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SOME FIXED POINT THEOREMS IN D^* METRIC SPACES

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Abstract. Recently Shaban Sedghi et al [16] introduced D^* metric space and proved some common fixed point in it. In this paper we improved the results of Shaban Sedghi et al [16] by introducing D^* compatible and semicompatible mappings in D^* metric spaces.

1. Introduction

The metric space is generalized by many authors see [11-17], one of its generalization is D-metric space initiated by B. C. Dhage [3]. He proved some fixed point theorems for self mappings satisfying different types of contractive conditions. Rhoades [6] generalized Dhage's contractive condition by increasing number of factors and prove existence and uniqueness of a fixed point in complete and bounded D-metric space. Ahmad et al [1], Dhage [4], Dhage et al [5] give some special contribution in D-metric space.

Jungck [11] introduced concept of compatible mappings. This concept extended to D-compatible mappings in D-metric space by Bijendra Singh and A. K. Sharma [7] and proved common fixed point theorems in it. Cho, Sharms and Sahu [2] introduced the concept of semi-compatible mappings in d-topological spaces. Bijendra Singh et al [8] used semicompability in D-metric and obtained some common fixed theorems in D-metric spaces.

Recently Shaban Sedghi et al [16] modify the D-metric space as follows.

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Definition 1.1. *Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function, $D^* : X^3 \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$:*

(1) $D^*(x, y, z) \geq 0$,
(2) $D^*(x, y, z) = 0$ *if and only if* $x = y = z$,
(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$, *(symmetry) where p is a permutation function,*
(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$,

The pair (X, D^*) is called a generalized metric (or D^*-metric) space.

Example 1.2. *Let (X, d) be a metric space. Define*

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$.
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.
(c) *If $X = R^n$, then we define*

$$D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p}$$

for every $p \in R^+$.

(d) *If $X = R$ then we define*

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

Then it is easy to verify that every D^* is a D^*-metric.

Remark 1.3. [16] *Let (X, D^*) be a D^* metric space. Then for all $x, y, z \in X$, we have $D^*(x, x, y) = D^*(x, y, y)$.*

Definition 1.4. [16] *Let (X, D^*) be a D^*-metric space and $A \subset X$.*

(1) *If for every $x \in A$ there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$, then subset A is called open subset of X.*
(2) *A is said to be D^*-bounded if there exists $r > 0$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.*
(3) *A sequence $\{x_n\}$ in X converges to x if there exists $n_0 \in N$ such that $D^*(x_n, x_n, x) < \epsilon$.*
(4) *A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $D^*(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$.*
(5) *A D^*-metric space (X, D^*) is said to be complete if every Cauchy sequence in X is convergent.*

Let τ be the set of all open subsets A of X. Then τ is a topology on X (induced by the D^*-metric D^*).

Lemma 1.5. [16] *Let (X, D^*) be a D^*-metric space. If $r > 0$, then the ball $B_{D^*}(x, r)$ with center $x \in X$ and radius r is open.*

Definition 1.6. [16] *Let (X, D^*) be a D^* metric space. D^* is said to be a continuous function on X^3 if* $\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$ *whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$ i.e.* $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$

Lemma 1.7. [16] *Let (X, D^*) be a D^*- metric space. Then D^* is a continuous function on X^3.*

Lemma 1.8. [16] *Let (X, D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Lemma 1.9. [16] *Let (X, D^*) be a D^*-metric space. Then the convergent sequence is Cauchy.*

Definition 1.10. [16] *Let A and S be two mappings from a D^*-metric space (X, D^*) into itself. Then the pair $\{A, S\}$ is said to be weakly commuting if*

$$D^*(ASx, SAx, SAx) \le D^*(Ax, Sx, Sx),$$

for all $x \in X$.

Clearly, a commuting pair is weakly commuting, but not conversely.

We extended D^*-compatible and semicompatible mappings as follows.

Definition 1.11. *Self maps S and T on a D^*-metric space (X, D^*) are said to be D^*-compatible if* $\lim_{n\to\infty} D^*(STx_n, TSx_n, z) = 0$, *where $z = STx_n$ or TSx_n, whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Clearly the pair of mappings (S, T) is D^*-compatible if and only if (T, S) is D^*-compatible.

