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ON λ-PSEUDO BI-STARLIKE FUNCTIONS IN PARABOLIC DOMAIN

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Abstract. In this paper we introduce a new class $\mathcal{L}^{\lambda}_{p,\Sigma}(\phi_\alpha)$ of λ-pseudo bi-starlike functions in parabolic domain and determine the bounds for $|a_2|$ and $|a_3|$ where a_2, a_3 are the initial Taylor coefficients of $f \in \mathcal{L}^{\lambda}_{p,\Sigma}(\phi_\alpha)$. Furthermore, we estimate the Fekete-Szegö functional for $\mathcal{L}^{\lambda}_{p,\Sigma}(\phi_\alpha)$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. Further, denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. One of the important and well-investigated subclass of $\mathcal{S}$ is the class $\mathcal{S}^*(\alpha)$ of starlike functions of

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order α $(0 \leq \alpha < 1)$ defined by the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

and also the class $\mathcal{K}(\alpha) \subset \mathcal{S}$ of convex functions of order α $(0 \leq \alpha < 1)$ is defined by the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

An analytic function f is said to be subordinate to an analytic function h, written by $f(z) \prec h(z)$, provided there is an analytic function ω with $\omega(0) = 0$ and such that $|\omega(z)| < 1$ in $\mathbb{U}$ and $f(z) = h(\omega(z))$.

Ma and Minda [11] unified approach to various subclasses of starlike and convex functions which are defined by a condition that either $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a function ϕ.

For this purpose, they considered a class Φ of analytic functions ϕ with positive real part in the unit disk $\mathbb{U}$, $\phi(0) = 1$, $\phi'(0) > 0$, such that ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions denoted by $\mathcal{S}^*(\phi)$, consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \phi(z).$$

Similarly, a function $f \in \mathcal{A}$ is in the class of Ma-Minda convex functions of functions denoted by $\mathcal{K}(\phi)$ if it satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z).$$

In the sequel, it is assumed that ϕ is in the class Φ.

Ali and Singh [1] introduced a new class of parabolic starlike functions denoted by $\mathcal{S}_p(\alpha)$ of order $\alpha(0 \leq \alpha < 1)$ salifies the following:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < (1 - 2\alpha) + \Re\left(\frac{zf'(z)}{f(z)}\right). \tag{1.2}$$

Equivalently,

$$f \in \mathcal{S}_p(\alpha) \Longleftrightarrow \left(\frac{zf'(z)}{f(z)}\right) \in \Omega_\alpha,$$

where Ω_α denotes the parabolic region in the right half-plane

$$\begin{aligned} \Omega_\alpha &= \{w = u + iv : v^2 < 4(1-\alpha)(u-\alpha)\} \\ &= \{w : |w - 1| < (1 - 2\alpha) + \Re(w)\}. \end{aligned} \tag{1.3}$$

Ali and Singh [1] showed that the normalized Riemann mapping function $\phi_\alpha(z)$ from the open unit disk $\mathbb{U}$ onto Ω_α is given by

$$\begin{aligned}\phi_\alpha(z) &= 1+\frac{4(1-\alpha)}{\pi^2}\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2\\ &= 1+\frac{16}{\pi^2}(1-\alpha)z+\frac{32}{3\pi^2}(1-\alpha)z^2+\frac{368}{45\pi^2}(1-\alpha)z^3+\cdots\\ &= 1+\sum_{k=1}^{\infty}B_k z^k, \qquad (1.4)\end{aligned}$$

where

$$B_k=\frac{16(1-\alpha)}{k\pi^2}\sum_{j=0}^{k-1}\frac{1}{2j+1} \quad (k\in\mathbb{N}). \qquad (1.5)$$

Due to Ma and Minda [11], we state the following lemma.

Lemma 1.1. *If a function $f\in\mathcal{S}_p(\alpha)$, then*

$$\left(\frac{zf'(z)}{f(z)}\right)\in\phi_\alpha(z),$$

where $\phi_\alpha(z)$ is given by (1.4).

