Nonlinear Functional Analysis and Applications
Vol. 24, No. 2 (2019), pp. 207-220
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

SYSTEM OF HIERARCHICAL NONLINEAR MIXED VARIATIONAL INEQUALITIES

Jong Kyu Kim[1] and Salahuddin[2]

[1]Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam 51767, Korea
e-mail: `jongkyuk@kyungnam.ac.kr`

[2]Department of Mathematics, Jazan University
Jazan, Kingdom of Saudi Arabia
e-mail: `drsalah12@hotmail.com`

Abstract. In this paper, we established the existence and convergence solution of a system of hierarchical nonlinear mixed variational inequalities in Hilbert spaces.

1. Introduction

The theory of variational inequality is well known and well developed because of its application in different areas of science, social science, engineering and management. The variational inequality problem provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity problems, optimal control theory, mathematical economics, equilibria and game theory, etc. It is well known that the variational inequality theory has emerged as an important tools in studying a wide class of obstacle, unilateral and equilibrium problem that areas in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems.

Hierarchical optimization was first defined by Bracken and McGill [3, 4] as a generalization of mathematical programming.

[0]Received October 18, 2018. Revised December 15, 2018.
[0]2010 Mathematics Subject Classification: 49J40, 47H06.
[0]Keywords: System of hierarchical nonlinear mixed variational inequalities, existence and convergence theorems, r-strongly monotone mappings, Hilbert spaces.
[0]Corresponding author: J. K. Kim(`jongkyuk@kyungnam.ac.kr`).

Inspired and motivated by the recent works [1, 2, 6, 8, 9, 10, 11, 17, 18], we established the existence and convergence solution of a system of hierarchical nonlinear mixed variational inequalities by using the Mainge′s schemes.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $\mathcal{D}$ be a nonempty closed convex subsets of $\mathcal{H}$, and a mapping $T : \mathcal{H} \to \mathcal{H}$ is called nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in \mathcal{H}.$$

We denotes $F(T)$ by the set of fixed points of T, that is,

$$F(T) = \{x \in \mathcal{H} : Tx = x\}.$$

A variational inequality problem is the problem of finding a point $x \in \mathcal{D}$ such that

$$\langle Ax, y - x \rangle \geq 0, \forall y \in \mathcal{D}, \tag{2.1}$$

where $A : \mathcal{D} \to \mathcal{D}$ is a nonlinear mapping and solution set of (2.1) is denoted by Ω.

The hierarchical fixed point problem [13, 14, 15, 19, 20] is the problem of finding a point $x^* \in F(T)$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \ \forall x \in F(T). \tag{2.2}$$

When the set $F(T)$ is replaced by the solution set of variational inequality (2.1), then (2.2) is known as a hierarchical variational inequality problem.

In this paper, we consider the following system of hierarchical nonlinear mixed variational inequalities for finding $x_i^* \in \Omega_i$ such that for given positive real number η_i $(i = 1, 2, \cdots, N)$, the following inequalities hold:

$$\begin{cases} \langle \eta_1 G(x_2^*) + x_1^* - x_2^*, \ x_1 - x_1^* \rangle \geq 0, \ \forall \ x_1 \in \Omega_1, \\ \langle \eta_2 G(x_3^*) + x_2^* - x_3^*, \ x_2 - x_2^* \rangle \geq 0, \ \forall \ x_2 \in \Omega_2, \\ \quad \vdots \\ \langle \eta_{N-1} G(x_N^*) + x_{N-1}^* - x_N^*, \ x_{N-1} - x_{N-1}^* \rangle \geq 0, \ \forall \ x_{N-1} \in \Omega_{N-1}, \\ \langle \eta_N G(x_1^*) + x_N^* - x_1^*, \ x_N - x_N^* \rangle \geq 0, \ \forall \ x_N \in \Omega_N. \end{cases} \tag{2.3}$$

Definition 2.1. Let $T, G : \mathcal{H} \to \mathcal{H}$ be the single-valued mappings. Then

