b3}) $p_b(x,y) = p_b(y,x)$;
(p_{b4}) $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

Observe that, if $s = 1$ in Definition 1.1, the pair (X, p_b) is called a partial metric and denoted by (X, p). In order to show that each partial b-metric p_b prompts a b-metric d_{p_b}, Mustafa et al. [9] revised the Definition 1.1 and replaced condition (p_{b4}) by ($p_{b4}^{'}$) as follows:

($p_{b4}^{'}$) $p_b(x,y) \leq s(p_b(x,z) + p_b(z,y) - p_b(z,z)) + (\frac{1-s}{2})(p_b(x,x) + p_b(y,y))$.

Remark 1.2. The class of partial b-metric space (X, p_b) is effectively larger than the class of partial metric space and b-metric space as well.

Proposition 1.3. ([9]) *Every partial b-metric p_b defines a b-metric d_{p_b}, where*

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y)$$

for all $x, y \in X$.

For covergence , Cauchy sequence and completeness, in the context of partial b-metric space, we refer [13].

Lemma 1.4. ([9]) *Let (X, p_b) be a partial b-metric space. Then*

(1) *a sequence $\{x_n\}$ is a p_b-Cauchy sequence in (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b});*
(2) *(X, p_b) is p_b-complete if and only if the b-metric space (X, d_{p_b}) is complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x_n, x) = 0$ if and only if $p_b(x,x) = \lim_{n\to\infty} p_b(x_n, x) = \lim_{n,m\to\infty} p_b(x_n, x_m)$.*

On the other hand, Wardowski [15] defined a new type of contraction $T : X \to X$ called F- contraction as follows:

$$\forall x, y \in X\Big(d(Tx,Ty) > 0 \implies \tau + F(d(Tx,Ty)) \leq F(d(x,y))\Big),$$

where F is defined in [15]. Later, Secelean et al. [12], Piri et al. [11] extended and refined the above definition of Wardowski [15].

Throughout for our succeeding discussion, we denote the set of all functions satisfying ($F1$) of [15] , ($F2^{'}$) of [12] and ($F3^{'}$) of [11] by $\aleph_F$.

Definition 1.5. ([7]) Let Θ denote the family of all functions $\theta : [0,\infty) \to [0,1)$ such that for any bounded sequence $\{p_n\}$ of positive reals, $\theta(p_n) \to 1 \Rightarrow p_n \to 0$.

For our further discussion subsequent family of functions will be employed.

(1) Let Ψ be a family of functions $\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and $\psi(p) = 0 \iff p = 0$.
(2) Also let Φ be the set of all functions $\phi : [0,\infty) \to [0,\infty)$ such that ϕ is non-decreasing, continuous and $\phi(p) = 0 \iff p = 0$.

2. Fixed point results for generalized F-contraction

We begin this section by introducing our very first and important definition.

Definition 2.1. Let (X, p_b) be a partial b-metric space. A mapping $T : X \to X$ is said to be a Geraghty type generalized F-Berinde contraction on X, if there exists $\theta \in \Theta$, $\phi \in \Phi$, $F \in \aleph_F$, $\psi \in \Psi$ and $L \geq 0$ such that for all $x, x^* \in X$ and $s > 1$ with $p_b(Tx, Tx^*) > 0$,

$$\begin{aligned} F(s^\epsilon p_b(Tx, Tx^*)) \leq \theta\Big(\phi\big(\mathcal{M}_\mathcal{T}(x, x^*)\big)\Big) F\Big(\mathcal{M}_\mathcal{T}(x, x^*)\Big) \\ + L\mathcal{N}_\mathcal{T}(x, x^*) - \psi\Big(\mathcal{M}_\mathcal{T}(x, x^*)\Big), \end{aligned} \tag{2.1}$$

where

$$\mathcal{M}_\mathcal{T}(x, x^*) = \max\Big\{p_b(x, x^*), p_b(x, Tx), p_b(x^*, Tx^*), \frac{p_b(x, Tx^*) + p_b(x^*, Tx)}{2s}\Big\},$$

$$\mathcal{N}_\mathcal{T}(x, x^*) = \min\Big\{p_b^m(x, Tx), p_b^m(x^*, Tx^*), p_b^m(x, Tx^*), p_b^m(x^*, Tx)\Big\},$$

$$p_b^m(x, x^*) = p_b(x, x^*) - \min\big\{p_b(x, x), p_b(x^*, x^*)\big\}$$

and $\epsilon > 1$ is a constant.

Following Example substantiates the validity of Definition 2.1

Example 2.2. Let $X = [0,\infty)$ be equipped with partial metric $p_b : X \times X \to [0,\infty)$ defined by

$$p_b(x, x^*) = [\max\{x, x^*\}]^2,$$

for all $x, x^* \in X$. It is obvious that, (X, p_b) is a complete partial b-metric space with $s = 2$.

Let the mapping $T : X \to X$ is defined by

$$Tx = 2x + \sin\Big(\frac{\sqrt{x^3}}{3x + \sqrt{2}}\Big).$$

Define $\theta : [0,\infty) \to [0,1)$ by $\theta(c) = \frac{1}{c+1}$ and $\phi, \psi : [0,\infty) \to [0,\infty)$ be given by $\phi(c) = log(1 + c)$, $\psi(c) = log(2^c)$. Also let $F(c) = log(c)$ for all $c \in \mathbb{R}^+$.

Without loss of generality we take $x, x^* \in X$ with $x < x^*$. Now L.H.S. of (2.1)

$$\begin{aligned}
F(s^\epsilon p_b(Tx, Tx^*)) &= F\Big(2^\epsilon \max(Tx, Tx^*)\Big) \\
&= F(2^\epsilon (Tx^*)^2) \\
&= F\left(2^\epsilon \Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^* + \sqrt{2}}\Big)\Big)^2\right) \\
&= log\left(2^\epsilon \Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^* + \sqrt{2}}\Big)\Big)^2\right).
\end{aligned}$$

Employing the definitions of $\mathcal{M}_\mathcal{T}(x, x^*)$ and $\mathcal{N}_\mathcal{T}(x, x^*)$, and with suitable calculations, we have

$$\mathcal{M}_\mathcal{T}(x, x^*) = (Tx^*)^2 \text{ and } \mathcal{N}_\mathcal{T}(x, x^*) = \min\big\{(Tx)^2 - x^2,\, p_b^m(x^*, Tx)\big\}$$