Definition 1.12. *A pair (S, T) of self-mappings of a D^*-metric space is said to be semicompatible if* $\lim_{n\to\infty} STx_n = Tx$, *whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x \in X.$$

It follows that (S, T) is semicompatible and $Sy = Ty$ then $STy = TSy$.

Proposition 1.13. *S and T are D^*-compatible self-maps on a D^*-metric space (X, D^*) and T is continuous then the pair (S, T) is semicompatible.*

Proof. Let $\{Sx_n\} \to u, \{Tx_n\} \to u$ for some $u \in X$. To show this, $STx_n \to Tu$. As T is continuous $TSx_n \to Tu$. Now, as (S, T) is D^*-compatible we have $\lim_{n\to\infty} D^*(STx_n, STx_n, TSx_n) = 0$. That is, $\lim_{n\to\infty} D^*(Tu, Tu, TSx_n) = 0$. That is, $\lim_{n\to\infty} STx_n = Tu$. Hence (S, T) is semicompatible. □

Proposition 1.14. *If S and T are semicompatible self-maps on a D^*-metric space (X, D^*) and T is continuous, then (S,T) is D^*-compatible.*

Proof. Let $\{Sx_n\} \to u, \{Tx_n\} \to u$ and T be continuous $TSx_n \to Tu$. Then semicompatibility of (S,T) gives $STx_n \to Tu$. Now,

$$\lim_{n\to\infty} D^*(STx_n, STx_n, TSx_n) = D^*(Tu, Tu, Tu) = 0.$$

Hence (S,T) is D^*-compatible. □

The following is an example of a pair of self-maps (S,T) which is semicompatible but not compatible. Further, it is shown that the semicompatibility of the pair (S,T) need not imply the semicompatibility of (T,S).

Example 1.15. *Let $X = [0,1]$ and consider the D^*-metric space (X, D^*), where D^* is defined by $D^*(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$, for all $x, y, z \in X$. Define a self-map as follows:*

$$Sx = \begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ 1 & \text{if } x \ge \frac{1}{2}. \end{cases}$$

Let I be the identity map on X and $x_n = 1/2 - 1/n$. Then $\{Ix_n\} = \{x_n\} \to 1/2$ and $\{Sx_n\} \to 1/2$. Again, $\{ISx_n\} = \{Sx_n\} \to 1/2 \neq S(1/2)$. Thus (I,S) is not semicompatible though it is compatible. Also for any sequence $\{x_n\}$ in X such that $\{x_n\} \to x$ and $\{Sx_n\} \to x$ we have $\{SIx_n\} = \{Sx_n\} \to x = Ix$. Thus (S,I) is always semicompatible.

Example 1.16. *Let $X = [0,2]$, define $D(x,y,z) = Max\{|x-y|, |y-z|, |z-x|\}$, for all $x, y, z \in X$. Define self-maps A and S on X as follows:*

$$Ax = \begin{cases} x & \text{if } x \in [0,1) \\ 2 & \text{if } x = 1 \\ \frac{x+3}{5} & \text{if } x \in (1,2]. \end{cases}$$

$$Sx = \begin{cases} 2 & \text{if } x \in [0,1] \\ \frac{x}{2} & \text{if } x \in [1,2]. \end{cases}$$

Taking $x_n = 2 - \frac{1}{2n}$, then we have $S(1) = A(1) = 2$ and $S(2) = A(2) = 1$. Also $SA(1) = AS(1) = 1$ and $SA(2) = AS(2) = 2$. Hence $Ax_n \to 1$ and $Sx_n \to 1$, $ASx_n \to 2$, and $SAx_n \to 1$. Now, $\lim_{n\to\infty} D^*(ASx_n, ASx_m, Sy) = D(2,2,2) = 0$, $\lim_{n\to\infty} D^*(ASx_n, SAx_n, ASx_n) = D(2,1,2) = 1 \neq 0$. Hence (A,S) is D^*-semicompatible but it is not D^*-compatible

2. Main Results

Let Φ denotes a family of mappings such that each $\phi \in \Phi$, $\phi : (R^+)^5 \to R^+$, and ϕ is continuous and increasing in each coordinate variable. Also $\phi(t, t, a_1 t, a_2 t, t) < t$ for every $t \in R^+$ where $a_1 + a_2 = 3$.

