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THE REPRESENTATION AND CONTINUITY OF A GENERALIZED METRIC PROJECTION ONTO HALF-SPACES IN BANACH SPACES

Ruijuan Liu[1], Chunyan Liu[2] and Zihou Zhang[3]

[1]School of Mathematics, Physics and Statistics
Shanghai University of Engineering Science, Shanghai 201620, China
e-mail: `ruirui0516@163.com`

[2]School of Mathematics, Physics and Statistics
Shanghai University of Engineering Science, Shanghai 201620, China
e-mail: `cyl@sues.edu.cn`

[3]School of Mathematics, Physics and Statistics
Shanghai University of Engineering Science, Shanghai 201620, China
e-mail: `zhz@sues.edu.cn`

Abstract. Let C be a closed bounded convex subset of a real Banach space X with 0 as its interior point and p_c the Minkowski functional generated by the set C. For a nonempty set G in X and $x \in X$, $g_0 \in G$ is called the generalized best approximation to x from G if $p_c(g_0 - x) \leq p_c(g - x)$ for all $g \in G$. In this paper, we will give a distance formula under p_C from a point to a half-space $K_{x_0^*,c} = \{x \in X : x_0^*(x) \leq c\}$ in Banach space and investigate the continuity of this generalized metric projection, extending corresponding results for the case of norm.

1. Introduction

In recent years, the generalized metric projection is concerned by more and more people. It has been used in many areas of mathematics such as the theories of optimization and approximation, fixed point theory, nonlinear programming, and variational inequalities. In practical application, giving the representations of generalized metric projection is very necessary. Generally

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[0]Corresponding author: R. Liu(`ruirui0516@163.com`).

speaking, this is very difficult. Wang and Yu [9] gave a representation of single-valued metric projection on a class of hyperplanes $H_{x_0^*,c} = \{x \in X : x_0^*(x) = c\}$ in reflexive, smooth, and strictly convex Banach Space X. Song and Cao [8] gave a representation of metric projection on a class of half-space $K_{x_0^*,c}$ in the reflexive, smooth, and strictly convex Banach Space X. Luo and Wang [5] discussed the generalized metric projection on closed hyperplane and investigate the continuity of this generalized metric projection.

Throughout this paper, $(X, \|\cdot\|)$ is a real Banach space with the closed unit ball $B(X)$, and X^* is its topological dual. For a nonempty subset A of X, as usual, $intA$ and bdA stand for the interior and the boundary of A, respectively.

Let C be a closed bounded convex subset of X with $0 \in int\ C$. Recall that the Minkowski function $p_C : X \to R$ with respect to the set C is defined by

$$p_C(x) = \inf\{t > 0 : x \in tC\},\ \forall x \in X. \tag{1.1}$$

Let G be a nonempty subset of X and $x \in X$. If there exists $g_0 \in G$ such that

$$p_C(g_0 - x) = \tau_C(x, G), \tag{1.2}$$

where

$$\tau_C(x, G) = \inf\{p_C(g - x) : g \in G\} \tag{1.3}$$

is the distance from the point x to the set G, then g_0 is called the generalized best approximation to x from G, see [9]. The set of all generalized best approximations to x from G is denoted by $P_G^C(x)$, that is,

$$P_G^C(x) = \{g_0 \in G : p_C(g_0 - x) = \tau_C(x, G)\}, \tag{1.4}$$

which is called the generalized metric projection onto G.

As is well known, p_C is the norm of X if C is a unit ball. The generalized best approximation is reduced to the classical best approximation, which has been studied deeply and extensively since the late 1950s; see [2,3,6,7] and reference therein.