Thus R.H.S. of (2.1) becomes

$$\begin{aligned}
&\theta\Big(\phi\big(\mathcal{M}_\mathcal{T}(x, x^*)\big)\Big)F\Big(\mathcal{M}_\mathcal{T}(x, x^*)\Big) + L\min\big\{(Tx)^2 - x^2,\, p_b^m(x^*, Tx)\big\} \\
&\quad - \psi\Big(\mathcal{M}_\mathcal{T}(x, x^*)\Big) \\
&= \theta\Big(\phi\big((Tx^*)^2\big)\Big)F\Big((T)^2\Big) + L\min\big\{(Tx)^2 - x^2,\, p_b^m(x^*, Tx)\big\} - \psi\Big((Tx^*)^2\Big) \\
&= \theta\bigg(\phi\bigg(\Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^* + \sqrt{2}}\Big)\Big)^2\bigg)\bigg)F\bigg(\Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^* + \sqrt{2}}\Big)\Big)^2\bigg) \\
&\quad + L\min\big\{(Tx)^2 - x^2,\, p_b^m(x^*, Tx)\big\} - \psi\bigg(\Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^* + \sqrt{2}}\Big)\Big)^2\bigg) \\
&= \frac{log\left\{\Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^*+\sqrt{2}}\Big)\Big)^2\right\}}{log\left\{1 + \Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^*+\sqrt{2}}\Big)\Big)^2\right\} + 1} \\
&\quad + L\min\left\{\Big(2x + \sin\Big(\frac{\sqrt{x^3}}{3x + \sqrt{2}}\Big)\Big)^2 - x^2, p_b^m(x^*, Tx)\right\} \\
&\quad - log\left\{2^{\Big(2x^* + \sin\Big(\frac{\sqrt{(x^*)^3}}{3x^*+\sqrt{2}}\Big)\Big)^2}\right\},
\end{aligned}$$

for all $x, x^* \in X = [0, \infty]$ and $L \geq 0$. Clearly for $\epsilon = 1.2$ and $L = 1$, one can see that R.H.S. of (2.1) dominates L.H.S. as shown in the following Figure.

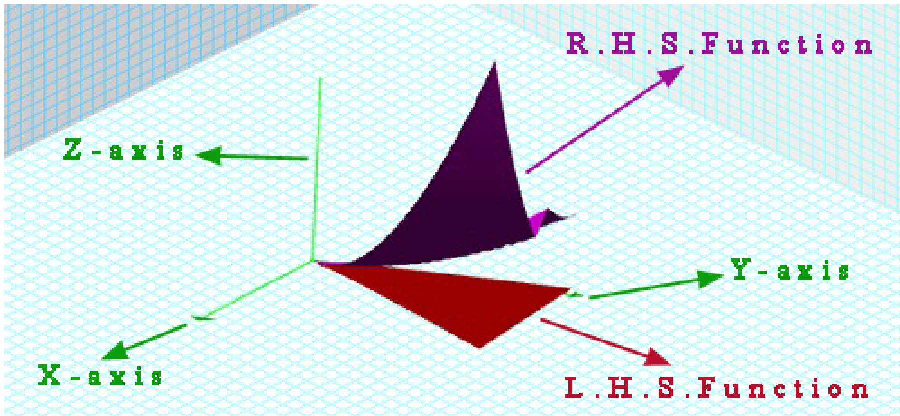

FIGURE 1. R.H.S. Superimposes L.H.S., 3D view.

Remark 2.3. Example 2.2 holds for all $\epsilon \in (1, 1.9]$ and $L \geq 0$.

Next theorem is proved for Geraghty type generalized F-Berinde contractive mappings in partial b-metric spaces.

Theorem 2.4. *Let (X, p_b) be a complete partial b-metric space and $T : X \to X$ be Geraghty type generalized F-Berinde contraction. If T is continuous, then T has a unique fixed point $w \in X$.*

Proof. For an arbitrary point $x_0 \in X$, we construct a sequence $\{x_n\}$ in X such that

$$x = x_0 \text{ and } x_n = Tx_{n-1}, \quad \forall\, n \in \mathbb{N}.$$

If there exists $n_o \in \mathbb{N}$ such that $p_b(x_{n_0}, Tx_{n_0}) = 0$, then x_{n_0} is the desired fixed point and we are through.

Consequently, for the subsequent discussion, we assume that $p_b(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By taking $x = x_{n-1}$ and $x^* = x_n$ in (2.1), we have

$$\begin{aligned} F(p_b(x_n, x_{n+1})) &\leq F(s^\epsilon p_b(Tx_{n-1}, Tx_n)) \\ &\leq \theta\big(\phi\big(\mathcal{M}_T(x_{n-1}, x_n)\big)\big) F\big(\mathcal{M}_\mathcal{T}(x_{n-1}, x_n)\big) \\ &\quad + L\mathcal{N}_\mathcal{T}(x_{n-1}, x_n) - \psi\big(\mathcal{M}_\mathcal{T}(x_{n-1}, x_n)\big), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{M}_\mathcal{T}(x_{n-1}, x_n) = \max\Big\{ & p_b(x_{n-1}, x_n), p_b(x_{n-1}, x_n), \\ & p_b(x_n, x_{n+1}), \frac{p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n)}{2s} \Big\}, \end{aligned}$$

and

$$\mathcal{N}_{\mathcal{T}}(x_{n-1},x_n)=\min\Big\{p_b^m(x_{n-1},x_n),p_b^m(x_n,x_{n+1}),\\ p_b^m(x_{n-1},x_{n+1}),p_b^m(x_n,x_n)\Big\}.$$

Since

$$\begin{aligned}\frac{p_b(x_{n-1},x_{n+1})+p_b(x_n,x_n)}{2s}&\le\frac{p_b(x_{n-1},x_n)+p_b(x_n,x_{n+1})}{2}\\&\le\max\Big\{p_b(x_{n-1},x_n),p_b(x_n,x_{n+1})\Big\}\end{aligned}$$

and

$$\begin{aligned}p_b^m(x_n,Tx_{n-1})&=p_b^m(x_n,x_n)\\&=p_b(x_n,x_n)-\min\Big\{p_b(x_n,x_n),p_b(x_n,x_n)\Big\}\\&=0,\end{aligned}$$

we have

$$\mathcal{M}_{\mathcal{T}}(x_{n-1},x_n)=\max\Big\{p_b(x_{n-1},x_n),p_b(x_n,x_{n+1})\Big\}\text{ and }\mathcal{N}_{\mathcal{T}}(x_{n-1},x_n)=0.$$