Lemma 2.1. *For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the composition of γ with itself n times.*

Theorem 2.2. *Let A be a self-mapping of a complete D^*-metric space (X, D^*), and let S, T be continuous self-mappings on X satisfying the following conditions:*

(i) *$\{A, S\}$ and $\{A, T\}$ are semicompatible pairs such that*

$$A(X) \subset S(X) \cup T(X),$$

(ii) *there exists a $\phi \in \Phi$ such that for all $x, y \in X$,*

$$D^*(Ax, Ay, Az) \le \phi(D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay)).$$

Then A, S, and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be given. Construct a sequence $\{x_n\}$, as follows

$$Sx_{2n+1} = Ax_{2n} = y_{2n}, n = 0, 1, 2, \cdots,$$
$$Tx_{2n+2} = Ax_{2n+1} = y_{2n+1}, n = 0, 1, 2, \cdots.$$

Denote $d_n = D^*(y_n, y_{n+1}, y_{n+1}), n = 0, 1, 2, \cdots$. We prove that $d_{2n} \le d_{2n-1}$. Now, if $d_{2n} > d_{2n-1}$ for some $n \in N$, since ϕ is an increasing function,

$$\begin{aligned} d_{2n} &= D^*(y_{2n}, y_{2n+1}, y_{2n+1}) = D^*(Ax_{2n}, Ax_{2n+1}, Ax_{2n+1}) \\ &= D^*(Ax_{2n+1}, Ax_{2n}, Ax_{2n}) \\ &\le \phi((D^*(Sx_{2n+1}, Tx_{2n}, Tx_{2n}), D^*(Sx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}), \\ &\quad D^*(Sx_{2n+1}, Ax_{2n}, Ax_{2n}), D^*(Tx_{2n}, Ax_{2n+1}, Ax_{2n+1}), \\ &\quad D^*(Tx_{2n}, Ax_{2n}, A_{2n})) \\ &= \phi(D^*(y_{2n}, y_{2n-1}, y_{2n-1}), D^*(y_{2n}, y_{2n+1}, y_{2n+1}), D^*(y_{2n}, y_{2n}, y_{2n}), \\ &\quad D^*(y_{2n-1}, y_{2n+1}, y_{2n+1}), D^*(y_{2n-1}, y_{2n}, y_{2n}). \end{aligned}$$

Since

$$\begin{aligned} D^*(y_{2n-1}, y_{2n+1}, y_{2n+1}) &\le D^*(y_{2n-1}, y_{2n-1}, y_{2n}) + D^*(y_{2n}, y_{2n+1}, y_{2n+1}) \\ &= d_{2n-1} + d_{2n}, \end{aligned}$$

from the above inequality we have

$$\begin{aligned} d_{2n} &\le \phi(d_{2n-1}, d_{2n}, 0, d_{2n-1} + d_{2n}, d_{2n-1}) \\ &\le \phi(d_{2n}, d_{2n}, d_{2n}, 2d_{2n}, d_{2n}) \\ &< d_{2n}, \end{aligned}$$

which is a contradiction. Hence $d_{2n} \le d_{2n-1}$. Similarly, one can prove that $d_{2n+1} \le d_{2n}$ for $n = 0, 1, 2, \cdots$. Consequently, $\{d_n\}$ is a nonincreasing sequence of nonnegative reals. Now,

$$\begin{aligned} d_1 &= D^*(y_1, y_2, y_2) = D^*(Ax_1, Ax_2, Ax_2) \\ &\le \phi(D^*(Sx_1, Tx_2, Tx_2), D^*(Sx_1, Ax_1, Ax_1), D^*(Sx_1, Ax_2, Ax_2), \\ &\quad D^*(Tx_2, Ax_1, Ax_1), D^*(Tx_2, Ax_2, A_{x2})) \\ &= \phi(D^*(y_0, y_1, y_1), D^*(y_0, y_1, y_1), D^*(y_0, y_2, y_2), D^*(y_1, y_1, y_1), D^*(y_1, y_2, y_2)) \\ &= \phi(d_0, d_0, d_0 + d_1, 0, d_0) \le \phi(d_0, d_0, 2d_0, d_0, d_0) \\ &= \gamma(d_0), \end{aligned}$$