In this paper, firstly, we gave two representations of the generalized metric projection $P_{K_{x_0^*,c}}$ and $P_{K_{x_0,c}}$. Secondly, by these representations, we prove that if X is weakly near strictly convex(resp.,weakly nearly smooth), then the generalized metric projection $P_{K_{x_0^*,c}}$(resp.,$P_{K_{x_0,c}}$) is norm-weakly upper semicontinuous, where

$$K_{x_0^*,c} = \{x \in X : x_0^*(x) \le c\},\ K_{x_0,c} = \{x^* \in X^* : x^*(x_0) \le c\}. \tag{1.5}$$

2. Preliminaries

Recall that $(X, \|\cdot\|)$ is a real Banach space with the topological dual X^*, C is a closed bounded convex subset of X with $0 \in intC$ and p_C is the Minkowski function given by (1). Define the polar C^0 of the set C by

$$C^0 = \{x^* \in X^* : x^*(x) \leq 1, \ \forall x \in C\}.$$

Then C^0 is a nonempty weakly* compact convex subset of X^* with $0 \in int\, C^0$.

We first list some useful properties of the Minkowski function p_C which can be proved easily by the definition.

Proposition 2.1. ([1]) *For every $x, y \in X$, we have*

(1) $p_C(x) \geq 0$, *and* $p_C(x) = 0$ *iff* $x = 0$;
(2) $p_C(x+y) \leq p_C(x) + p_C(y)$;
(3) $-p_C(y-x) \leq p_C(x) - p_C(y) \leq p_C(x-y)$;
(4) $p_C(\lambda x) = \lambda p_C(x)$, *if* $\lambda \geq 0$;
(5) $p_C(-x) = p_{-C}(x)$;
(6) $p_C(x) = 1$ *iff* $x \in \partial C$;
(7) $p_C(x) < 1$ *iff* $x \in int\ C$;
(8) $\mu\|x\| \leq p_C(x) \leq \nu\|x\|$, *where*

$$\mu = \inf_{x \in \partial B} p_C(x) \ and \ \nu = \sup_{x \in \partial B} p_C(x).$$

Let $x \in X$, from the definition in [5],

$$\sigma(x) = \{x^* \in X^* : x^*(x) = p_C(x) p_C(x^*) = p_C(x)^2 = p_{C^0}(x^*)^2\},$$

which is analogous to the dual mapping in Banach Spaces. Then, for $x^* \in X^*$, one obtains that

$$\sigma^{-1}(x^*) = \{x \in X : x^* \in \sigma(x)\}$$

and

$$\sigma^{-1}(\lambda x^*) = \lambda \sigma^{-1}(x^*), \ \forall \lambda \in R.$$

Hence, for $0 \neq x^* \in X^*$ and $x \in X$, noting that $p_{C^0}(x^*) \neq 0$, one has that

$$x \in \sigma^{-1}(x^*) \Leftrightarrow x^*(x) = p_C(x) P_{C^0}(x^*) = P_{C^0}(x^*)^2$$

and so

$$x \in \sigma^{-1}\left(\frac{x^*}{P_{C^0}(x^*)}\right) \Leftrightarrow \frac{x^*(x)}{P_{C^0}(x^*)} = p_C(x) = 1.$$

We then give the following definitions which will be used in the rest of this paper.

Definition 2.2. ([4]) C is called strictly convex if $\partial C = extC$, the set of all extreme points of C.

From the definition, it follows that C is strictly convex if and only if for any $x, y \in \partial C$, $p_C(x+y) = p_C(x) + p_C(y)$ implies $x = y$.

Remark 2.3. ([5]) The set C is said to be strictly convex(resp.,nearly strictly convex and weakly nearly strictly convex) if each convex subset of ∂C is a singleton (res., relatively compact and relatively weakly compact.)

Definition 2.4. ([5]) Let T be a set-valued mapping from X into 2^X, where 2^X is the set of all subsets of X.

(1) Let $x \in X$ with $Tx \neq \emptyset$. Then T is said to be norm-to-norm (res., norm-to-weak) upper semicontinuous at x if, for each open set (resp., weakly open set) $W \supseteq T(x)$ there exists an open neighborhood V of x such that $T(y) \subseteq W$ whenever $y \in V$.