If $\max\Big\{p_b(x_{n-1},x_n),p_b(x_n,x_{n+1})\Big\}=p_b(x_n,x_{n+1})$, for all $n\in\mathbb{N}$. From (2.2) and by the hypothesis of θ and ψ, we obtain

$$\begin{aligned}F(p_b(x_n,x_{n+1}))&\le F(s^\epsilon p_b(Tx_{n-1},Tx_n))\\&\le\theta\big(\phi\big(\mathcal{M}_{\mathcal{T}}(x_{n-1},x_n)\big)\big)F\big(\mathcal{M}_{\mathcal{T}}(x_{n-1},x_n)\big)\\&\quad+L\mathcal{N}_{\mathcal{T}}(x_{n-1},x_n)-\psi\big(\mathcal{M}_{\mathcal{T}}(x_{n-1},x_n)\big)\\&\le\theta\big(\phi\big(p_b(x_n,x_{n+1})\big)\big)F\big(p_b(x_n,x_{n+1})\big)-\psi\big(p_b(x_n,x_{n+1})\big)\\&\le F\big(p_b(x_n,x_{n+1})\big)-\psi\big(p_b(x_n,x_{n+1})\big)\\&<F\big(p_b(x_n,x_{n+1})\big),\end{aligned}$$

which is a contradiction. Thus we conclude that $\mathcal{M}_{\mathcal{T}}(x_{n-1},x_n)=p_b(x_{n-1},x_n)$. From (2.2) and again by the definitions of θ and ψ, we have

$$\begin{aligned}F(p_b(x_n,x_{n+1}))&\le F(s^\epsilon p_b(Tx_{n-1},Tx_n))\\&\le F\big(p_b(x_{n-1},x_n)\big)-\psi\big(p_b(x_{n-1},x_n)\big)\\&<F\big(p_b(x_{n-1},x_n)\big).\end{aligned}\tag{2.3}$$

Thus the sequence $\{p_b(x_n,x_{n+1})\}$ is a non increasing sequence of non-negative numbers and is bounded below and hence it is convergent to some point, say $\gamma\ge0$. i.e.,

$$\lim_{n\to\infty}p_b(x_n,x_{n+1})=\gamma.$$

Letting $n \to \infty$ in (2.3), we obtain

$$F(\gamma) \leq F(\gamma) - \psi(\gamma).$$

This implies that $\psi(\gamma) = 0$ and thus $\gamma = 0$. Hence,

$$\lim_{n\to\infty} p_b(x_n, Tx_n) = \lim_{n\to\infty} p_b(x_n, x_{n+1}) = 0. \tag{2.4}$$

Further, by using the property (p_{b_2}) of partial b-metric space, we have the following equality

$$\lim_{n\to\infty} p_b(x_n, x_n) = 0. \tag{2.5}$$

Next, we claim that $\{x_n\}$ is a p_b-Cauchy sequence in X. By applying Lemma 1.4 we need to prove that $\{x_n\}$ is a b-Cauchy sequence in the b-metric space (X, d_{p_b}). Suppose, on the contrary, that there exists $\delta > 0$ such that for an integer k there exist integer $m'(k) > m(k) \geq k$ such that

$$d_{p_b}(x_{m(k)}, x_{m'(k)}) \geq \delta. \tag{2.6}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $m'(k)$ satisfying (2.6) such that

$$d_{p_b}(x_{m(k)}, x_{m'(k)-1}) < \delta. \tag{2.7}$$

From (2.6), we get

$$\delta \leq d_{p_b}(x_{m(k)}, x_{m'(k)}) \leq s d_{p_b}(x_{m(k)}, x_{m'(k)-1}) + s d_{p_b}(x_{m'(k)-1}, x_{m'(k)}). \tag{2.8}$$

Which on letting limit $k \to \infty$ and using (2.7) give rise to

$$\frac{\delta}{s} \leq \lim_{k\to\infty} \inf d_{p_b}(x_{m(k)}, x_{m'(k)-1}) \leq \lim_{k\to\infty} \sup d_{p_b}(x_{m(k)}, x_{m'(k)-1}) \leq \delta. \tag{2.9}$$

Also from (2.7), (2.8) and (2.9), we have

$$\delta \leq \lim_{k\to\infty} \sup d_{p_b}(x_{m(k)}, x_{m'(k)}) \leq s\delta. \tag{2.10}$$

Moreover,

$$d_{p_b}(x_{m(k)+1}, x_{m'(k)-1}) \leq s d_{p_b}(x_{m(k)+1}, x_{m(k)}) + s d_{p_b}(x_{m(k)}, x_{m'(k)-1}).$$

Which yields

$$\lim_{k\to\infty} \sup d_{p_b}(x_{m(k)+1}, x_{m'(k)-1}) \leq s\delta. \tag{2.11}$$

Further, by using triangle inequality, we obtain

$$\begin{aligned} \delta &\leq d_{p_b}(x_{m'(k)}, x_{m(k)}) \\ &\leq s d_{p_b}(x_{m'(k)}, x_{m(k)+1}) + s d_{p_b}(x_{m(k)+1}, x_{m(k)}) \end{aligned} \tag{2.12}$$

and

$$d_{p_b}(x_{m'(k)}, x_{m(k)+1}) \leq s d_{p_b}(x_{m'(k)}, x_{m(k)}) + s d_{p_b}(x_{m(k)}, x_{m(k)+1}). \tag{2.13}$$

It follows from (2.4), (2.10), (2.12), and (2.13) that

$$\frac{\delta}{s} \leq \lim_{k\to\infty} \sup d_{p_b}(x_{m'(k)}, x_{m(k)+1}) \leq s^2\delta. \tag{2.14}$$

Now, by using Proposition (1.3) in (2.9), (2.10), (2.11) and (2.14), one will get

$$\frac{\delta}{2s} \leq \lim_{k\to\infty} \sup p_b(x_{m(k)}, x_{m'(k)-1}) \leq \frac{\delta}{2}. \tag{2.15}$$

$$\frac{\delta}{2} \leq \lim_{k\to\infty} \sup p_b(x_{m(k)}, x_{m'(k)}) \leq \frac{s\delta}{2}. \tag{2.16}$$

$$\lim_{k\to\infty} \sup p_b(x_{m(k)+1}, x_{m'(k)-1}) \leq \frac{s\delta}{2}. \tag{2.17}$$

$$\frac{\delta}{2s} \leq \lim_{k\to\infty} \sup p_b(x_{m'(k)}, x_{m(k)+1}) \leq \frac{s^2\delta}{2}. \tag{2.18}$$

