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STUDY ON MEROMORPHIC HURWTIZ-ZETA FUNCTION DEFINED BY LINEAR OPERATOR

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Abstract. The aim of this paper is to define a new class of Hurwitz-Lerch-Zeta functions by itroducing two classes of meromorphic functions in terms of the Srivastava-Attiya operator. Coefficicient inequalities, growth and distortion inequalities, as well as radii of meromorphically starlikeness are obtained. In addition, some intresting properties depending on some integral representations are discussed.

1. Introduction

Let M denote the class of meromorphic functions $f(z)$ defined by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For $0 \le \beta < 1$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of M consisting of all meromorphic functions which are respectively, starlike of order β and convex of order β in U^*.

The functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n. \tag{1.2}$$

We denote the Hadamard product (or convolution) of f_1 and f_2 by

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$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n, \tag{1.3}$$

where

$$Z_0^{-1} = \{0, -1, -2, ...\}, U = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\partial U = \{z \in \mathbb{C} : |z| = 1\}.$$

Now, we recall a general Hurwitz-Lerch zeta function which, as many authors do, see for example ([6, 12 - 15]), we define by the following series.

$$\Phi(z, t, a) = \frac{1}{a^t} + \sum_{n=1}^{\infty} \frac{z^n}{(n+a)^s}, \tag{1.4}$$

$a \in \mathbb{C} \backslash Z_0^-$, $t \in \mathbb{C}$ when $z \in U = U^* \cup \{0\}$; $\mathbb{R}(t) > 1$ when $z \in \partial U$. Several interesting properties and characteristics of the Hurwitz-Lerch zeta function $\Phi(z, t, a)$ can be found in the recent investigations by Choi and Srivastava [2], Ferreira and Lopez [4], Garg et al. [7], Lin and Srivastava [8], Lin et al. [10], and others. Recent results on $\Phi(z, t, a)$, can be found in the expositions [22], [23].

In [5] (see also [18] and [19]) Ghanim defined

$$\begin{aligned} G_{t,a}(z) &= (1+a)^s \left[\Phi(z, t, a) - a^s + \frac{1}{z(1+a)^s} \right] \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1+a}{n+a} \right)^s z^n, \ (z \in U^*). \end{aligned} \tag{1.5}$$

Corresponding to the functions $G_{s,a}(z)$ and using the Hadamard product for $f(z) \in M$, we define a new linear operator

$$\begin{aligned} L_a^s(\alpha, \beta) f(z) &= \Phi(z, t, a) * G_{t,a}(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} C_a^s(n) a_n z^n, \quad (z \in U^*), \end{aligned} \tag{1.6}$$

for $\beta \neq 0, -1, -2, ...$ and $\alpha \in \mathbb{C} \backslash \{0\}$. Here $C_a^s(n) = \left| \left(\frac{1+a}{n+1+a} \right)^s \right|$ and unless indicated othewise, throughout this paper the parameter a is constrained to $a \in \mathbb{C} \backslash \{Z_0^-\}$, and s belongs $\mathbb{C}$. Meromorphic functions in terms of guessean and generalized hypergeometric functions were considered recently by many others (see for example [15-22] and the others therein).

It follows from (1.6) that

$$z \left(L_a^s(\alpha, \beta) f(z) \right)' = \alpha \left(L_a^s(\alpha+1, \beta) f(z) \right) - (\alpha+1) \left(L_a^s(\alpha, \beta) f(z) \right). \tag{1.7}$$

Now, for univalently meromorphic function $f(z) \in M$ the normalization

$$z^2 f(z)\mid_{z=o} = 0 \text{ and } zf(z)\mid_{z=o} = 1, \tag{1.8}$$

is classical. One can obtain interesting results by applying Montel's normalization [12] of the form

$$z^2 f(z)\mid_{z=o} = 0 \text{ and } zf(z)\mid_{z=\rho} = 1, \tag{1.9}$$

where ρ is a fixed point from the unit circle. Note that if $\rho = 0$ the normalization (1.9) is the classical normalization (1.8)

Meromorphic multivalent functions have been studied by Mogra[11], Raina and Ganigi[20], Uralegaddi and Somanatha [21], Aouf and Hossen [1], Srivastava et. al [16]. We define the following new subclass $M_a^s(\alpha, \beta)$ of meromorphic starlike function in the parabolic region of function M by making use of the generalized operator L_a^t with Montel's normalization. We study its characteristic properties: for example coefficient inequalities, growth and distortion inequalities, radii of starlikeness are obtained. And we also establish some new results concerning the convolution products.