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [7] ensures that the image of U under every univalent function $f\in\mathcal{S}$ of the form (1.1), contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f\in\mathcal{S}$ has an inverse f^{-1} which is defined by

$$f^{-1}(f(z))=z \quad (z\in\mathbb{U})$$

and

$$f(f^{-1}(w))=w \quad \left(|w|<r_0(f); r_0(f)\geq\frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots. \qquad (1.6)$$

A function $f\in\mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if there exists a function $g\in\mathcal{S}$ such that $g(z)$ is an univalent extension of f^{-1} to $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$. The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class Σ (see details in [14]). However, the familiar Koebe function is not bi-univalent. Lewin [10] investigated the class of *bi-univalent* functions σ and obtained a bound $|a_2| \leqq 1.51$.

Motivated by the work of Lewin [10], Brannan and Clunie [4] conjectured that $|a_2| \leqq \sqrt{2}$. The coefficient estimate problem for $|a_n|$ $(n \in \mathbb{N}, \ \ n \geqq 3)$ is still open(see, [14]). Brannan and Taha [5] also worked on certain subclasses of the bi-univalent function class Σ and obtained estimates for their initial coefficients. Various classes of bi-univalent functions were introduced and studied in recent times, the study of *bi-univalent* functions gained momentum mainly due to the work of Srivastava *et al.*[14]. Motivated by this, many researchers (see [3, 8, 12, 14, 15, 16] also the references cited there in) recently investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients.

Recently for some $\lambda \geq 1$, in [2] Babalola introduced and investigated the class of λ-pseudo-starlike functions of order $\alpha, (0 \leq \alpha < 1)$ denoted by $\mathcal{L}_\lambda(\alpha)$ as defined below.

Definition 1.2. ([2]) A function $f \in \mathcal{A}$ is in the class $\mathcal{L}_\lambda(\alpha)$ if it satisfies

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \alpha \qquad (\lambda \geq 1),$$

where α $(0 \leq \alpha < 1)$ and $z \in \mathbb{U}$.

Further in [2] it was showed that all pseudo-starlike functions are Bazilevič functions of type $(1 - 1/\lambda)$ and of order $\alpha^{1/\lambda}$ and univalent in open unit disk $\mathbb{U}$. We note that $\mathcal{L}_1(\alpha) \equiv \mathcal{S}^*(\alpha)$.

Making use of the above definition, due to Bulut [6], Joshi *et al.* [9] and Ali and Singh [1], in this paper we define a new class $\mathcal{L}^\lambda_{p,\Sigma}(\phi_\alpha)$, λ-bi-pseudo-parabolic starlike functions of Σ and determine the bounds for the initial Taylor-Maclaurin coefficients of $|a_2|$ and $|a_3|$ for $f \in \mathcal{L}^\lambda_{p,\Sigma}(\phi_\alpha)$. Further we consider the Fekete-Szegö problem in this class.

Definition 1.3. Assume that $f \in \Sigma$, $\lambda \geq 1$ and $(f'(z))^\lambda$ is analytic in $\mathbb{U}$ with $(f'(0))^\lambda = 1$. Furthermore, assume that $g(z)$ is an extension of f^{-1} to $\mathbb{U}$, and $(g'(z))^\lambda$ is analytic in $\mathbb{U}$ with $(g'(0))^\lambda = 1$. Then $f(z)$ is said to be in the class $\mathcal{L}^\lambda_{p,\Sigma}(\phi_\alpha)$ of λ-bi-pseudo-starlike functions if the following conditions are satisfied:

$$\left|\frac{z(f'(z))^\lambda}{f(z)} - 1\right| < (1-2\alpha) + \Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) \quad (z \in \mathbb{U}) \tag{1.7}$$

and

$$\left|\frac{w(g'(w))^\lambda}{g(w)} - \right| < (1-2\alpha) + \Re\left(\frac{w(g'(w))^\lambda}{g(w)}\right) \quad (w \in \mathbb{U}). \tag{1.8}$$

Due to Lemma 1.1 and by the above the definition we can state

$$\frac{z(f'(z))^{\lambda}}{f(z)} \prec \phi_{\alpha}(z) \quad (z \in \mathbb{U}) \tag{1.9}$$

and

$$\frac{w(g'(w))^{\lambda}}{g(w)} \prec \phi_{\alpha}(w) \quad (w \in \mathbb{U}), \tag{1.10}$$

where $\phi_{\alpha}(z)$ is given by (1.4).