On the other hand

$$\begin{aligned} p_b^m(x_{m(k)}, x_{m(k)+1}) &= p_b(x_{m(k)}, x_{m(k)+1}) \\ &\quad - \min\left\{p_b(x_{m(k)}, x_{m(k)}), p_b(x_{m(k)+1}, x_{m(k)+1})\right\} \\ &\leq p_b(x_{m(k)}, x_{m(k)+1}) \end{aligned}$$

and thanks to (2.4), we obtain

$$\lim_{k\to\infty} p_b^m(x_{m(k)}, x_{m(k)+1}) = 0. \tag{2.19}$$

Since $F\Big(p_b(x_{m(k)+1}, x_{n(k)})\Big) = F\big(p_b(Tx_{m(k)}, Tx_{m'(k)-1})\big) > 0$, by the assumption of the theorem, we have

$$\begin{aligned} &F(p_b(x_{m(k)+1}, x_{m'(k)})) \\ &\leq F\big(s^\epsilon p_b(Tx_{m(k)}, Tx_{m'(k)-1})\big) \\ &\leq \theta\big(\phi\big(\mathcal{M}_\mathcal{T}(x_{m(k)}, x_{m'(k)-1})\big)\big) F\big(\mathcal{M}_\mathcal{T}(x_{m(k)}, x_{m'(k)-1})\big) \\ &\quad + L\mathcal{N}_\mathcal{T}(x_{m(k)}, x_{m'(k)-1}) - \psi\big(\mathcal{M}_\mathcal{T}(x_{m(k)}, x_{m'(k)-1})\big). \end{aligned} \tag{2.20}$$

By using definitions of $\mathcal{M}_\mathcal{T}$ and $\mathcal{N}_\mathcal{T}$ along with inequalities (2.15), (2.16), (2.17) and (2.19), one will

$$\lim_{k\to\infty} \sup \mathcal{M}_\mathcal{T}(x_{m(k)}, x_{m'(k)-1}) \leq \frac{\delta}{2} \tag{2.21}$$

and

$$\lim_{k\to\infty} \sup \mathcal{N}_\mathcal{T}(x_{m(k)}, x_{m'(k)-1}) = 0. \tag{2.22}$$

Indeed,

$$\mathcal{M}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1}) = \max\Big\{p_b(x_{m(k)}, x_{m'(k)-1}), p_b(x_{m(k)}, x_{m(k)+1}), p_b(x_{m'(k)-1}, x_{m'(k)}), \frac{p_b(x_{m(k)}, x_{m'(k)}) + p_b(x_{m'(k)-1}, x_{m(k)+1})}{2s}\Big\}$$

and

$$\mathcal{N}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1}) = \min\Big\{p_b^m(x_{m(k)}, x_{m(k)+1}), p_b^m(x_{m'(k)-1}, x_{m'(k)}), p_b^m(x_{m(k)}, x_{m'(k)}), p_b^m(x_{m'(k)-1}, x_{m(k)+1})\Big\},$$

Hence, we have

$$\lim_{k\to\infty} \sup \mathcal{M}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1}) \le \max\Big\{\frac{\delta}{2}, 0, 0, \frac{1}{2s}\big[\frac{s\delta}{2} + \frac{s\delta}{2}\big]\Big\} \le \frac{\delta}{2}.$$

and

$$\lim_{k\to\infty} \sup \mathcal{N}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1}) = 0.$$

Taking lim sup as $n \to \infty$ in (2.20) and using (2.18), (2.21) and (2.22), we have

$$\begin{aligned} F(\frac{\delta}{2}) &= F\big(s\frac{\delta}{2s}\big) \\ &\le \lim_{k\to\infty} \sup F\Big(s^{\epsilon} p_b(Tx_{m(k)}, Tx_{m'(k)-1})\Big) \\ &\le \lim_{k\to\infty} \sup \theta\Big(\phi\Big(\mathcal{M}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1})\Big)\Big) F\big(\mathcal{M}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1})\big) \\ &\quad + L \lim_{k\to\infty} \sup \mathcal{N}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1}) - \lim_{k\to\infty} \sup \psi\big(\mathcal{M}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1})\big) \\ &\le \lim_{k\to\infty} \sup\, \theta\Big(\phi\Big(\mathcal{M}_{\mathcal{T}}(x_{m(k)}, x_{m'(k)-1})\Big)\Big) F(\frac{\delta}{2}) - F(\frac{\delta}{2}), \end{aligned}$$

which is a contradiction, since $\theta \in [0, 1)$. Thus $\{x_n\}$ is a b-Cauchy sequence in the b-metric space (X, d_{p_b}), hence from Lemma 1.4 $\{x_n\}$ is a p_b-Cauchy sequence in the partial b-metric space (X, p_b). As (X, p_b) is complete so that by Lemma 1.4, b-metric space (X, d_{p_b}) is b-complete. Therefore, the sequence $\{x_n\}$ converges to some point $w \in X$, that is, $\lim_{n\to\infty} d_{p_b}(x_n, w) = 0$.

Again, from Lemma 1.4

$$\lim_{n\to\infty} p_b(x_n, w) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(w, w) = 0. \tag{2.23}$$

Next, We will prove that $p_b(w,Tw)=0$. Assume to the contrary that $p_b(w,Tw)>0$, then from inequality (2.1), we obtain

$$\begin{aligned} F(p_b(Tw,x_{n+1})) &\leq F(s^{\epsilon}p_b(Tw,Tx_n)) \\ &\leq \theta\big(\phi(\mathcal{M}_{\mathcal{T}}(w,x_n))\big)F\big(\mathcal{M}_{\mathcal{T}}(w,x_n)\big) \\ &\quad + L\mathcal{N}_{\mathcal{T}}(w,x_n)-\psi\big(\mathcal{M}_{\mathcal{T}}(w,x_n)\big), \end{aligned} \tag{2.24}$$

where

$$\mathcal{M}_{\mathcal{T}}(w,x_n)=\max\Big\{p_b(w,x_n),p_b(w,Tw),p_b(x_n,x_{n+1}), \frac{p_b(w,x_{n+1})+p_b(x_n,Tw)}{2s}\Big\}$$

and

$$\mathcal{N}_{\mathcal{T}}(x_n,u)=\min\Big\{p_b^m(w,Tw),p_b^m(x_n,x_{n+1}),p_b^m(w,x_{n+1}),p_b^m(x_n,Tw)\Big\}.$$