For fixed parameters $\alpha \geq \frac{1}{2+\beta}$; $0 \leq \beta < 1$, denote the set $M_a^s(\alpha, \beta)$ consisting of those meromorphic function $f(z) \in M$ with two fixed points (or classical normalization) which satisfy

$$\left| \frac{z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha + \alpha\beta \right| \leq \mathbb{R}\left\{ \frac{-z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha - \alpha\beta \right\}, \tag{1.10}$$

where $L_a^t(\alpha, \beta)f(z)$ given by (1.6). In addition the text further, more let the subclass $M_a^s(\alpha, \beta)$ satisfying the condition (1.10) with Montel's (1.9) is denoted by $M_a^s(\alpha, \beta, \rho)$.

2. Main results

In this section we will discuss certain characterization properties for $f(z) \in M_a^s(\alpha, \beta)$.

Theorem 2.1. *Let $f \in M$. Then f is in the class $M_a^s(\alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} d_n(\alpha, \beta)\,|a_n| \leq (1 - \alpha\beta), \tag{2.1}$$

where

$$d_n(\alpha, \beta) = \left[(n - 1 + \alpha\beta)\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right] C_a^s(n) \tag{2.2}$$

and

$$\alpha > \frac{1}{2+\beta};\ 0 \le \beta < 1,\ n \in \mathbb{N}_0.$$

Proof. Let $f(z) \in M_a^s(\alpha,\beta)$. Then by the inequality (1.10), we have

$$\left|\frac{z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha + \alpha\beta\right| \le \mathbb{R}\left\{\frac{-z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha - \alpha\beta\right\},$$

that is,

$$\begin{aligned}\mathbb{R}\left\{\frac{z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha + \alpha\beta\right\} &\le \left|\frac{z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha + \alpha\beta\right| \\ &\le \mathbb{R}\left\{\frac{-z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha - \alpha\beta\right\}.\end{aligned}$$

Hence

$$\mathbb{R}\left\{\frac{z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha\beta\right\} \le 0.$$

Substituting for $L_a^s(\alpha,\beta)f(z)$ and $\left(L_a^s(\alpha,\beta)f(z)\right)'$, we get

$$\mathbb{R}\left\{\frac{\frac{\alpha}{z}+\alpha\sum_{n=1}^{\infty}\frac{(\alpha+1)_{n+1}}{(\beta)_{n+1}}C_a^s(n)a_nz^n-\frac{\alpha+1}{z}-(\alpha+1)\sum_{n=1}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}C_a^s(n)a_nz^n}{\frac{1}{z}+\sum_{n=1}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}C_a^s(n)a_nz^n} + \alpha\beta\right\}$$
$$\le 0.$$

Since $\mathbb{R}(z) \le |z|$, we have

$$\left|-(1-\alpha\beta)+\sum_{n=1}^{\infty}(n-1+\alpha\beta)\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}C_a^s(n)a_nz^{n+1}\right| \le 0$$

and by letting $|z| \longrightarrow 1^-$, we get

$$\sum_{n=1}^{\infty}(n-1+\alpha\beta)\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}C_a^s(n)\left|a_n\right| \le (1-\alpha\beta).$$

Now conversely, we assume that the inequality holds. Then, if we let $z \in \partial U$, we find the following from (1.1) and (2.1),

$$\mathbb{R}\left\{\frac{z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha\beta\right\} \le 0$$

or

$$\mathbb{R}\left\{\frac{\frac{\alpha}{z}+\alpha\sum_{n=1}^{\infty}\frac{(\alpha+1)_{n+1}}{(\beta)_{n+1}}C_a^s(n)a_nz^n-\frac{\alpha+1}{z}-(\alpha+1)\sum_{n=1}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}C_a^s(n)a_nz^n}{\frac{1}{z}+\sum_{n=1}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}C_a^s(n)a_nz^n}+\alpha\beta\right\}$$
$$\leq 0.$$