Remark 1.4. For $\lambda = 1$, a function $f \in \Sigma$ is in the class $\mathcal{L}^1_{p,\Sigma}(\phi_\alpha) \equiv \mathcal{S}^p_{\Sigma}(\phi_\alpha)$ [6] if the following conditions are satisfied:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < (1-2\alpha) + \Re\left(\frac{zf'(z)}{f(z)}\right) \quad (z \in \mathbb{U})$$

and

$$\left|\frac{wg'(w)}{g(w)} - 1\right| < (1-2\alpha) + \Re\left(\frac{wg'(w)}{g(w)}\right) \quad (w \in \mathbb{U}),$$

where $z, w \in \mathbb{U}$ and the function g is described in Definition 1.3.

Further, in particular, we set $\mathcal{L}^1_{p,\Sigma}(\phi_{\frac{1}{2}}) \equiv \mathcal{S}^p_{\Sigma}(\phi_{\frac{1}{2}}) \equiv \mathcal{S}^p_{\Sigma}$ for the class of parabolic bi-starlike functions.

Remark 1.5. For $\lambda = 2$, a function $f \in \Sigma$ is in the class $\mathcal{L}^2_{p,\Sigma}(\phi_\alpha) \equiv \mathcal{G}_{p,\Sigma}(\phi_\alpha)$ if the following conditions are satisfied:

$$\left|f'(z)\frac{zf'(z)}{f(z)} - 1\right| < (1-2\alpha) + \Re\left(f'(z)\frac{zf'(z)}{f(z)}\right) \quad (z \in \mathbb{U})$$

and

$$\left|g'(w)\frac{wg'(w)}{g(w)} - 1\right| < (1-2\alpha) + \Re\left(g'(w)\frac{wg'(w)}{g(w)}\right) \quad (w \in \mathbb{U}),$$

where $z, w \in \mathbb{U}$ and the function g is described in Definition 1.3.

2. Coefficient estimates for $f \in \mathcal{L}^{\lambda}_{p,\Sigma}(\phi_\alpha)$.

Using the following lemma, we obtain the initial coefficients $|a_2|$ and $|a_3|$ for $f \in \mathcal{L}^{\lambda}_{p,\Sigma}(\phi_\alpha)$.

Lemma 2.1. ([13]) *If $p \in \mathcal{P}$, and*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad (z \in \mathbb{U}) \tag{2.1}$$

then $|p_n| \leq 2$ for $n \geq 1$, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which

$$\Re\left(p(z)\right) > 0 \quad (z \in \mathbb{U}). \tag{2.2}$$

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{L}^{\lambda}_{p,\Sigma}(\phi_\alpha)$. Then*

$$|a_2| \leq \frac{|B_1|\sqrt{B_1}}{\sqrt{|(2\lambda^2-\lambda)B_1^2-(B_2-B_1)(2\lambda-1)^2|}} \tag{2.3}$$

and

$$|a_3| \leq \frac{2(3\lambda-1)B_1^3+B_1|B_2-B_1|(2\lambda-1)^2}{2(3\lambda-1)\left|(2\lambda^2-\lambda)\,B_1^2-(B_2-B_1)(2\lambda-1)^2\right|}, \tag{2.4}$$

where

$$B_k = \frac{16(1-\alpha)}{k\pi^2}\sum_{j=0}^{k-1}\frac{1}{2j+1}, \quad (k \in \mathbb{N}). \tag{2.5}$$

Proof. Let g be of the form

$$g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

Since $f \in \mathcal{L}^{\lambda}_{p,\Sigma_\alpha}(\phi)$, there exist two analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = 0 = v(0)$, such that $|u(z)| < 1$, $|v(z)| < 1$ and

$$\frac{z[f'(z)]^\lambda}{f(z)} = \phi_\alpha(u(z)), \tag{2.6}$$

$$\frac{w[g'(w)]^\lambda}{g(w)} = \phi_\alpha(v(w)). \tag{2.7}$$

Assume that $p(z)$ and $q(z)$ are in $\mathcal{P}$ and they are such that

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1z + p_2z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1z + q_2z^2 + \cdots .$$