(1) T is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in \mathcal{H}.$$

(2) T is said to be quasi nonexpansive, if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \ \forall x \in \mathcal{H}, p \in F(T).$$

It should be noted that T is said to be quasi nonexpansive if and only if for all $x \in \mathcal{H}, p \in F(T)$

$$\langle x - Tx, \ x - p \rangle \geq \frac{1}{2}\|x - Tx\|^2.$$

(3) T is said to be strongly quasi nonexpansive, if T is quasi nonexpansive and

$$x_n - Tx_n \to 0$$

whenever $\{x_n\}$ is a bounded sequence in $\mathcal{H}$ and for some $p \in F(T)$,

$$\|x_n - p\| - \|Tx_n - p\| \to 0.$$

(4) G is said to be μ-Lipschitzian, if there exists $\mu > 0$ such that

$$\|G(x) - G(y)\| \leq \mu\|x - y\|, \ \forall x, y \in \mathcal{H}.$$

(5) G is said to be r-strongly monotone, if there exists $r > 0$ such that

$$\langle G(x) - G(y), x - y \rangle \geq r\|x - y\|^2, \ \forall x, y \in \mathcal{H}.$$

(6) G is said to be α-inverse strongly monotone, if there exists $\alpha > 0$ such that

$$\langle G(x) - G(y), x - y \rangle \geq \alpha\|G(x) - G(y)\|^2, \ \forall x, y \in \mathcal{H}.$$

Lemma 2.2. ([21]) *Let $A : \mathcal{H} \to \mathcal{H}$ be an α-inverse strongly monotone mapping. Then*

(1) *A is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping;*
(2) *$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2$ for $\lambda > 0$;*
(3) *for $\lambda \in (0, 2\alpha]$, $I - \lambda A$ is a nonexpansive mapping where I is the identity mapping on $\mathcal{H}$.*

Lemma 2.3. *Let $x \in \mathcal{H}$ and $z \in \mathcal{D}$ be any points and $P_{\mathcal{D}}$ be the metric projection of $\mathcal{H}$ onto $\mathcal{D}$. Then*

(1) *$z = P_{\mathcal{D}}(x)$ if and only if $\langle x - z, \ y - z \rangle \geq 0, \ \forall y \in \mathcal{D}$;*
(2) *$z = P_{\mathcal{D}}[x]$ if and only if $\|x - z\|^2 \geq \|x - y\|^2 - \|y - z\|^2, \ \forall y \in \mathcal{D}$;*
(3) *$\langle P_{\mathcal{D}}(x) - P_{\mathcal{D}}(y), \ x - y \rangle \geq \|P_{\mathcal{D}}(x) - P_{\mathcal{D}}(y)\|^2, \ \forall x, y \in \mathcal{H}$;*
(4) *$\|P_{\mathcal{D}}(x) - P_{\mathcal{D}}(y)\| \leq \|x - y\|^2 - \|(x - P_{\mathcal{D}}(x)) - (y - P_{\mathcal{D}}(y))\|^2$.*

Lemma 2.4. ([16]) *For $x, y \in \mathcal{H}$ and $\omega \in (0, 1)$, the following statements hold:*

(1) $\big|\langle x, y \rangle\big| \leq \|x\| \ \|y\|$;

(2) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;
(3) $\|(1-\omega)x+\omega y\|^2 = (1-\omega)\|x\|^2 + \omega\|y\|^2 - \omega(1-\omega)\|x-y\|^2$.

Lemma 2.5. ([12]) *Let $\{a_n\}$ be a sequence of real numbers and there exists a subsequence $\{a_{m_j}\}$ of $\{a_n\}$ such that $a_{m_j} < a_{m_j+1}$ for all $j \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers. Then there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that*

$$\lim_{k\to\infty} n_k = \infty$$

and the following properties are satisfied for all (sufficiently large) number $k \in \mathbb{N}$,

$$a_{n_k} \le a_{n_k+1}, \ a_k \le a_{n_k+1}. \tag{2.4}$$

In fact n_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that

$$a_n < a_{n+1}.$$

Lemma 2.6. ([7]) *Let $\{a_n\} \subset [0,\infty)$, $\{\alpha_n\} \subset [0,1), \{b_n\} \subset (-\infty,+\infty)$ and $\tau \in [0,1]$ be such that*