Since $\mathbb{R}(z) \leq |z|$, we have

$$\sum_{n=1}^{\infty}\frac{(n-1+\alpha\beta)\,\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}C_a^s(n)\,|a_n|}{(1-\alpha\beta)}\leq 1.$$

This completes the proof. □

Theorem 2.2. *If* $f\in M_a^s(\alpha,\beta)$, *then*

$$\left(1-\frac{(1-\alpha\beta)r}{d_1}\right)r^{-1}\leq|f(z)|\leq\left(1+\frac{(1-\alpha\beta)r}{d_1}\right)r^{-1},\;\;(0<|z|=r<1)\,.$$

Proof. Using classical normalization (that is by taking $\rho = 0$ in Theorem 2.2) it is very simple to prove the theorem . □

Theorem 2.3. *Let the function* $f(z)$ *defined by equation* (1.1) *in the class* $M_a^s(\alpha,\beta)$. *Then* $f(z)$ *is meramorphically valent starlike of order* μ $(0\leq\mu<1)$ *in the disk* $|z|<r$; $(0\leq\mu<1)$, *that is,*

$$\mathbb{R}\left(-\frac{zf'(z)}{f(z)}\right)>\mu,$$

where

$$r=\left(\frac{d_n(1-\mu)}{(n+\mu)\left((1-\alpha\beta)\right)}\right)^{\frac{1}{n+1}}.$$

Proof. From equation (1.1) we have

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,$$

and we can easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{\frac{zf'(z)}{f(z)}-1+2\mu}\right| \leq \frac{\sum\limits_{n=1}^{\infty}(n+1)a_n\,|z|^{n+1}}{-2(1-\mu)+\sum\limits_{n=1}^{\infty}(n-1+2\mu)a_n\,|z|^{n+1}}.$$

Thus, the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{\frac{zf'(z)}{f(z)}-1+2\mu}\right| \leq 1, \quad \text{if} \quad \sum_{n=1}^{\infty}\frac{(n+\mu)}{1-\mu}a_n\ \ |z|^{n+1} \leq 1. \tag{2.3}$$

Since $f \in M_a^s(\alpha,\beta)$ from Theorem 2.1, we have

$$\sum_{n=1}^{\infty}\frac{d_n\,|a_n|}{(1-\alpha\beta)} \leq 1. \tag{2.4}$$

Then from (2.3) and (2.4), we get

$$\frac{n+\mu}{1-\mu}\,|z|^{n+1} \leq \frac{d_n}{1-\alpha\beta},$$

and then

$$|z|^{n+1} \leq \frac{d_n\,(1-\mu)}{(1-\alpha\beta)\,(n+\mu)},$$

from which we conclude

$$|z| \leq \left(\frac{d_n\,(1-\mu)}{(1-\alpha\beta)\,(n+\mu)}\right)^{\frac{1}{n+1}}.$$

This completes the proof. □

3. Convolution properties

For the functions

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty}|a_{n,j}|\,z^n, (j=1,2), \tag{3.1}$$

we denote by $(f_1 * f_2)(z)$ the Hadamard product or (convolution) of the functions $f_1(z)$ and $f_2(z)$, that is

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty}|a_{n,1}|\,|a_{n,2}|\,z^n. \tag{3.2}$$

Theorem 3.1. *Let the function* $f_j(z)$, $(j = 1, 2)$ *defined by* (3.1) *be in the class* $M_a^s(\alpha, \beta)$. *Then it follows that* $(f_1 * f_2)(z) \in M_a^s(\alpha, \delta)$ *with*

$$\delta \leq \left(\frac{d_1^2}{(1-\alpha\beta)^2 C_a^s(1) + d_1^2} \right),$$

where $d_n(\alpha, \beta) = \left[(n-1+\alpha\beta) \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right] C_a^s(n)$, $C_a^s(n) = \left(\frac{1+a}{n+1+a} \right)^s$.