Passing limit as $n\to\infty$ and using (2.4) and (2.23) , we have

$$\lim_{n\to\infty}\mathcal{M}_{\mathcal{T}}(x_n,w)=\max\Big\{0,p_b(w,Tw),0,\frac{p_b(w,Tw)}{2s}\Big\}=p_b(w,Tw). \tag{2.25}$$

Moreover, from (2.4) and (2.5), we have

$$\lim_{n\to\infty}p_b^m(x_n,Tx_n)=0.$$

Consequently

$$\lim_{n\to\infty}\mathcal{N}_{\mathcal{T}}(x_n,u)=0. \tag{2.26}$$

Now, letting $n\to\infty$ in (2.24) and utilizing (2.25) and (2.26) togather with continuity of T and property of function F, we obtain

$$\begin{aligned} F(p_b(Tw,w)) &\leq F(s^{\epsilon}p_b(Tw,w)) \\ &\leq \lim_{n\to\infty}\theta\big(\phi(\mathcal{M}_{\mathcal{T}}(w,x_n))\big)F\big(p_b(w,Tw)\big)-\psi\big(p_b(w,Tw)\big) \\ &\leq \lim_{n\to\infty}\theta\big(\phi(\mathcal{M}_{\mathcal{T}}(w,x_n))\big)F\big(p_b(w,Tw)\big). \end{aligned}$$

Which implies

$$1\leq\lim_{n\to\infty}\theta\big(\phi(\mathcal{M}_{\mathcal{T}}(w,x_n))\big).$$

Using the hypothesis of θ and ϕ, the above inequality turns into

$$\lim_{n\to\infty}\theta\big(\phi(\mathcal{M}_{\mathcal{T}}(w,x_n))\big)=1$$

or

$$\lim_{n\to\infty}\phi\big(\mathcal{M}_{\mathcal{T}}(w,x_n)\big)=0$$

which further implies that

$$\lim_{n\to\infty}\mathcal{M}_{\mathcal{T}}(w,x_n)=0,$$

which is a contradiction. Hence $p_b(w,Tw)=0$, that is, T has a fixed point $w\in X$. For the uniqueness of fixed point, suppose u and v are two fixed points of T, such that $p_b(u,v)>0$, then from (2.1), we obtain

$$\begin{aligned}F(p_b(u,v)) &= F(p_b(Tu,Tv))\\ &\leq F(s^{\epsilon}p_b(p_b(Tu,Tv)))\\ &\leq \theta\Big(\phi\Big(\mathcal{M}_{\mathcal{T}}(u,v)\Big)F(\mathcal{M}_{\mathcal{T}}(u,v))+L\mathcal{N}_{\mathcal{T}}(u,v)-\psi(\mathcal{M}_{\mathcal{T}}(u,v))\\ &\leq F(p_b(u,v))-\psi(p_b(u,v))<F(p_b(u,v)).\end{aligned}$$

Which is a contradiction. Hence $(p_b(u,v))=0$. Thus fixed point of T is unique. □

Following Example authenticates the validity of the hypothesis of Theorem 2.4.

Example 2.5. Let $X=[0,\frac{1}{2}]$ be equipped with partial metric $p_b:X\times X\to[0,\infty)$ defined by

$$p_b(x,x^*)=[\max\{x,x^*\}]^2,$$

for all $x,x^*\in X$. Then, (X,p_b) is a complete partial b-metric space with $s=2$. Let the mapping $T:X\to X$ is defined by

$$Tx=\frac{x}{3}+xlog\Big(1+x^{\frac{1}{3}}\Big).$$

Define $\theta:[0,\infty)\to[0,1)$ by $\theta(c)=\frac{1}{1+log(2^c)}$ and $\phi,\psi:[0,\infty)\to[0,\infty)$ be given by $\phi(c)=log(2^c)$ and $\psi(c)=log(1+c)$. Also $F(c)=log(c)$ for all $c\in\mathbb{R}^+$. Without loss of generality we take $x,x^*\in X$ with $x>x^*$.
Now L.H.S.

$$\begin{aligned}F(s^{\epsilon}p_b(Tx,Tx^*)) &= F\Big(2^{\epsilon}\max(Tx,Tx^*)\Big)\\ &= F(2^{\epsilon}(Tx)^2)\\ &= F\left(2^{\epsilon}\Big(\frac{x}{3}+xlog\Big(1+x^{\frac{1}{3}}\Big)\Big)^2\right)\\ &= log\left(2^{\epsilon}\Big(\frac{x}{3}+xlog\Big(1+x^{\frac{1}{3}}\Big)\Big)^2\right).\end{aligned}$$

Utilizing the definitions of $\mathcal{M}_{\mathcal{T}}(x,x^*)$ and $\mathcal{N}_{\mathcal{T}}(x,x^*)$, and with suitable calculations, we acquire

$$\mathcal{M}_{\mathcal{T}}(x,x^*)=x^2 \text{ and } \mathcal{N}_{\mathcal{T}}(x,x^*)=\min\big\{(x^*)^2-(Tx^*)^2,\, p_b^m(x^*,Tx)\big\}$$

Thus R.H.S. becomes

$$\begin{aligned}
&\theta\Big(\phi\big(\mathcal{M}_{\mathcal{T}}(x,x^*)\big)\Big)F\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)+L\mathcal{N}_{\mathcal{T}}(x,x^*)-\psi\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)\\
&=\theta\big(\phi(x^2)\big)F(x^2)+L\min\big\{(x^*)^2-(Tx^*)^2,\, p_b^m(x^*,Tx)\big\}-\psi(x^2)\\
&=\frac{logx^2}{1+log\Big(2^{log(2^{x^2})}\Big)}\\
&\quad+L\min\left((x^*)^2-\Big(\frac{x^*}{3}+x^*log\Big(1+(x^*)^{\frac{1}{3}}\Big)\Big)^2, p_b^m(x^*,Tx)\right)-log(1+x^2)
\end{aligned}$$

for all $x,x^*\in X=[0,\frac{1}{2}]$ and $L\geq 0$. Clearly for $\epsilon=1.1$ and $L=100$, one can see that R.H.S. dominates L.H.S. as shown in the following Figure.