(1) *$\{a_n\}$ is a bounded sequence;*
(2) $a_{n+1} \le (1-\alpha_n)^2 a_n + 2\alpha_n\tau\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n, \ \forall n \ge 1$;
(3) *whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying*

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0, \tag{2.5}$$

it follows that

$$\limsup_{k\to\infty} b_{n_k} \le 0;$$

(4) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. ([5]) *Let $A : \mathcal{H} \to \mathcal{H}$ be an α-inverse strongly monotone mapping and $\Omega \neq \emptyset$ be a solution set of (2.1). Then the following statements hold:*

(1) *If $\lambda \in (0, 2\alpha]$, then the mapping $\mathcal{K} : \mathcal{H} \to \mathcal{D}$ defined by*

$$\mathcal{K} = P_{\mathcal{D}}(I - \lambda A)$$

is quasi nonexpansive, where I is an identity mapping;

(2) *The mapping $I - \mathcal{K} : \mathcal{H} \to \mathcal{H}$ is demiclosed at zero, that is, for any sequence $\{x_n\} \subset \mathcal{H}$ if*

$$x_n \rightharpoonup x \text{ and } (I-\mathcal{K})x_n \to 0$$

then $x = \mathcal{K}x$;

(3) *The mapping* $\mathcal{K}_\beta$ *defined by*

$$\mathcal{K}_\beta = (I-\beta)I + \beta\mathcal{K}, \quad \text{for } \beta \in (0,1)$$

is strongly quasi nonexpansive mapping and $F(\mathcal{K}_\beta) = F(\mathcal{K})$;

(4) $I - \mathcal{K}_\beta,\ \beta \in (0,1)$ *is demiclosed at zero.*

3. Existence and convergence analysis

Throughout this section, we always assume that the following conditions are satisfied:

(C1) $A_i : \mathcal{H} \to \mathcal{H}$ is an α_i-inverse strongly monotone mapping and Ω_i is the solution set of (2.1) with $A = A_i$ and $\Omega_i \neq \emptyset (i = 1, 2, \cdots, N)$.

(C2) $\mathcal{K}_i$ and $\mathcal{K}_{i,\beta}, \beta \in (0,1),\ (i = 1, 2, \cdots, N)$ are the mappings defined by

$$\begin{cases} \mathcal{K}_i = P_{\mathcal{D}}(I - \lambda A_i),\ \lambda \in (0, 2\alpha_i], \\ \mathcal{K}_{i,\beta} = (1-\beta)I + \beta\mathcal{K}_i,\ \beta \in (0,1). \end{cases} \tag{3.1}$$

Theorem 3.1. *Assume that* $A_i, \Omega_i, \mathcal{K}_i$ *and* $\mathcal{K}_{i,\beta}$ *are satisfying the conditions* $(C1)$-$(C2)$ *and* $g_i : \mathcal{H} \to \mathcal{H}$ *be contraction with a contractive constant* $\tau_i \in (0,1)(i = 1, 2, \cdots, N)$. *Then there exists a unique element* $(x_1^*, x_2^*, \cdots, x_N^*) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ *such that for each* $x_i \in \Omega_i$, *the following inequalities are satisfied:*

$$\begin{cases} \langle x_1^* - g_1(x_2^*),\ x_1 - x_1^* \rangle \geq 0,\ \forall x_1 \in \Omega_1, \\ \langle x_2^* - g_2(x_3^*),\ x_2 - x_2^* \rangle \geq 0,\ \forall x_2 \in \Omega_2, \\ \qquad \vdots \\ \langle x_{N-1}^* - g_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^* \rangle \geq 0,\ \forall x_{N-1} \in \Omega_{N-1}, \\ \langle x_N^* - g_N(x_1^*),\ x_N - x_N^* \rangle \geq 0,\ \forall x_N \in \Omega_N. \end{cases} \tag{3.2}$$