(2) T is said to be norm-to-norm (res., norm-to-weak)upper semicontinuous on X if, for each $x \in X$, $Tx \neq \emptyset$ and T is norm-to-norm (resp., norm-to-weak) upper semicontinuous at x.

(3) T is said to be norm-to-norm continuous on X if, for each $x \in X$, Tx is single valued and T is norm-to-norm upper semicontinuous at X.

3. Main results

Lemma 3.1. *Let X be a Banach space and let $x_0^* \in X^*\backslash\{\theta\}$. Then*

$$\tau_C(x, K_{x_0^*,c}) = \frac{x_0^*(x) - c}{p_{C^0}(x_0^*)},$$

for all $x \in X\backslash K_{x_0^*,c}$.

Proof. Firstly, suppose that $p_{C^0}(x_0^*) = 1$, let $x \in X\backslash K_{x_0^*,c}$. For any $y \in K_{x_0^*,c}$, Since

$$p_C(x-y) = p_C(x-y)p_{C^0}(x_0^*) \geq x_0^*(x-y) \geq x_0^*(x) - c > 0,$$

we deduce that

$$\tau_C(x, K_{x_0^*,c}) \geq x_0^*(x) - c.$$

On the other hand, for any $\varepsilon > 0, (\varepsilon < 1/4)$, there exists u_ε in ∂C such that $1 - \varepsilon < x_0^*(u_\varepsilon) \leq 1$. Set $y_\varepsilon = x - (1+2\varepsilon)(x_0^*(x) - c)u_\varepsilon$. Then

$$\begin{aligned} x_0^*(y_\varepsilon) &= x_0^*(x) - (1+2\varepsilon)(x_0^*(x) - c)x_0^*(u_\varepsilon) \\ &< x_0^*(x) - (1+2\varepsilon)(x_0^*(x) - c)(1-\varepsilon) \\ &= x_0^*(x) - (1+\varepsilon-2\varepsilon^2)(x_0^*(x) - c) \\ &\leq x_0^*(x) - (x_0^*(x) - c) = c. \end{aligned} \tag{3.1}$$

Consequently, $y_\varepsilon \in K_{x_0^*,c}$ and $p_{C^0}(x - y_\varepsilon) = (1+2\varepsilon)(x_0^*(x) - c)$, It follows that

$$\tau_C(x, K_{x_0^*,c}) \leq (1+2\varepsilon)(x_0^*(x) - c).$$

By arbitrariness of ε, we deduce that

$$\tau_C(x, K_{x_0^*,c}) \leq x_0^*(x) - c.$$

This means that

$$\tau_C(x, K_{x_0^*,c}) = x_0^*(x) - c.$$

Secondly, for $x^* \in X^*\backslash\{\theta\}$ and $p_{C^0}(x_0^*) \neq 1$, since

$$K_{x_0^*,c} = \{x \in X : x_0^*(x) \leq c\} = \{x \in X : \frac{x_0^*(x)}{p_{C^0}(x_0^*)} \leq \frac{c}{p_{C^0}(x_0^*)}\}. \tag{3.2}$$

From (3.2), we may obtain that

$$\tau_C(x, K_{x_0^*,c}) = \frac{x_0^*(x) - c}{p_{C^0}(x_0^*)},$$

for all $x \in X\backslash K_{x_0^*,c}$. □

Remark 3.2. For given $x_0^* \in X^*\backslash\{\theta\}$ and $c \in R$, by the proof of Lemma 3.1, we have that

$$\tau_C(x, K_{x_0^*,c}) = \tau_C(x, H_{x_0^*,c})$$

for any $x \in X\backslash K_{x_0^*}, c$.

Remark 3.3. ([10]) Let X be a Banach space and let $x_0^* \in X^*\backslash\{\theta\}$. Then

$$d(x, K_{x_0^*,c}) = \frac{x_0^*(x) - c}{\|x_0^*\|},$$

for all $x \in X\backslash K_{x_0^*,c}$.