which implies thate $d_n \le \gamma^n(d_0)$. So if $d_0 > 0$, then $\lim_{n\to\infty} d_n = 0$. For $d_0 = 0$, we clearly have $\lim_{n\to\infty} d_n = 0$, since then $d_n = 0$ for each n. Now we prove that sequence $\{Ax_n = y_n\}$ is Cauchy. Since $\lim_{n\to\infty} d_n = 0$, it is sufficient to show that the sequence $\{Ax_{2n} = y_{2n}\}$ is Cauchy. Suppose that $\{Ax_{2n} = y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, for $k = 0, 1, 2, \cdots$, there exist even integers $2n(k)$ and $2m(k)$ with $2k \le 2n(k) < 2m(k)$ such that $D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon$. Let, for each even integer $2k, 2m(k)$ be the least integer exceeding $2n(k)$ satisfying the above inequality. Therefore $D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) \le \epsilon, D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon$. Then, for each even integer $2k$ we have

$$\begin{aligned} \epsilon &< D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \\ &\le D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + D^*(Ax_{2m(k)-2}, Ax_{2m(k)-2}), Ax_{2m(k)-1}) \\ &\quad + D^*(Ax_{2m(k)-1}, Ax_{2m(k)-1}, Ax_{2m(k)}) \\ &= D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \end{aligned}$$

From, $d_n \to 0$, we obtain $\lim_{k\to\infty} D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) = \epsilon$. It follows immediately from the triangular inequality that

$$|D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)})| \le d_{2m(k)-1},$$

$|D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)})|$
$< d_{2m(k)-1} + d_{2n(k)}$. Hence as $k \to \infty$,

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}) \to \epsilon,$$

$$D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1})) \to \epsilon.$$

Now

$$\begin{aligned}
&D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \\
&\leq D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2n(k)+1}) + D^*(Ax_{2n(k)+1}, Ax_{2m(k)}, Ax_{2m(k)}) \\
&\leq d_{2n(k)} + \phi((D^*(Ax_{2n(k)}, Ax_{2m(k)-1}, Ax_{2m(k)-1}), d_{2n(k)}, \\
&\quad D^*(Ax_{2n(k)}, Ax_{2m(k)}, Ax_{2m(k)}), \\
&\quad D^*(Ax_{2m(k)-1}, Ax_{2n(k)+1}, Ax_{2n(k)+1}), d_{2m(k)-1})).
\end{aligned}$$

Using, $\lim_{k\to\infty} d_n = 0$, and continuity and nondecreasing property of ϕ in each coordinate variable, we have

$$\epsilon \leq \phi(\epsilon, 0, \epsilon, \epsilon, 0) \leq \phi(\epsilon, \epsilon, 2\epsilon, \epsilon, \epsilon) = \phi(\epsilon) < \epsilon$$

as $k \to \infty$, which is a contradiction. Thus $\{Ax_n = y_n\}$ is a Cauchy sequence and hence by completeness of X, it converges to $z \in X$. That is, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} y_n = z$. Since the sequences $\{Sx_{2n+1} = y_{2n+1}\}$ and $\{Tx_{2n} = y_{2n}\}$ are subsequences of $\{Ax_n = y_n\}$; they have the same limit z. As S and T are continuous, we have $STx_{2n} \to Sz$ and $TSx_{2n+1} \to Tz$. Since $\{A, S\}$ is semicompatible, hence $ASx_{2n+1} \to Sz$. Put $x = Sx_{2n+1}, y = Sx_{2n+1}, z = Tx_{2n}$ in (ii) we have

$$\begin{aligned}
&D^*(ASx_{2n+1}, ASx_{2n+1}, ATx_{2n}) \\
&\leq \phi(D^*(SSx_{2n+1}, TSx_{2n+1}, TTx_{2n}), D^*(SSx_{2n+1}, ASx_{2n+1}, ASx_{2n+1}), \\
&\quad D^*(SSx_{2n+1}, ASx_{2n+1}, ASx_{2n+1}), D^*(TSx_{2n+1}, ASx_{2n+1}, ASx_{2n+1}), \\
&\quad D^*(TSx_{2n+1}, ASx_{2n+1}, ASx_{2n+1})).
\end{aligned}$$