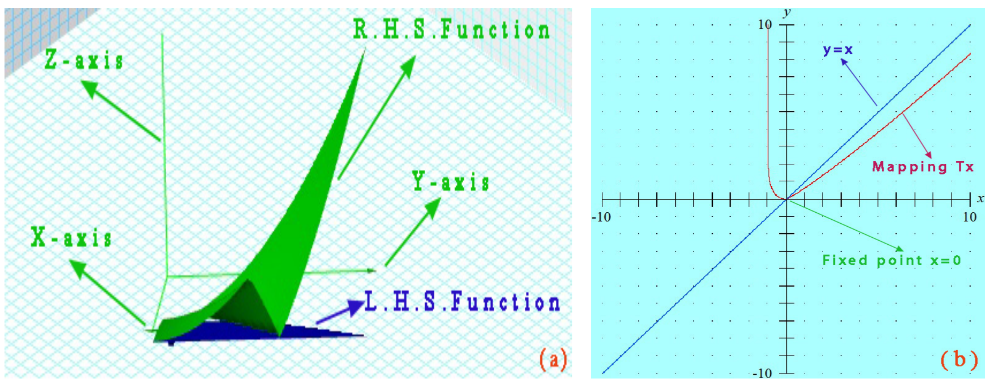

Figure 2. (a) Domination of R.H.S over L.H.S., 3D view & (b) Fixed point of T.

Moreover the mapping $T:X\to X$ is continous. Thus all the conditions of Theorem 2.4 are fulfilled. Hence T is Geraghty type generalized F-Berinde contraction and has a unique fixed point $0\in X$.

Remark 2.6. Theorem 2.4 generalizes and extends F-contraction version of Theorem 1 and Theroem 2 of V. Berinde [4] in the setting of partial b-metric space by setting $\phi(\mathcal{M}_{\mathcal{T}}(x,x^*))=\delta$, $\mathcal{M}_{\mathcal{T}}(x,x^*)=p_b(x,x^*)$, $\mathcal{N}_{\mathcal{T}}(x,x^*)$ $=p_b(x,Tx)$ and $\psi(t)=\tau$ in Theorem 2.4.

Remark 2.7. Theorem 2.4 generalizes and extends F-contraction version of Theorem 2.1 of Altun and Sadarangani [1] in the setting of partial b-metric space by setting $\phi(t)=t$ and $L=0$ in Theorem 2.4.

3. Some consequences

By choosing $\psi\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)=\tau>0$ in Theorem 2.4, Berinde-Wardowiski type result is obtained in the setting of partial b-metric spaces as follows.

Corollary 3.1. *Theorem* 2.4 *remains true, if we replace the assumption* (2.1) *by the following (besides retaining the rest of the hypotheses):*

$$F(s^{\epsilon}p_b(Tx,Tx^*))\leq\theta\Big(\phi\big(\mathcal{M}_{\mathcal{T}}(x,x^*)\big)\Big)F\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)+L\mathcal{N}_{\mathcal{T}}(x,x^*)-\tau, \quad (3.1)$$

Taking $L=0$ in Theorem 2.4, we have the following corollary.
If $\psi\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)=\tau>0$ and $L=0$ in Theorem 2.4, we have the following corollary.

Corollary 3.2. *Theorem* 2.4 *remains true, if we replace the assumption* (2.1) *by the following (besides retaining the rest of the hypotheses):*

$$F(s^{\epsilon}p_b(Tx,Tx^*))\leq\theta\Big(\phi\big(\mathcal{M}_{\mathcal{T}}(x,x^*)\big)\Big)F\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)-\tau, \quad (3.2)$$

Further by taking $\phi(t)=t$ in Corollary 3.2, we have the following corollary as a consequence of Theorem 2.4.

Corollary 3.3. *Let (X,p_b) be a complete partial b-metric space with $s>1$. Let T be a continuous self mapping on X. If there exist , $\theta\in\Theta$, $F\in\aleph_F$, $\tau>0$ and $L\geq 0$ such that for all $x,y\in X$ with $p_b(Tx,Tx^*)>0$,*

$$F(s^{\epsilon}p_b(Tx,Tx^*))\leq\theta\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)F\Big(\mathcal{M}_{\mathcal{T}}(x,x^*)\Big)-\tau, \quad (3.3)$$

where

$$\mathcal{M}_{\mathcal{T}}(x,x^*)=\max\Big\{p_b(x,x^*),p_b(x,Tx),p_b(x^*,Tx^*),\frac{p_b(x,Tx^*)+p_b(x^*,Tx)}{2s}\Big\},$$

and $\epsilon>1$ is a constant, then T has a unique fixed point.

Remark 3.4. By taking $\theta(t)=c$, where $c\in[0,1)$ in Corollary 3.3 reduces to F-contraction version of Theorem 5.3 of Mathews [8] in the framework of partial metric space and partial b-metric space along with Geraghty type contraction.

Remark 3.5. Corollary 3.3 generalizes Theorem 2.4 of Wardowski and Dung [14] for parial b-metric space along with Geraghty type contraction.

Remark 3.6. Corollary 3.3 generalizes the result of Wardowski [15] for partial metric space and partial b-metric space along with Geraghty type contraction.

Following Corollary is another version of Corollary 3.3.

Corollary 3.7. *Let (X, p_b) be a complete partial b-metric space with $s > 1$. Let T be a continuous self mapping on X. If there exist , $\theta \in \Theta$, $F \in \aleph_F$, $\tau > 0$ and $L \geq 0$ such that for all $x, y \in X$ with $p_b(Tx, Tx^*) > 0$,*

$$F(s^{\epsilon} p_b(Tx, Tx^*)) \leq F\left(\theta\Big(\mathcal{M}_{\mathcal{T}}(x, x^*)\Big)\Big(\mathcal{M}_{\mathcal{T}}(x, x^*)\Big)\right) - \tau, \qquad (3.4)$$

where

$$\mathcal{M}_{\mathcal{T}}(x, x^*) = \max\Big\{p_b(x, x^*), p_b(x, Tx), p_b(x^*, Tx^*), \frac{p_b(x, Tx^*) + p_b(x^*, Tx)}{2s}\Big\},$$

and $\epsilon > 1$ is a constant, then T has a unique fixed point.