Proof. Let $f_1(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} |a_{n,1}| z^n$ and $f_2(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} |a_{n,2}| z^n$ be in the class $M_a^s(\alpha, \beta)$. Then, by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{d_n |a_{n,1}|}{(1-\alpha\beta)} \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{d_n |a_{n,2}|}{(1-\alpha\beta)} \leq 1.$$

Employing the technique used earlier by many authors, we need to find smallest δ such that

$$\sum_{n=1}^{\infty} \frac{(n-1+\alpha\delta)\, C_a^s(n)}{(1-\alpha\delta)} |a_{n,1}| \, |a_{n,2}| \leq 1, \tag{3.3}$$

where $C_a^s(n) = \left(\frac{1+a}{n+1+a} \right)^s$.

By Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{d_n}{(1-\alpha\beta)} \sqrt{|a_{n,1}| \, |a_{n,2}|} \leq 1, \tag{3.4}$$

where $d_n(\alpha, \beta) = \left[(n-1+\alpha\beta) \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right] C_a^s(n)$, and then

$$\frac{(n-1+\alpha\delta)\, C_a^s(n)}{(1-\alpha\delta)} |a_{n,1}| \, |a_{n,2}| \leq \frac{d_n}{(1-\alpha\beta)} \sqrt{|a_{n,1}| \, |a_{n,2}|}.$$

It implies that

$$\sqrt{|a_{n,1}| \, |a_{n,2}|} \leq \frac{d_n \,(1-\alpha\delta)}{(n-1+\alpha\delta)\, C_a^s(n)(1-\alpha\beta)}. \tag{3.5}$$

We know that

$$\sqrt{|a_{n,1}| \, |a_{n,2}|} \leq \frac{(1-\alpha\beta)}{d_n}. \tag{3.6}$$

Now from (3.5) and (3.6), we see that it suffices to choose $\beta > 0$ in such a way that

$$\frac{(1-\alpha\beta)}{d_n} \leq \frac{d_n\,(1-\alpha\delta)}{(n-1+\alpha\delta)\,C_a^s(n)(1-\alpha\beta)},$$

it follows from this inequality hat

$$\delta = \frac{1}{\alpha}\left(1 - \frac{n(1-\alpha\beta)^2 C_a^s(n)}{(1-\alpha\beta)^2 C_a^s(n) + d_n^2}\right).$$

Now define a function $\Psi(n)$ by

$$\Psi(n) = \frac{1}{\alpha}\left(1 - \frac{n(1-\alpha\beta)^2 C_a^s(n)}{(1-\alpha\beta)^2 C_a^s(n) + d_n^2}\right),\ n \geq 1.$$

We observe that $\Psi(n)$ is an increasing function of n, we thus conclude that

$$\delta = \Psi(1) = \frac{1}{\alpha}\left(1 - \frac{(1-\alpha\beta)^2 C_a^s(1)}{(1-\alpha\beta)^2 C_a^s(1) + d_1^2}\right). \tag{3.7}$$

Since $d_n(\alpha,\beta) = \left[(n-1+\alpha\beta)\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right] C_a^s(n)$, then $d_1 = \left[\alpha\beta\frac{(\alpha)_2}{(\beta)_2}\right] C_a^s(1)$ and $C_a^s(1) = \left(\frac{1+a}{2+a}\right)^s$. Substituting in equation (3.7) and simplifing we get

$$\delta \leq \left(\frac{d_1^2}{(1-\alpha\beta)^2 C_a^s(1) + d_1^2}\right).$$

This complete the proof. □

Theorem 3.2. *If $f_1(z) \in M_a^s(\alpha,\beta)$ and $f_2(z) \in M_a^s(\alpha,\gamma)$ then $(f_1 * f_2)(z) \in M_a^s(\alpha,\eta)$ with*

$$\eta \leq \frac{1}{\alpha}\left(\frac{d_1(\alpha,\beta)d_1(\alpha,\gamma)}{d_1(\alpha,\beta)d_1(\alpha,\gamma)C_a^s(1) + (1-\alpha\beta)(1-\alpha\gamma)}\right),$$

where $d_1(\alpha,\beta) = \alpha\beta\frac{(\alpha)_2}{(\beta)_2}C_a^s(1)$ and $d_1(\alpha,\gamma) = \alpha\gamma\frac{(\alpha)_2}{(\gamma)_2}C_a^s(1)$.