As $n \to \infty$ and if $D^*(Sz, Sz, Tz) \neq 0$, then

$$\begin{aligned}
D^*(Sz, Sz, Tz) &\leq \phi(D^*(Sz, Tz, Tz), D^*(Sz, Sz, Sz), D^*(Sz, Sz, Sz), \\
&\quad D^*(Tz, Sz, Sz), D^*(Tz, Sz, Sz)) \\
&\leq \phi(D^*(Sz, Sz, Tz), D^*(Sz, Sz, Sz), D^*(Sz, Sz, Sz), \\
&\quad D^*(Sz, Sz, Tz), D^*(Sz, Sz, Tz)) \\
&< D^*(Sz, Sz, Tz),
\end{aligned}$$

which is a contradiction. Hence $D^*(Sz, Sz, Tz) = 0$ that is, $Sz = Tz$.
Now

$$\begin{aligned}
D^*(SAx_{2n+1}, Az, Az) &\leq D^*(SAx_{2n+1}, ASx_{2n+1}, ASx_{2n+1}) \\
&\quad + D^*(Az, Az, ASx_{2n+1}).
\end{aligned}$$

Using (ii) and the semicompatibility of (A,S), we have

$$\begin{aligned}&D^*(SAx_{2n+1}, Az, Az)\\&\leq D^*(SAx_{2n+1}, ASx_{2n+1}, ASx_{2n+1}) + \phi(D^*(Sz, Tz, TSx_{2n+1}),\\&D^*(Sz, Az, Az), D^*(Sz, Az, Az), D^*(Tz, Az, Az), D^*(Tz, Az, Az)).\end{aligned}$$

Letting $n \to \infty$, then we have

$$\begin{aligned}D^*(Sz, Az, Az) &\leq D^*(Sz, Sz, Dz) + \phi(D^*(Sz, Tz, Tz), D^*(Sz, Az, Az),\\&\quad D^*(Sz, Az, Az), D^*(Tz, Az, Az), D^*(Tz, Az, Az))\\&= \phi(0, D^*(Sz, Az, Az), D^*(Sz, Az, Az), D^*(Sz, Az, Az),\\&\quad D^*(Sz, Az, Az))\\&< D^*(Sz, Az, Az).\end{aligned}$$

Since $Sz = Az$, $Az = Sz = Tz$. It now follows that

$$\begin{aligned}&D^*(Az, Ax_{2n}, Ax_{2n})\\&\leq \phi(D^*(Sz, Tx_{2n}, Tx_{2n}), D^*(Sz, Az, Az), D^*(Sz, Ax_{2n}, Ax_{2n}),\\&\quad D^*(Tx_{2n}, Az, Az), D^*(Tx_{2n}, Ax_{2n}, Ax_{2n}))\end{aligned}$$

Then as $n \to \infty$, we get

$$\begin{aligned}D^*(Az, z, z) &\leq \phi(D^*(Sz, z, z), 0, D^*(Sz, z, z), D^*(z, Az, Az), 0)\\&< D^*(Az, z, z),\end{aligned}$$

which is a contradiction, and therefore $Az = z = Sz = Tz$. Thus z is a common fixed point of A, S and T. The uniqueness is easy one. This completes the proof. □

Theorem 2.3. *Let A be a self-mapping of complete D^*-metric space (X, D^*), and let S, T be continuous self-mappings on X satisfying the following conditions:*

(i) *$\{A, S\}$ and $\{A, T\}$ are D^*-compatible pairs such that $A(X) \subset S(X) \cup T(X)$;*

(ii) *there exists a $\phi \in \Phi$ such that for all $x, y \in X$,*

$$\begin{aligned}D^*(Ax, Ay, Az) \leq \phi(&D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay),\\&D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay)).\end{aligned}$$

Then A, S, and T have a unique common fixed point in X.

Proof. Since the semicompatibility implies D^*-compatibility, the result is obvious. □

Example 2.4. *Let $X = [0,1]$ and consider the D^*-metric space (X, D^*), where D^* is defined by $D^*(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$, for all $x,y,z \in X$. Define a self-map as follows:*

$$Ax = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Let $Sx = x$ and $Tx = 1 - x$. Let $x_n = 1/2 - 1/n$ be sequence in X. Then $Ax_n = \frac{1}{2}$, $\{Sx_n\} = \{\frac{1}{2} - \frac{1}{2}\} \to 1/2$ and $\{Tx_n\} = \{1 - \frac{1}{2} + \frac{1}{n}\} \to 1/2$. Again, $\{ASx_n\} = A(\frac{1}{2}) = \frac{1}{2} = S(\frac{1}{2})$ and $\{ATx_n\} = A(\frac{1}{2}) = \frac{1}{2} = T(\frac{1}{2})$. Thus (A,S) and (A,T) is semicompatible. For all $x,y,z \in X$, we have $D^*(\frac{1}{2}, \frac{1}{2}, 1) = D^*(\frac{1}{2}, 1, 1) = D^*(1, \frac{1}{2}, \frac{1}{2}) = D^*(1, \frac{1}{2}, 1) = D^*(1, 1, \frac{1}{2}) = D^*(\frac{1}{2}, 1, \frac{1}{2}) = \frac{1}{2}$ and $D*(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = D^*(1,1,1) = 0$. we easily verified that

$$D^*(Ax, Ay, Az) \leq \phi(D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay)).$$