Proposition 3.4. *Let X be a Banach space, let $x_0^* \in X^*\backslash\{\theta\}$ and let $c \in R$. Then*

$$\tau_C(x^*, K_{x_0,c}) = \frac{x^*(x_0) - c}{p_{C^0}(x_0)},$$

for any $x^* \in X^*\backslash K_{x_0,c}$.

The following result is a characteristic of $K_{x_0^*,c}$ being a proximinal set.

Theorem 3.5. *Let X be a Banach Space, let $x_0^* \in X^*\backslash\{\theta\}$. Then $P^C_{K_{x_0^*,c}}(x) \neq \emptyset$ for any $x \in X$ if and only if $\sigma^{-1}(x_0^*) \neq \emptyset$.*

Proof. On necessary: take $x \in K_{x_0^*,c}$, then there exists a $y \in P_{K_{x_0^*,c}}(x)$. Set $u = \frac{p_{c^0}(x_0^*)^2}{x_0^*(x)-c}(x - y)$; by Lemma 3.1, we have that

$$P_{C^0}(u) = \frac{P^2_{C^0}(x_0^*)}{x_0^*(x) - c} P_{C^0}(x - y) = \frac{P^2_{C^0}(x_0^*)}{x_0^*(x) - c} \frac{x_0^*(x) - c}{P_{C^0}(x_0^*)} = P_{C^0}(x_0^*).$$

Hence, $x_0^*(u) \leq P_{C^0}(x_0^*)P_{C^0}(u) = P_{C^0}(x_0^*)^2$.

On the other hand,

$$x_0^*(u) = \frac{P_{C^0}^2(x_0^*)}{x_0^*(x) - c}(x_0^*(x) - x_0^*(y)) \geq \frac{P_{C^0}^2(x_0^*)}{x_0^*(x) - c}(x_0^*(x) - c) = P_{C^0}^2(x_0^*).$$

This shows that $x_0^*(u) = P_{C^0}^2(x_0^*) = P_{C^0}^2(u)$, that is, $u \in \sigma^{-1}(x_0^*)$, and $\sigma^{-1}(x_0^*) \neq \emptyset$.

On sufficiency: take $x \in \partial C$ such that

$$x_0^*(x) = P_{C^0}(x_0^*)P_{C^0}(x) = P_{C^0}^2(x_0^*) = P_{C^0}^2(x).$$

we discuss that in two cases.

Case I. If $x \in K_{x_0^*,c}$, then $x \in P_{K_{x_0^*,c}}(x)$.

Case II. If $x \notin K_{x_0^*,c}$, since

$$x_0^*(x - \frac{x_0^*(x) - c}{P_{C^0}^2(x_0^*)}x_0) = x_0^*(x) - (x_0^*(x) - c) = c;$$

then we have that

$$x - \frac{x_0^*(x) - c}{P_{C^0}^2(x_0^*)}x_0 \in K_{x_0^*,c}.$$

By Lemma 3.1,

$$P_{C^0}\{x - (x - \frac{x_0^*(x) - c}{P_{C^0}^2(x_0^*)}x_0)\} = \frac{x_0^*(x) - c}{P_{C^0}(x_0^*)} = \tau(x, K_{x_0}^*).$$

It follows that $x - ((x_0^*(x) - c)/P_{C^0}^2(x_0^*)x_0 \in P_{K_{x_0^*,c}}(x)$. $\square$

Theorem 3.6. *Let $x^* \in X^*\backslash\{\theta\}$. Then the following assertion holds*

$$P_{K_{x_0^*,c}}(x) = x - \max\big\{0,\ \frac{x_0^*(x) - c}{p_{c^0}(x^*)^2}\big\}\sigma^{-1}(x_0^*)$$

Proof. Take $x \in X$. we discuss that in two cases.

Case I. If $x \in K_{x_0^*,c}$, then $P_{K_{x_0^*,c}}(x) = \{x\}$.