It follows that,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1z + \left(q_2 - \frac{q_1^2}{2}\right)z^2 + \cdots\right],$$

so we have

$$\phi(u(z)) = 1 + \frac{1}{2}B_1p_1z + \left[\frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right]z^2 + \cdots \tag{2.8}$$

and

$$\phi(v(w)) = 1 + \frac{1}{2}B_1q_1w + \left[\frac{B_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2q_1^2\right]w^2 + \cdots . \tag{2.9}$$

On the other hand, we have

$$\frac{z[f'(z)]^{\lambda}}{f(z)} = 1 + (2\lambda-1)a_2z + [(3\lambda-1)a_3 + \left(2\lambda^2-4\lambda+1\right)a_2^2]z^2 + \cdots \quad (2.10)$$

and

$$\frac{w[g'(w)]^{\lambda}}{g(w)} = 1-(2\lambda-1)a_2w+[\left(2\lambda^2+2\lambda-1\right)a_2^2-(3\lambda-1)a_3]w^2+\cdots. \quad (2.11)$$

Using (2.8), (2.9), (2.10) and (2.11) and comparing the like coefficients of z and z^2, we get

$$(2\lambda-1)a_2 = \frac{1}{2}B_1p_1, \quad (2.12)$$

$$(3\lambda-1)a_3 + \left(2\lambda^2-4\lambda+1\right)a_2^2 = \frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2, \quad (2.13)$$

$$-(2\lambda-1)a_2 = \frac{1}{2}B_1q_1 \quad (2.14)$$

and

$$\left(2\lambda^2+2\lambda-1\right)a_2^2-(3\lambda-1)a_3 = \frac{1}{2}B_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}B_2q_1^2 \quad (2.15)$$

From (2.12) and (2.14), we find that

$$a_2 = \frac{B_1p_1}{2(2\lambda-1)} = -\frac{B_1q_1}{2(2\lambda-1)}.$$

Therefore, it follows that

$$p_1 = -q_1 \quad (2.16)$$

and

$$8(2\lambda-1)^2a_2^2 = B_1^2(p_1^2+q_1^2). \quad (2.17)$$

Thus,

$$a_2^2 = \frac{B_1^2(p_1^2+q_1^2)}{8(2\lambda-1)^2} \quad \text{or} \quad p_1^2+q_1^2 = \frac{8(2\lambda-1)^2}{B_1^2}a_2^2. \quad (2.18)$$

Adding (2.13) and (2.15), we have

$$\begin{aligned}\left(4\lambda^2-2\lambda\right)a_2^2 &= \frac{1}{2}B_1(p_1+q_1)+\frac{1}{2}B_1\left[(p_2+q_2)-\frac{1}{2}\left(p_1^2+q_1^2\right)\right] \\ &\quad +\frac{1}{4}B_2\left(p_1^2+q_1^2\right) \\ &= \frac{1}{2}B_1(p_2+q_2)+\frac{1}{4}(B_2-B_1)\left(p_1^2+q_1^2\right). \quad (2.19)\end{aligned}$$

Substituting (2.16) and (2.18) in (2.19), we get

$$\left(4\lambda^2 - 2\lambda\right) a_2^2 = \frac{1}{2}B_1(p_2+q_2) + \frac{1}{4}(B_2 - B_1)\frac{8(2\lambda-1)^2}{B_1^2}a_2^2,$$

$$\left[\left(4\lambda^2 - 2\lambda\right) - \frac{2(B_2-B_1)(2\lambda-1)^2}{B_1^2}\right] a_2^2 = \frac{1}{2}B_1(p_2+q_2)$$

and

$$\left[\left(4\lambda^2 - 2\lambda\right) B_1^2 - 2(B_2-B_1)(2\lambda-1)^2\right] a_2^2 = B_1^3(p_2+q_2).$$

Hence

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{2\left[\left(2\lambda^2-\lambda\right) B_1^2 - (B_2-B_1)(2\lambda-1)^2\right]}. \tag{2.20}$$

Applying Lemma 2.1 in (2.20), we get the desired inequality (2.3).