That requirement of Theorem 2.2 is fulfil and clearly A, S and T have unique fixed point.

As Theorem 2.2 it is easy to prove following theorem.

Theorem 2.5. *Let A be a self-mapping of a complete D^*-metric space (X, D^*), and let S, T be continuous self-mappings on X satisfying the following conditions:*

(i) *$\{A, S\}$ and $\{A, T\}$ are semicompatible pairs such that $A(X) \subset S(X) \cup T(X)$;*

(ii) *there exists a $\phi : R^+ \to R^+$ such that $\phi(0) = 0, \phi(t) < t$ and for all $x, y \in X$,*

$$D^*(Ax, Ay, Az) \leq \phi(\max\{(D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay))\}).$$

Then $A, S,$ and T have a unique common fixed point in X.

Theorem 2.6. *Let A be a self-mapping of a complete D^*-metric space (X, D^*), and let S, T be continuous self-mappings on X satisfying the following conditions:*

(i) *$\{A, S\}$ and $\{A, T\}$ are D^*-compatible pairs such that $A(X) \subset S(X) \cup T(X)$;*

(ii) *there exists a* $\phi : R^+ \to R^+$ *such that* $\phi(0) = 0, \phi(t) < t$ *and for all* $x, y \in X$,

$$D^*(Ax, Ay, Az) \leq \phi(\max\{(D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay))\}).$$

Then $A, S,$ *and* T *have a unique common fixed point in* X.

Theorem 2.7. *Let* A *be a self-mapping of a complete* D^**-metric space* (X, D^*), *and let* S *be continuous self-mappings on* X *satisfying the following conditions:*

(i) $\{A, S\}$ *is a semicompatible pair such that* $A(X) \subset S(X)$;

(ii) *there exists a* $\phi \in \Phi$ *such that for all* $x, y \in X$,

$$D^*(Ax, Ay, Az) \leq \phi(D^*(Sx, Ay, Az), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ay, Ax, Ax)).$$

Then A *and* S *have a unique common fixed point in* X.

Proof. The proof follows from Theorem 3.1 by putting $T = A$. □

Theorem 2.8. *Let* A *be a self-mapping of a complete* D^**-metric space* (X, D^*), *and let* S *be continuous self-mappings on* X *satisfying the following conditions:*

(i) $\{A, S\}$ *is a* D^**-compatible pair such that* $A(X) \subset S(X)$;

(ii) *there exists a* $\phi \in \Phi$ *such that for all* $x, y \in X$,

$$D^*(Ax, Ay, Az) \leq \phi(D^*(Sx, Ay, Az), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ay, Ax, Ax)).$$

Then A *and* S *have a unique common fixed point in* X.

Proof. The proof follows from Theorem 2.3 by putting $T = A$. □

Corollary 2.9. *Let* $A, R, S, T,$ *and* H *be self-mappings of a complete* D^**-metric space* (X, D^*), *and let* SR, TH *be continuous self-mappings on* X *satisfying the following conditions:*

(i) $\{A, SR\}$ *and* $\{A, TH\}$ *are semicompatible pairs such that* $A(X) \subset SR(X) \cap TH(X)$;

(ii) *there exists a* $\phi \in \Phi$ *such that for all* $x, y \in X$,

$$D^*(Ax, Ay, Az) \leq \phi(D^*(SRx, THy, THz), D^*(SRx, Ax, Ax), D^*(SRx, Ay, Ay), D^*(THy, Ax, Ax), D^*(THy, Ay, Ay)).$$

If $SR = RS, TH = HT, AH = HA,$ *and* $AR = RA$, *then* $A, S, R, H,$ *and* T *have a unique common fixed point in* X.