4. Applications

4.1. Application to solution of differential equation of first order.

As an application, we obtain the solution of the following first-order periodic boundary value problem in this section:

$$\begin{cases} v'(t) = g(t, v(t)), & t \in [0, T]; \\ v(0) = v(T). \end{cases} \qquad (4.1)$$

Where $T > 0$ and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Above problem is equivalent to the integral equation:

$$v(t) = \int_0^T \mathcal{G}(t, s)[g(s, v(s)) + \eta v(s)]ds, \quad t \in [0, T], \qquad (4.2)$$

in which $\mathcal{G}(t, s)$ is the Green's function, given by

$$\mathcal{G}(t, s) = \begin{cases} \frac{e^{\eta(T+s-t)}}{e^{\eta T}-1} & 0 \leq s \leq t \leq T; \\ \frac{e^{\eta(s-t)}}{e^{\eta T}-1} & 0 \leq t \leq s \leq T. \end{cases} \qquad (4.3)$$

Let $X = \mathbb{C}([0, T], \mathbb{R})$ be the set of all real continuous functions on $[0, T]$ and $p_b : X \times X \to [0, \infty)$ be defined by

$$p_b(v, w) = \max_{t \in [0,T]} |v(t) - w(t)|^2, \qquad (4.4)$$

for all $v, w \in X$. Then clearly, the space (X, p_b) is a complete partial b-metric space with $s = 2$.

Consider the self map $f : X \to X$ given by

$$fw(t) = \int_0^T \mathcal{G}(t, s)[g(s, w(s)) + \eta w(s)]ds, \quad t \in [0, T].$$

then w is a solution of (4.2) if and only if it is a fixed point of f.

Subsequent Theorem is furnished for the assertion of the existence of fixed point of f.

Theorem 4.1. *Suppose there exists $\eta > 0$ such that for any $x, y \in \mathbb{R}$, with $\tau > 0$*

$$|g(t,x(t))+\eta x(t)-g(t,x^*(t))-\eta x^*(t)| \leq \frac{\eta}{2^{\frac{\epsilon}{2}+1}}\sqrt{(|x(t)-x^*(t)|^2-r)e^{-\tau}} \quad (4.5)$$

where $|x(t)-x^*(t)| > r$, $t \in [0,T]$ and $r \geq 0$. Then the differential equation (4.1) has a solution.

Proof. One can easily note that $\big(\mathbb{C}([0,T],\mathbb{R}), p_b\big)$ is a complete partial b-metric space on account of (4.4).

For $v, w \in X$ and $\tau > 0$, we have

$$\begin{aligned}
&\Big|fv(t) - fw(t)\Big| \\
&= \Big|\int_0^T \mathcal{G}(t,s)[g(s,v(s)) + \eta v(s)]ds - \int_0^T \mathcal{G}(t,s)[g(s,w(s)) + \eta w(s)]\ ds\Big| \\
&\leq \int_0^T \mathcal{G}(t,s)\Big|g(s,v(s)) + \eta v(s) - g(s,w(s)) - \eta w(s)\Big|\ ds \\
&\leq \max_{t\in[0,T]}\Big|g(s,v(t)) + \eta v(t) - g(t,w(t)) - \eta w(t)\Big|\int_0^T \mathcal{G}(t,s)\,ds \\
&\leq \frac{\eta}{2^{\frac{\epsilon}{2}+1}}\max_{t\in[0,T]}\sqrt{(|v(t)-w(t)|^2-r)e^{-\tau}}\int_0^T \mathcal{G}(t,s)\,ds \\
&= \frac{\eta}{2^{\frac{\epsilon}{2}+1}}\max_{t\in[0,T]}\sqrt{(|v(t)-w(t)|^2-r)e^{-\tau}}\Big[\int_0^t \frac{e^{\eta(T+s-t)}}{e^{\eta T}-1}ds + \int_t^T \frac{e^{\eta(s-t)}}{e^{\eta T}-1}ds\Big] \\
&= \frac{\eta}{2^{\frac{\epsilon}{2}+1}}\max_{t\in[0,T]}\sqrt{(|v(t)-w(t)|^2-r)e^{-\tau}} \\
&\quad\times\Big[\frac{1}{\eta(e^{\eta T}-1)}\Big(e^{\eta T} - e^{\eta(T-t)} + e^{\eta(T-t)} - 1\Big)\Big] \\
&\leq \frac{1}{2^{\frac{\epsilon}{2}+1}}\max_{t\in[0,T]}\sqrt{(|v(t)-w(t)|^2-r)e^{-\tau}},
\end{aligned}$$

which yields

$$2^\epsilon \max_{t\in[0,T]}\Big|fv(t) - fw(t)\Big|^2 \leq \frac{1}{4}\max_{t\in[0,T]}\big\{(|v(t)-w(t)|^2-r)e^{-\tau}\big\}$$

or

$$2^\epsilon p_b(fv, fw) \leq \frac{1}{4}p_b(v,w)e^{-\tau}.$$

Which amounts to say that

$$2^\epsilon p_b(fv, fw) \leq \frac{1}{4}\mathcal{M}_{\mathcal{T}}(v,w)e^{-\tau},$$

where

$$\mathcal{M}_{\mathcal{T}}(v,w)=\max\{p_b(v,w),p_b(v,fv),p_b(w,fw),\frac{1}{4}[p_b(v,Tw)+p_b(w,fv)]\}.$$

Since $\theta \in \Theta$, taking $\theta\Big(\mathcal{M}_{\mathcal{T}}(v,w)\Big)=\frac{1}{4}$, we obtain

$$2^{\epsilon}p_b(fv,fw)\leq \theta\Big(\mathcal{M}_{\mathcal{T}}(v,w)\Big)\mathcal{M}_{\mathcal{T}}(v,w)e^{-\tau},$$

Passing to logarithms, we get

$$\ln\Big(2^{\epsilon}p_b(fv,fw)\Big)\leq \ln\Bigg(\theta\Big(\mathcal{M}_{\mathcal{T}}(v,w)\Big)\Big(\mathcal{M}_{T}(v,w)\Big)\Bigg)-\tau.$$

For $\ln(k)=k$, $k>0$, above inequality turn into

$$F\Big(2^{\epsilon}p_b(fv,fw)\Big)\leq F\Bigg(\theta\Big(\mathcal{M}_{\mathcal{T}}(v,w)\Big)\Big(\mathcal{M}_{T}(v,w)\Big)\Bigg)-\tau.$$