Proof. Given that $\Omega_1, \cdots, \Omega_N$ are nonempty closed and convex. Therefore, the metric projection P_{Ω_i} is well defined for each $i = 1, 2, \cdots, N$. Since g_i is a contraction mapping for each $i = 1, 2, \cdots, N$, $P_{\Omega_i} g_i$ is a contraction mapping for each $i = 1, 2, \cdots, N$. Therefore

$$P_{\Omega_1} g_1 \circ P_{\Omega_2} g_2 \circ \cdots \circ P_{\Omega_N} g_N \tag{3.3}$$

is also a contraction. Hence there exists a unique element $x^* \in \mathcal{H}$ such that

$$x^* = (P_{\Omega_1} g_1 \circ P_{\Omega_2} g_2 \circ \cdots \circ P_{\Omega_N} g_N) x^*. \tag{3.4}$$

Putting $x_N^* = P_{\Omega_N} g_N(x_1^*), \cdots,\ x_2^* = P_{\Omega_2} g_2(x_3^*),\ x_1^* = P_{\Omega_1} g_1(x_2^*),\ x_N^* \in \Omega_N, \cdots, x_1^* \in \Omega_1$.

Suppose that there is an element $(\bar{x}_1, \cdots, \bar{x}_N) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ such that for all $x_i \in \Omega_i$ the following inequalities are satisfied:

$$\begin{cases} \langle \bar{x}_1 - g_1(\bar{x}_2),\ x_1 - \bar{x}_1 \rangle \geq 0,\ \forall x_1 \in \Omega_1, \\ \langle \bar{x}_2 - g_2(\bar{x}_3),\ x_2 - \bar{x}_2 \rangle \geq 0,\ \forall x_2 \in \Omega_2, \\ \quad \vdots \\ \langle \bar{x}_{N-1} - g_{N-1}(\bar{x}_N),\ x_{N-1} - \bar{x}_{N-1} \rangle \geq 0,\ \forall x_{N-1} \in \Omega_{N-1}, \\ \langle \bar{x}_N - g_N(\bar{x}_1),\ x_N - \bar{x}_N \rangle \geq 0,\ \forall x_N \in \Omega_N. \end{cases} \tag{3.5}$$

Then, we have

$$\begin{cases} \bar{x}_1 = P_{\Omega_1} g_1(\bar{x}_2), \\ \bar{x}_2 = P_{\Omega_2} g_2(\bar{x}_3), \\ \quad \vdots \\ \bar{x}_{N-1} = P_{\Omega_{N-1}} g_{N-1}(\bar{x}_N). \\ \bar{x}_N = P_{\Omega_N} g_N(\bar{x}_1). \end{cases} \tag{3.6}$$

Therefore, we have

$$\bar{x}_1 = (P_{\Omega_1} g_1 \circ P_{\Omega_2} g_2 \circ \cdots \circ P_{\Omega_N} g_N)\bar{x_1}. \tag{3.7}$$

This implies that $\bar{x}_1 = x_1^*, \bar{x}_2 = x_2^*, \cdots, \bar{x}_N = x_N^*$. The proof is completed. □

Theorem 3.2. *Let $A_i, \Omega_i, \mathcal{K}_i$ and $\mathcal{K}_{i,\beta}$ satisfying the conditions $(C1)$-$(C2)$ and $g_i : \mathcal{H} \to \mathcal{H}$ be a contraction with a contractive constant $\tau_i \in (0,1), (i = 1, 2, \cdots, N)$. Let $\{x_i^n\}$ be the sequence defined by $x_i^0 \in \mathcal{H}$ and*

$$\begin{cases} x_1^{n+1} = (1-\alpha_n)\mathcal{K}_{1,\beta} x_1^n + \alpha_n g_1(\mathcal{K}_{2,\beta} x_2^n), \\ x_2^{n+1} = (1-\alpha_n)\mathcal{K}_{2,\beta} x_2^n + \alpha_n g_2(\mathcal{K}_{3,\beta} x_3^n), \\ \quad \vdots \\ x_N^{n+1} = (1-\alpha_n)\mathcal{K}_{N,\beta} x_N^n + \alpha_n g_N(\mathcal{K}_{1,\beta} x_1^n), \end{cases} \tag{3.8}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_i^n\}$ generated by (3.8) converges to x_i^* for each $i = 1, 2, \cdots, N$, where $(x_1^*, \cdots, x_N^*)$ is the unique element in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ verifying (3.2).