From (2.13) and from (2.15), we get

$$a_3 = a_2^2 + \frac{B_1(p_2-q_2)}{4(3\lambda-1)}. \tag{2.21}$$

By using (2.20) and Lemma 2.1, by simple computation, we obtain

$$|a_3| \leq \frac{2(3\lambda-1)B_1^3 + B_1|B_2-B_1|(2\lambda-1)^2}{2(3\lambda-1)\left|\left(2\lambda^2-\lambda\right) B_1^2 - (B_2-B_1)(2\lambda-1)^2\right|}. \tag{2.22}$$

This completes the proof of Theorem 2.2. □

Remark 2.3. We note that $B_1 = \frac{16}{\pi^2}(1-\alpha)$ and $B_2 = \frac{32}{3\pi^2}(1-\alpha)$ from (2.5).

By taking $\lambda = 1$, we state the following result.

Corollary 2.4. *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{L}^1_{p,\Sigma}(\phi_\alpha) \equiv \mathcal{S}_{p,\Sigma}(\phi_\alpha)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}}$$

and

$$|a_3| \leq \frac{4B_1^3 + B_1|B_2-B_1|}{4|B_1^2 - (B_2-B_1)|}.$$

By taking $\lambda = 2$ we state the following new result.

Corollary 2.5. *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{L}^2_{p,\Sigma}(\phi_\alpha) \equiv \mathcal{G}_{p,\Sigma}(\phi_\alpha)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|6B_1^2 - 9(B_2-B_1)|}}$$

and

$$|a_3| \leq \frac{10B_1^3 + 9B_1|B_2 - B_1|}{10|B_1^2 - 9(B_2 - B_1)|}.$$

3. Fekete-Szegö inequalities for the Function Class $\mathcal{L}_{p,\Sigma}^{\lambda}(\phi_\alpha)$

Making use of the values of a_2^2 and a_3, and motivated by the recent work of Zaprawa [17], we prove the following Fekete-Szegö result for the function class $\mathcal{L}_{p,\Sigma}^{\lambda}(\phi_\alpha)$.

Theorem 3.1. *Let the function $f(z)$ be in the class $\mathcal{L}_{\Sigma}^{\lambda}(\phi)$ and $\mu \in \mathbb{C}$. Then*

$$|a_3 - \mu a_2^2| \leq 2B_1 \left| \left(\Theta(\mu) + \frac{1}{4(3\lambda - 1)}\right) + \left(\Theta(\mu) - \frac{1}{4(3\lambda - 1)}\right) \right|, \qquad (3.1)$$

where

$$\Theta(\mu) = \frac{B_1^2(1-\mu)}{2\left[(2\lambda^2 - \lambda)\, B_1^2 - (B_2 - B_1)(2\lambda - 1)^2\right]}, B_1 > 0.$$

Proof. From (2.21), we have

$$a_3 = a_2^2 + \frac{B_1(p_2 - q_2)}{4(3\lambda - 1)}.$$

Using (2.20), by simple calculation we get

$$a_3 - \mu a_2^2 = B_1 \left[\left(\Theta(\mu) + \frac{1}{4(3\lambda - 1)}\right) p_2 + \left(\Theta(\mu) - \frac{1}{4(3\lambda - 1)}\right) q_2 \right],$$

where

$$\Theta(\mu) = \frac{B_1^2(1-\mu)}{2\left[(2\lambda^2 - \lambda)\, B_1^2 - (B_2 - B_1)(2\lambda - 1)^2\right]}.$$

Since all B_j are real and $B_1 > 0$, we have

$$|a_3 - \mu a_2^2| \leq 2B_1 \left| \left(\Theta(\mu) + \frac{1}{4(3\lambda - 1)}\right) + \left(\Theta(\mu) - \frac{1}{4(3\lambda - 1)}\right) \right|,$$

which completes the proof. □

Remark 3.2. Specializing $\lambda = 1$ and $\lambda = 2$, we can obtain the Fekete-Szegö inequality for the function class $\mathcal{S}_{p,\Sigma}(\phi_\alpha)$ and $\mathcal{G}_{p,\Sigma}(\phi_\alpha)$, respectively.

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