Case II. If $x \notin K_{x_0^*,c}$, we arbitrarily take $x_0 \in \sigma^{-1}(x_0^*)$. Let $y = x - (\frac{x_0^*(x)-c}{p_{c^0}(x^*)^2})x_0$. Similar to the proof of Lemma 3.1, we may obtain that $y \in P_{K_{x_0^*,c}}(x)$. Therefore,

$$x - \frac{x_0^*(x) - c}{p_{c^0}(x^*)^2}\sigma^{-1}(x_0^*) \subset P_{K_{x_0^*,c}}(x).$$

On the other hand, we arbitrarily take $y \in P_{K_{x_0^*,c}}(x)$. Let $u = (\frac{p_{c^0}(x^*)^2}{x_0^*(x)-c})(x - y)$; Similar to the proof of Lemma 3.1, we may obtain that $u \in \sigma^{-1}(x_0^*)$.

Therefore,

$$y = x - \frac{x_0^*(x) - c}{p_{c^0}(x^*)^2} u \in x - \frac{x_0^*(x) - c}{p_{c^0}(x^*)^2} \sigma^{-1}(x_0^*).$$

That is,

$$P_{K_{x_0^*,c}}(x) \subset x - \frac{x_0^*(x) - c}{p_{c^0}(x^*)^2} \sigma^{-1}(x_0^*)$$

By Case 1 and Case 2, we have

$$P_{K_{x_0^*,c}}(x) = x - \frac{x_0^*(x) - c}{p_{c^0}(x^*)^2} \sigma^{-1}(x^*),$$

for any $x \in X$. □

By a similar proof to that in Theorem 3.6, we can also prove that the following result according to Lemma 3.1.

Proposition 3.7. *Let X be a Banach Space, let $x_0 \in X \backslash \{\theta\}$, and let $c \in R$. Then*

$$P_{K_{x_0,c}}(x^*) = x^* - \max\{0, \ \frac{x^*(x_0) - c}{p_{c^0}(x_0)^2}\} \sigma(x_0),$$

for any $x^* \in X^*$.

The second main result of this section is as follows, which describes the continuity of the generalized metric projection $P_{K_{x_0^*,c}}$ onto the half spaces $K_{x_0^*,c}$ under the condition that the set C is weakly nearly strictly convex.

Theorem 3.8. *Let $x^* \in X^* \backslash \{\theta\}$, x^* attains its supremum on ∂C. Let the set C be weakly nearly strictly convex. Then the generalized metric projection $P^C_{K_{x_0^*,c}}$ is norm-weakly upper semicontinuous.*

Proof. We assume that $x^* \in X^* \backslash \{\theta\}$, x^* attains its supremum on ∂C. firstly, $\sigma^{-1}(x^*)$ is convex. In fact, let $y_1, y_2 \in \sigma^{-1}(x^*)$, and $\lambda \in [0, 1]$, then we obtain that

$$\begin{aligned} x^*(y_1) &= P_C(y_1) P_{C^0}(x^*) = P_{C^0}(x^*)^2 \\ &= P_C(y_2) P_{C^0}(x^*) = x^*(y_2). \end{aligned} \tag{3.3}$$

This, together with Proposition 3.7, implies that

$$\begin{aligned} P_{C^0}(x^*)^2 &= x^*(\lambda y_1 + (1 - \lambda) y_2) \\ &\leq P_{C^0}(\lambda y_1 + (1 - \lambda) y_2)) P_{C^0}(x^*) \\ &\leq (\lambda P_C(y_1) + (1 - \lambda) P_C(y_2)) P_{C^0}(x^*) \\ &= P_{C^0}(x^*)^2. \end{aligned} \tag{3.4}$$