Proof. (i) We first prove that the sequences $\{x_1^n\}, \cdots, \{x_N^n\}$ are bounded. It follows from Lemma 2.7 that $\mathcal{K}_{i,\beta}$ is strongly quasi nonexpansive and $F(\mathcal{K}_{i,\beta}) = F(\mathcal{K}_i) = \Omega_i$ $(i = 1, \cdots, N)$. Since g_i is a contraction with coefficient τ_i $(i =$

$1, \cdots, N$) and $x_1^* \in F(\mathcal{K}_{1,\beta})$, $x_2^* \in F(\mathcal{K}_{2,\beta})$, $\cdots$, $x_N^* \in F(\mathcal{K}_{N,\beta})$, it follows that

$$\begin{aligned} \|x_1^{n+1} - x_1^*\| &\leq (1-\alpha_n)\|\mathcal{K}_{1,\beta}x_1^n - x_1^*\| + \alpha_n\|g_1(\mathcal{K}_{2,\beta}x_2^n) - x_1^*\| \\ &\leq (1-\alpha_n)\|x_1^n - x_1^*\| + \alpha_n\|g_1(\mathcal{K}_{2,\beta}x_2^n) - g_1(x_2^*)\| \\ &\quad +\alpha_n\|g_1(x_2^*) - x_1^*\| \\ &\leq (1-\alpha_n)\|x_1^n - x_1^*\| + \alpha_n\tau_1\|\mathcal{K}_{2,\beta}x_2^n - x_2^*\| \\ &\quad +\alpha_n\|g_1(x_2^*) - x_1^*\| \\ &\leq (1-\alpha_n)\|x_1^n - x_1^*\| + \alpha_n\tau_1\|x_2^n - x_2^*\| + \alpha_n\|g_1(x_2^*) - x_1^*\|. \end{aligned} \tag{3.9}$$

Similarly, we can also compute that

$$\begin{aligned} &\|x_2^{n+1} - x_2^*\| \leq (1-\alpha_n)\|x_2^n - x_2^*\| + \alpha_n\tau_2\|x_3^n - x_3^*\| + \alpha_n\|g_2(x_3^*) - x_2^*\|; \\ &\qquad\vdots \\ &\|x_N^{n+1} - x_N^*\| \leq (1-\alpha_n)\|x_N^n - x_N^*\| + \alpha_n\tau_N\|x_1^n - x_1^*\| + \alpha_n\|g_N(x_1^*) - x_N^*\|. \end{aligned} \tag{3.10}$$

This implies that

$$\begin{aligned} &\|x_1^{n+1} - x_1^*\| + \|x_2^{n+1} - x_2^*\| + \cdots + \|x_N^{n+1} - x_N^*\| \\ &\leq (1-\alpha_n)[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|] \\ &\quad + \alpha_n[\tau_1\|x_1^n - x_1^*\| + \cdots + \tau_N\|x_N^n - x_N^*\|] + \alpha_n[\|g_1(x_2^*) - x_1^*\| \\ &\quad + \|g_2(x_3^*) - x_2^*\| + \cdots + \|g_N(x_1^*) - x_N^*\|] \\ &\leq (1-\alpha_n)[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|] + \alpha_n\tau[\|x_1^n - x_1^*\| \\ &\quad + \cdots + \|x_N^n - x_N^*\|] \\ &\quad + \alpha_n[\|g_1(x_2^*) - x_1^*\| + \|g_2(x_3^*) - x_2^*\| + \cdots + \|g_N(x_1^*) - x_N^*\|] \\ &\leq (1-\alpha_n(1-\tau))[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|] \\ &\quad + \alpha_n(1-\tau)\frac{\|g_1(x_2^*) - x_1^*\| + \|g_2(x_3^*) - x_2^*\| + \cdots + \|g_N(x_1^*) - x_N^*\|}{1-\tau} \\ &\leq \max\left\{\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|, \frac{\|g_1(x_2^*) - x_1^*\| + \cdots + \|g_N(x_1^*) - x_N^*\|}{1-\tau}\right\}, \end{aligned} \tag{3.11}$$