Proof. Since $f_1(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_{n,1}|\, z^n \in M_a^s(\alpha,\beta)$ and $f_2(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_{n,2}|\, z^n \in M_a^s(\alpha,\gamma)$, from Theorem 2.1 we have

$$\sum_{n=1}^{\infty} \frac{d_n(\alpha,\beta)C_a^s(n)}{(1-\alpha\beta)}\,|a_{n,1}| \leq 1$$

and

$$\sum_{n=1}^{\infty} \frac{d_n(\alpha,\gamma)C_a^s(n)}{(1-\alpha\gamma)}\,|a_{n,2}| \leq 1,$$

where

$$d_n(\alpha,\beta) = \left[(n-1+\alpha\beta)\frac{(\alpha)_{n+1}}{(\beta)\,n+1}\right] C_a^s(n) \tag{3.8}$$

and

$$d_n(\alpha,\delta) = \left[(n-1+\alpha\gamma)\frac{(\alpha)_{n+1}}{(\gamma)\,n+1}\right] C_a^s(n). \tag{3.9}$$

Morover, we have $(f_1 * f_2)(z) \in M_a^s(\alpha,\eta)$ then, from Theorem 2. 1, we have

$$\sum_{n=1}^{\infty} \frac{d_n(\alpha,\eta)C_a^s(n)}{(1-\eta\beta)} |a_{n,1}|\,|a_{n,2}| \leq 1, \tag{3.10}$$

where

$$d_n(\alpha,\eta) = \left[(n-1+\alpha\eta)\frac{(\alpha)_{n+1}}{(\eta)_{n+1}}\right] C_a^s(n). \tag{3.11}$$

Now, using the Caucy-Schwarz inequality, we get

$$\sum_{n=1}^{\infty} \frac{C_a^s(n)\sqrt{d_n(\alpha,\beta)d_n(\alpha,\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}} \sqrt{|a_{n,1}|\,|a_{n,2}|} \leq 1. \tag{3.12}$$

From equations (3.10) and (3.12), we get

$$\frac{d_n(\alpha,\eta)C_a^s(n)}{(1-\eta\beta)} |a_{n,1}|\,|a_{n,2}| \leq \frac{C_a^s(n)\sqrt{d_n(\alpha,\beta)d_n(\alpha,\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}} \sqrt{|a_{n,1}|\,|a_{n,2}|},$$

from which we get

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{(1-\eta\beta)C_a^s(n)\sqrt{d_n(\alpha,\beta)d_n(\alpha,\gamma)}}{d_n(\alpha,\eta)C_a^s(n)\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}. \tag{3.13}$$

But from (3.12) we have

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}{C_a^s(n)\sqrt{d_n(\alpha,\beta)d_n(\alpha,\gamma)}}. \tag{3.14}$$

Therefore from equations (3.13) and (3.14) we will have

$$\frac{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}{C_a^s(n)\sqrt{d_n(\alpha,\beta)d_n(\alpha,\gamma)}} \leq \frac{(1-\eta\beta)C_a^s(n)\sqrt{d_n(\alpha,\beta)d_n(\alpha,\gamma)}}{d_n(\alpha,\eta)C_a^s(n)\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}.$$

Solving this inquality for η, we get

$$\eta \leq \frac{1}{\alpha}\left\{1 - \frac{n(1-\alpha\beta)(1-\alpha\gamma)}{d_n(\alpha,\beta)d_n(\alpha,\gamma)C_a^s(n) + (1-\alpha\beta)(1-\alpha\gamma)}\right\}.$$

Define the function $\Psi(n)$ by

$$\Psi(n)=\frac{1}{\alpha}\left\{1-\frac{n(1-\alpha\beta)(1-\alpha\gamma)}{d_n(\alpha,\beta)d_n(\alpha,\gamma)C_a^s(n)+(1-\alpha\beta)(1-\alpha\gamma)}\right\}, \tag{3.15}$$

then it is clear that $\Psi(n)$ is an inceasing function of n. Hence we have

$$\Psi(1)=\frac{1}{\alpha}\left\{1-\frac{(1-\alpha\beta)(1-\alpha\gamma)}{d_1(\alpha,\beta)d_1(\alpha,\gamma)C_a^s(1)+(1-\alpha\beta)(1-\alpha\gamma)}\right\},$$

and then, for $n=1$, we have $C_a^s(1)=\left(\frac{1+a}{2+a}\right)^s$. Substituting in (3.15) and simplifing we get

$$\eta\leq\frac{1}{\alpha}\left(\frac{d_1(\alpha,\beta)d_1(\alpha,\gamma)}{d_1(\alpha,\beta)d_1(\alpha,\gamma)C_a^s(1)+(1-\alpha\beta)(1-\alpha\gamma)}\right).$$