Hence, $\lambda y_1 + (1-\lambda)y_2 \in \sigma^{-1}(x^*)$ and $\sigma^{-1}(x^*)$ is convex. We then show that $\sigma^{-1}(x^*)$ is weakly compact. Since $\sigma^{-1}(x^*/P_{C^0}(x^*)) \subset \partial C$ and

$$\sigma^{-1}(x^*/P_{C^0}(x^*)) = (1/P_{C^0}(x^*))\sigma^{-1}(x^*),$$

we see that $\sigma^{-1}(x^*/P_{C^0}(x^*))$ is a convex subset of ∂C. It follows that it is relatively weakly compact because C is weakly nearly strictly convex. Thus, to complete the proof, it suffices to show that $\sigma^{-1}(x^*)$ is weakly closed. To do this, let $\{x_\delta\}$ be a net in $\sigma^{-1}(x^*)$ convergent weakly to some $x \in X$. Since

$$x^*(x_\delta) = P_C(x_\delta)P_{C^0}(x^*) = P_{C^0}(x^*)^2, \ \forall \delta.$$

and P_C is weakly lower semicontinuous, we have that

$$\begin{aligned} P_{C^0}(x^*)^2 &= x^*(x) \\ &= \lim_\delta x^*(x_\delta) \\ &= \lim_\delta P_C(x_\delta)P_{C^0}(x^*) \\ &\geq P_C(x)P_{C^0}(x^*). \end{aligned} \tag{3.5}$$

Noting that $x^*(x) \leq P_C(x)P_{C^0}(x^*)$, we get that

$$x^*(x) = P_C(x)P_{C^0}(x^*) = P_{C^0}(x^*)^2.$$

Hence, $x \in \sigma^{-1}(x^*)$ and the weak closedness of $\sigma^{-1}(x^*)$ is proved.

Finally, we show that $P^C_{K_{x_0^*,c}}$ is norm-to-weak upper semicontinuous at x. Otherwise, there exists a weakly open set

$$W \supseteq P^C_{K_{x_0^*,c}}(x)$$

and a sequence $\{x_n\} \subset X$ such that $p_{C^0}(x_n - x) \to 0$, $P^C_{K_{x_0^*,c}}(x_n) \nsubseteq W$. Without loss of generation, we may assume that $x^*(x) < c$. Since $p_{C^0}(x_n - x) \to 0$, we may assume that each $x^*(x_n) < c$ for all n. Now take $y_n \in P^C_{K_{x_0^*,c}}(x_n) \setminus W$ for each n. There exists $z_n \in \sigma^{-1}(x_0^*)$ such that

$$y_n = x_n + ((c - x_0^*(x_n))/P_{C^0}(x_0^*))z_n$$

for all n, using the weak compactness of $\sigma^{-1}(x_0^*)$, one has a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to z$ weakly for some $z \in \sigma^{-1}(x_0^*)$. Therefore,

$$y_{n_k} = x + \frac{c - x_0^*(x_{n_k})}{P_{C^0}(x_0^*)} z_{n_k} \xrightarrow{w} x + \frac{c - x_0^*(x)}{P_{C^0}(x_0^*)} z \in P^C_{K_{x_0^*,c}}(x) \subset W.$$

This implies that $y_{n_k} \in W$ for sufficiently large k, which contradicts the choice of y_{n_k}. The proof of assertion is complete. □

Similar to the proof of Theorem 3.8, we may prove the following theorem.

Theorem 3.9. (1) *Let the set* C *be nearly strictly convex,* $x^* \in X^*\backslash\{\theta\}$*, suppose that* $x \in K_{x_0^*,c}$ *and that* x^* *attains its supremum on* bdC*. Then* $P^C_{K_{x_0^*,c}}$ *is norm-to-norm upper semicontinuous.*

(2) *Let the set* C *be weakly nearly smooth,* $x_0 \in X\backslash\{\theta\}$*. Then* $P^C_{K_{x_0,c}}$ *is norm-weakly upper semicontinuous.*

(3) *Let the set* C *be nearly smooth,* $x_0 \in X\backslash\{\theta\}$*. Then* $P^C_{K_{x_0,c}}$ *is norm-norm upper semicontinuous.*

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