where $\tau = \max\{\tau_1, \tau_2, \cdots, \tau_N\}$. By induction, we have

$$\begin{aligned} &\|x_1^{n+1} - x_1^*\| + \|x_2^{n+1} - x_2^*\| + \cdots + \|x_N^{n+1} - x_N^*\| \\ &\leq \max\left\{\|x_1^0 - x_1^*\| + \cdots + \|x_N^0 - x_N^*\|, \frac{\|g_1(x_2^*) - x_1^*\| + \cdots + \|g_N(x_1^*) - x_N^*\|}{1-\tau}\right\}, \end{aligned} \tag{3.12}$$

for all $n \geq 1$. Hence $\{x_1^n\}, \cdots, \{x_N^n\}$ are bounded, consequently $\{\mathcal{K}_{1,\beta}x_1^*\}, \cdots,$ $\{\mathcal{K}_{N,\beta}x_N^*\}$ are also bounded.

(ii) Next, we prove that for each $n \geq 1$, the following inequalities hold:

$$\begin{aligned}
&\|x_1^{n+1} - x_1^*\|^2 + \|x_2^{n+1} - x_2^*\|^2 + \cdots + \|x_N^{n+1} - x_N^*\|^2 \\
&\leq (1-\alpha_n)^2(\|x_1^n - x_1^*\|^2 + \|x_2^n - x_2^*\|^2 + \cdots + \|x_N^n - x_N^*\|^2) \\
&\quad + 2\alpha_n\tau(\|x_1^{n+1} - x_1^*\|\|x_2^n - x_2^*\| + \|x_2^{n+1} - x_2^*\|\|x_3^n - x_3^*\| \\
&\quad + \cdots + \|x_N^{n+1} - x_N^*\|\|x_1^n - x_1^*\|) + 2\alpha_n(\langle g_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\
&\quad + \langle g_2(x_3^*) - x_2^*,\ x_2^{n+1} - x_2^*\rangle + \cdots + \langle g_N(x_1^*) - x_N^*,\ x_N^{n+1} - x_N^*\rangle).
\end{aligned} \tag{3.13}$$

From (3.8) and Lemma 2.4, we have

$$\begin{aligned}
&\|x_1^{n+1} - x_1^*\|^2 \\
&= \|(1-\alpha_n)(\mathcal{K}_{1,\beta}(x_1^n) - x_1^*) + \alpha_n(g_1(\mathcal{K}_{2,\beta}(x_2^n)) - x_1^*)\|^2 \\
&\leq \|(1-\alpha_n)(\mathcal{K}_{1,\beta}(x_1^n) - x_1^*)\|^2 + 2\alpha_n\langle g_1(\mathcal{K}_{2,\beta}(x_2^n)) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\
&\leq (1-\alpha_n)^2\|\mathcal{K}_{1,\beta}(x_1^n) - x_1^*\|^2 \\
&\quad + 2\alpha_n\langle g_1(\mathcal{K}_{2,\beta}(x_2^n)) - g_1(x_2^*),\ x_1^{n+1} - x_1^*\rangle \\
&\quad + 2\alpha_n\langle g_1(x_2^n) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\
&\leq (1-\alpha_n)^2\|x_1^n - x_1^*\|^2 + 2\alpha_n\|g_1(\mathcal{K}_{2,\beta}(x_2^n)) - g_1(x_2^*)\|\|x_1^{n+1} - x_1^*\| \\
&\quad + 2\alpha_n\langle g_1(x_2^n) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\
&\leq (1-\alpha_n)^2\|x_1^n - x_1^*\|^2 + 2\alpha_n\tau_1\|\mathcal{K}_{2,\beta}(x_2^n) - x_2^*\|\|x_1^{n+1} - x_1^*\| \\
&\quad + 2\alpha_n\langle g_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\
&\leq (1-\alpha_n)^2\|x_1^n - x_1^*\|^2 + 2\alpha_n\tau_1\|x_2^n - x_2^*\|\|x_1^{n+1} - x_1^*\| \\
&\quad + 2\alpha_n\langle g_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle.
\end{aligned} \tag{3.14}$$