Hence all the hypothesis of Corollary 3.7 are satisfied, we conclude that f has a fixed point w^* in X, which is the solution of integral equation 4.2. Consequently, the differential equation (4.1) has a solution. □

4.2. **Application to solution of differential equation of second order.** In this section, the existence of solution for the following second order boundary value problem is established:

$$\begin{cases} u''(t)=g(t,u(t)), & t\in[0,1];\\ u(0)=u_0, u(1)=u_1. \end{cases} \tag{4.6}$$

Where $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Firstly consider the space $X=C(\mathbb{H})(\mathbb{H}=[0,1],\mathbb{R})$ of continuous functions defined on $\mathbb{H}$. Obviously this space with metric given by

$$p_b(u,v)=\sup_{t\in[0,1]}|u(t)-v(t)|^2,\ \text{for all}\ u,v\in X, \tag{4.7}$$

is a complete partial b-metric space with $s=2$.

Theorem 4.2. *Consider problem* (4.6) *and suppose there exists* $\xi>0$ *such that for any* $x,y\in C(\mathbb{H})$*, with* $\tau>0$

$$|x(t)-y(t)|\leq \frac{1}{\xi 2^{\frac{\epsilon}{2}}}\sqrt{(|x(t)-y(t)|^2-r)e^{-\tau}} \tag{4.8}$$

where $|x(t)-y(t)|>r$ *and* $r\geq 0$. *Then the differential equation* (4.6) *has a solution.*

Proof. Problem (4.6) is equivalent to the second kind Fredholm integral equation

$$u(t) = F(t) + \xi \int_0^1 G^*(t,s)u(s)ds, \quad t \in [0,1], \tag{4.9}$$

in which $F(t) = u_0 + t(u_1 - u_0)$ and $G^*(t,s)$ is the Green's function, given by

$$G^*(t,s) = \begin{cases} s(1-s) & 0 \le s \le t; \\ t(1-s) & t \le s \le 1. \end{cases} \tag{4.10}$$

Note that if $u \in C(\mathbb{H})$ is a fixed point of f, then u is a solution of (4.6). For $u, v \in C(\mathbb{H})$ and $\tau > 0$, we have

$$\begin{aligned} \Big|fu(t) - fv(t)\Big| &= \Big|F(t) + \xi \int_0^1 G^*(t,s)u(s)ds - \big[F(t) + \xi \int_0^1 G^*(t,s)v(s)ds\big]\ \Big| \\ &\le \xi \int_0^1 G^*(t,s)\Big|u(s) - v(s)\Big|\ ds \\ &\le \sup_{t\in[0,1]} \Big|u(t) - v(t)\Big| \int_0^1 G^*(t,s)\ \ ds \\ &\le \frac{1}{2^{\frac{\epsilon}{2}}} \sup_{t\in[0,1]} \sqrt{(|u(t)-v(t)|^2 - r)e^{-\tau}} \int_0^1 G^*(t,s)\ \ ds \\ &= \frac{1}{2^{\frac{\epsilon}{2}}}\Big[\frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{2}\Big] \sup_{t\in[0,1]} \sqrt{(|u(t)-v(t)|^2 - r)e^{-\tau}} \\ &\le \left(\frac{1}{6}\right)\left(\frac{1}{2^{\frac{\epsilon}{2}}}\right) \sup_{t\in[0,1]} \sqrt{(|u(t)-v(t)|^2 - r)e^{-\tau}}. \end{aligned}$$

Which implies that

$$2^{\epsilon} \sup_{t\in[0,1]} \Big|fu(t) - fv(t)\Big|^2 \le \frac{1}{36} \sup_{t\in[0,1]} \big\{(|u(t)-v(t)|^2 - r)e^{-\tau}\big\}$$

or

$$2^{\epsilon} p_b(fu, fv) \le \frac{1}{36}\, p_b(u,v)e^{-\tau}.$$

Which amounts to say that

$$2^{\epsilon} p_b(fu, fv) \le \frac{1}{36} \mathcal{M}_{\mathcal{T}}(u,v)e^{-\tau},$$

where

$$\mathcal{M}_{\mathcal{T}}(u,v) = \max\{p_b(u,v), p_b(u,fu), p_b(v,fv), \frac{1}{4}[p_b(u,Tv) + p_b(v,fu)]\}.$$

Taking $\theta\Big(\mathcal{M}_{\mathcal{T}}(u,v)\Big) = \frac{1}{36}$, we obtain

$$2^{\epsilon} p_b(fu, fv) \le \theta\Big(\mathcal{M}_{\mathcal{T}}(u,v)\Big)\mathcal{M}_{\mathcal{T}}(u,v)e^{-\tau}.$$

Consequently, passing to logarithms, one can get

$$\ln\Big(2^{\epsilon}p_b(fu,fv)\Big) \leq \ln\left(\theta\Big(\mathcal{M}_{\mathcal{T}}(u,v)\Big)\Big(\mathcal{M}_{T}(u,v)\Big)\right) - \tau,$$

or

$$F\Big(2^{\epsilon}p_b(fu,fv)\Big) \leq F\left(\theta\Big(\mathcal{M}_{\mathcal{T}}(u,v)\Big)\Big(\mathcal{M}_{T}(u,v)\Big)\right) - \tau,$$

for $\ln(k) = k$, $k > 0$. Hence, we conclude that the contractive condition of Corollary 3.7 is satisfied with $s = 2$, authenticating f has a fixed point v^* in $C(\mathbb{H})$, which is the solution of integral equation (4.9). Consequently, the differential equation (4.6) has a solution. □

Open Problem: As an open problem, one can establish the existence of solution of storekeeper's control problem, which is stated as:

For the use of optimal storage space, a storekeeper wants to keep the stores stock of goods constant. This can be mathematically modeled as a Volterra integral equation of the first kind as follows:

$$\zeta h(\nu) + \int_0^{\nu} h(\nu - \vartheta)g(\vartheta)d\vartheta = m_0,$$

where ζ is anumber of products in stock at time ν=0, $h(\nu)$ is remainder of products in stock (in percent) at the time ν, $g(\nu)$ is the velocity (products/time unit) with which new products are purchased, $g(\vartheta)\Delta\vartheta$ is trhe amount of purchased products during the time interval ϑ.

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