Proof. By Theorem 2.2, A, TH, and SR have a unique common fixed point in X. That is, there exists $a \in X$, such that $A(a) = TH(a) = SR(a) = a$. We prove that $R(a) = a$. By (ii), we get

$$\begin{aligned} D^*(ARa, Aa, Aa) \leq \phi(&D^*(SRRa, THa, THa), D^*(SRRa, ARa, ARa), \\ &D^*(SRRa, Aa, Aa), D^*(THa, ARa, ARa), \\ &D^*(THa, Aa, Aa)). \end{aligned}$$

Hence if $Ra \neq a$, then we have

$$\begin{aligned} D^*(Ra, a, a) &\leq \phi((D^*(Ra, a, a), D^*(Ra, Ra, Ra), D^*(Ra, a, a), D^*(a, Ra, Ra), \\ &\quad D^*(a, a, a)) \\ &\leq \phi(D^*(Ra, a, a), D^*(Ra, a, a), D^*(Ra, a, a), 2D^*(Ra, a, a), \\ &\quad D^*(Ra, a, a)) \\ &< D^*(Ra, a, a), \end{aligned}$$

which is a contradiction. Therefore it follows that $Ra = a$. Hence $S(a) = SR(a) = a$. Similarly, we get that $T(a) = H(a) = a$. □

Corollary 2.10. *Let A, R, S, T, and H be self-mappings of a complete D^*-metric space (X, D^*), and let SR, TH be continuous self-mappings on X satisfying the following conditions:*

(i) *$\{A, SR\}$ and $\{A, TH\}$ are D^*-compatible pairs such that $A(X) \subset SR(X) \cap TH(X)$;*

(ii) *there exists a $\phi \in \Phi$ such that for all $x, y \in X$,*

$$\begin{aligned} D^*(Ax, Ay, Az) \leq \phi(&D^*(SRx, THy, THz), D^*(SRx, Ax, Ax), D^*(SRx, Ay, Ay), \\ &D^*(THy, Ax, Ax), D^*(THy, Ay, Ay)). \end{aligned}$$

If $SR = RS, TH = HT, AH = HA$, and $AR = RA$, then A, S, R, H, and T have a unique common fixed point in X.

Corollary 2.11. *Let A_i be a sequence self-mapping of complete D^*-metric space (X, D^*) for each $i \in N$, and let S, T be continuous self-mappings on X satisfying the following conditions:*

(i) *there exists $i_0 \in N$ such that $\{A_{i0}, S\}$ and $\{A_{i0}, T\}$ are semicompatible pairs such that $A_{i0}(X) \subset S(X) \cap T(X)$;*

(ii) *there exists a $\phi \in \Phi$ and $i, j, k \in N$ such that for all $x, y \in X$,*

$$\begin{aligned} D^*(A_i x, A_j y, A_k z) \leq \phi(&D^*(Sx, Ty, Tz), D^*(Sx, A_i x, A_i x), D^*(Sx, A_j y, A_j y), \\ &D^*(Ty, A_i x, A_i x), D^*(Ty, A_j y, A_j y). \end{aligned}$$

Then A_i, S, and T have a unique common fixed point in X for every $i \in N$.

Proof. By Theorem 2.3, S, T, and A_{i0}, for some $i = j = k = i_0 \in N$, have a unique common fixed point in X. That is, there exists a unique $a \in X$ such that $S(a) = T(a) = A_{i0}(a) = a$ and using Corollary 2.6 in [16] A_i, S, and T have a unique common fixed point in X for every $i \in N$. □

Corollary 2.12. *Let A_i be a sequence self-mapping of complete D^*-metric space (X, D^*) for each $i \in N$, and let S, T be continuous self-mappings on X satisfying the following conditions:*

(i) *there exists $i_0 \in N$ such that $\{A_{i0}, S\}$ and $\{A_{i0}, T\}$ are D^*-compatible pairs such that $A_{i0}(X) \subset S(X) \cap T(X)$;*

(ii) *there exists a $\phi \in \Phi$ and $i, j, k \in N$ such that for all $x, y \in X$,*

$$D^*(A_i x, A_j y, A_k z) \leq \phi(D^*(Sx, Ty, Tz), D^*(Sx, A_i x, A_i x), D^*(Sx, A_j y, A_j y), D^*(Ty, A_i x, A_i x), D^*(Ty, A_j y, A_j y).$$

Then A_i, S, and T have a unique common fixed point in X for every $i \in N$.

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