Similarly, we can also prove that

$$\begin{aligned}
\|x_2^{n+1} - x_2^*\|^2 &\leq (1-\alpha_n)^2\|x_2^n - x_2^*\|^2 + 2\alpha_n\tau_2\|x_3^n - x_3^*\|\|x_2^{n+1} - x_2^*\| \\
&\quad +2\alpha_n\langle g_2(x_3^*) - x_2^*,\ x_2^{n+1} - x_2^*\rangle, \\
&\ \ \vdots \\
\|x_N^{n+1} - x_N^*\|^2 &\leq (1-\alpha_n)^2\|x_N^n - x_N^*\|^2 + 2\alpha_n\tau_N\|x_1^n - x_1^*\|\|x_N^{n+1} - x_N^*\| \\
&\quad +2\alpha_n\langle g_N(x_1^*) - x_N^*,\ x_N^{n+1} - x_N^*\rangle.
\end{aligned} \tag{3.15}$$

Adding up (3.14) and (3.15), and take $\tau = \max\{\tau_1, \cdots, \tau_N\}$, inequality (3.13) is proved.

(iii) Next, we prove that if there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\liminf_{k\to\infty}\{(\|x_1^{n_k+1}-x_1^*\|^2+\cdots+\|x_N^{n_k+1}-x_N^*\|^2)-(\|x_1^{n_k}-x_1^*\|^2 \\ +\cdots+\|x_N^{n_k}-x_N^*\|^2)\}\geq 0, \tag{3.16}$$

then

$$\limsup_{k\to\infty}\{\langle g_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle+\langle g_2(x_3^*)-x_2^*,\ x_2^{n_k+1}-x_2^*\rangle \\ +\cdots+\langle g_N(x_1^*)-x_N^*,\ x_N^{n_k+1}-x_N^*\rangle\}\leq 0. \tag{3.17}$$

Since the norm $\|\cdot\|^2$ is convex and $\lim_{n\to\infty}\alpha_n=0$, by (3.8) we have

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}\{(\|x_1^{n_k+1}-x_1^*\|^2+\cdots+\|x_N^{n_k+1}-x_N^*\|^2) \\
&\quad -(\|x_1^{n_k}-x_1^*\|^2+\cdots+\|x_N^{n_k}-x_N^*\|^2)\} \\
&\leq \liminf_{k\to\infty}\{(1-\alpha_{n_k})\|\mathcal{K}_{1,\beta}x_1^{n_k}-x_1^*\|^2 \\
&\quad +\alpha_{n_k}\|g_1(\mathcal{K}_{2,\beta}(x_2^{n_k}))-x_1^*\|^2+(1-\alpha_{n_k})\|\mathcal{K}_{2,\beta}x_2^{n_k}-x_2^*\|^2 \\
&\quad +\alpha_{n_k}\|g_2(\mathcal{K}_{3,\beta}(x_3^{n_k}))-x_2^*\|^2+\cdots+(1-\alpha_{n_k})\|\mathcal{K}_{N,\beta}x_N^{n_k}-x_N^*\|^2 \\
&\quad +\alpha_{n_k}\|g_N(\mathcal{K}_{1,\beta}(x_1^{n_k}))-x_N^*\|^2-(\|x_1^{n_k}-x_1^*\|^2+\cdots+\|x_N^{n_k}-x_N^*\|^2)\} \\
&\leq \liminf_{k\to\infty}\{(\|\mathcal{K}_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2) \\
&\quad +(\|\mathcal{K}_{2,\beta}(x_2^{n_k})-x_2^*\|^2-\|x_2^{n_k}-x_2^*\|^2) \\
&\quad +\cdots+(\|\mathcal{K}_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)\} \\
&\leq \limsup_{k\to\infty}\{(\|\mathcal{K}_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2)+(\|\mathcal{K}_{2,\beta}(x_2^{n_k})-x_2^*\|^2 \\
&\quad -\|x_2^{n_k}-x_2^*\|^2)+\cdots+(\|\mathcal{K}_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)\} \\
&\leq 0.
\end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned}
&\lim_{k\to\infty}(\|\mathcal{K}_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2) \\
&=\lim_{k\to\infty}(\|\mathcal{K}_{2,\beta}x_2^{n_k}-x_2^*\|^2-\|x_2^{n_k}-x_2^*\|^2) \\
&\quad\vdots \\
&=\lim_{k\to\infty}(\|\mathcal{K}_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2) \\
&=0.
\end{aligned} \tag{3.19}$$