This complet the proof. □

Theorem 3.3. *If the function $f_j(z)(j=1,2)$ defined by*

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}|a_{n,j}|\,z^n$$

is in the class $M_a^s(\alpha,\beta)$, then the function $h(z)$ defined by

$$h(z)=\frac{2}{z}+\sum_{n=1}^{\infty}\left(|a_{n,1}|^2+|a_{n,2}|^2\right)z^n \tag{3.16}$$

belonges to the class $M_a^s(\alpha,\gamma)$ with

$$\gamma\leq\frac{1}{\alpha}\left\{\frac{C_1^2+\rho}{C_1^2+2\left(1-\alpha\beta\right)^2\left(C_a^s(1)-\rho\right)}\right\},$$

where $C_a^s(1)=\left(\frac{1+a}{2+a}\right)^s$ and $C_1=\alpha\beta C_a^s(1)+(1-\alpha\beta)\,\rho$.

Proof. Noting that

$$\sum_{n=1}^{\infty}\left[\frac{C_n}{(1-\alpha\beta)}\right]^2|a_{n,j}|^2\leq\sum_{n=1}^{\infty}\left[\frac{C_n}{(1-\alpha\beta)}\,|a_{n,j}|\right]^2\leq 1, \tag{3.17}$$

where

$$C_n=\left[n-(1-\alpha\beta)\right]C_a^s(n)+(1-\alpha\beta)\,\rho^n.$$

Since $f_j(z)\epsilon M_a^s(\alpha,\beta),(j=1,2)$, we have

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{C_n}{(1-\alpha\beta)}\right]^2\left(|a_{n,1}|^2+|a_{n,2}|^2\right)\le 1. \tag{3.18}$$

Now we have to find largest γ such that

$$\sum_{n=1}^{\infty}\left[\frac{[n-(1-\alpha\gamma)]\,C_a^s(n)+(1-\alpha\gamma)}{(1-\alpha\gamma)}\right]\left(|a_{n,1}|^2+|a_{n,2}|^2\right)\le 1. \tag{3.19}$$

From equations (3.18) and (3.19) we get

$$\left[\frac{[n-(1-\alpha\gamma)]\,C_a^s(n)+(1-\alpha\gamma)}{(1-\alpha\gamma)}\right]\le\frac{1}{2}\left[\frac{C_n}{(1-\alpha\beta)}\right]^2,(n\ge 1).$$

Solving this inequality for γ and simplifing we get

$$\gamma\le\frac{1}{\alpha}\left\{\frac{C_n^2-2(n-1)(1-\alpha\beta)^2C_a^s(n)+\rho^n}{C_n^2+2\,(1-\alpha\beta)^2\,(C_a^s(n)-\rho^n)}\right\},(n\ge 1).$$

Define a function $\Psi(n)$ by

$$\Psi(n)=\frac{1}{\alpha}\left\{\frac{C_n^2-2(n-1)(1-\alpha\beta)^2C_a^s(n)+\rho^n}{C_n^2+2\,(1-\alpha\beta)^2\,(C_a^s(n)-\rho^n)}\right\},(n\ge 1),$$

then we know that $\Psi(n)$ is an inceasing function of n and for $n=1$, we have

$$\Psi(1)=\frac{1}{\alpha}\left\{\frac{C_1^2+\rho}{C_1^2+2\,(1-\alpha\beta)^2\,(C_a^s(1)-\rho)}\right\}. \tag{3.20}$$

We conlude that

$$\gamma\le\frac{1}{\alpha}\left\{\frac{C_1^2+\rho}{C_1^2+2\,(1-\alpha\beta)^2\,(C_a^s(1)-\rho)}\right\}.$$

This complets the proof. □

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