Since the sequences $\{\|\mathcal{K}_{1,\beta}x_1^{n_k}-x_1^*\|+\|x_1^{n_k}-x_1^*\|\}$, $\{\|\mathcal{K}_{2,\beta}x_2^{n_k}-x_2^*\| + \|x_2^{n_k}-x_2^*\|\}$, $\cdots$, $\{\|\mathcal{K}_{N,\beta}x_N^{n_k}-x_N^*\|+\|x_N^{n_k}-x_N^*\|\}$ are bounded, we have

$$\begin{aligned}
&\lim_{k\to\infty}(\|\mathcal{K}_{1,\beta}x_1^{n_k}-x_1^*\|-\|x_1^{n_k}-x_1^*\|)\\
&=\lim_{k\to\infty}(\|\mathcal{K}_{2,\beta}x_2^{n_k}-x_2^*\|-\|x_2^{n_k}-x_2^*\|)\\
&\ \ \vdots\\
&=\lim_{k\to\infty}(\|\mathcal{K}_{N,\beta}x_N^{n_k}-x_N^*\|-\|x_N^{n_k}-x_N^*\|)\\
&=0.
\end{aligned}\tag{3.20}$$

From Lemma 2.7, $\mathcal{K}_{1,\beta},\cdots,\mathcal{K}_{N,\beta}$ are strongly quasi nonexpansive, we have

$$\mathcal{K}_{1,\beta}x_1^{n_k}-x_1^{n_k}\to 0,\ \mathcal{K}_{2,\beta}x_2^{n_k}-x_2^{n_k}\to 0,\ \cdots,\ \mathcal{K}_{N,\beta}x_N^{n_k}-x_N^{n_k}\to 0. \tag{3.21}$$

Consequently, we obtain that

$$x_1^{n_k}-x_1^{n_k+1}\to 0,\ x_2^{n_k}-x_2^{n_k+1}\to 0,\ \cdots,\ x_N^{n_k}-x_N^{n_k+1}\to 0. \tag{3.22}$$

It follows from the boundedness of $\{x_1^{n_k}\}$ in Hilbert space $\mathcal{H}$ that there exists a subsequence $\{x_1^{n_{k_\ell}}\}$ of $\{x_1^{n_k}\}$ such that $x_1^{n_{k_\ell}}\rightharpoonup p$ and

$$\begin{aligned}
\lim_{\ell\to\infty}\langle g_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle &= \limsup_{k\to\infty}\langle g_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle\\
&=\limsup_{k\to\infty}\langle g_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle.
\end{aligned}\tag{3.23}$$

From Lemma 2.7, since $I-\mathcal{K}_{1,\beta}$ is demiclosed at zero, $p\in F(\mathcal{K}_{1,\beta})=\Omega_1$. Hence from (3.2) we have

$$\lim_{\ell\to\infty}\langle g_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle=\langle g_1(x_2^*)-x_1^*,\ p-x_1^*\rangle\le 0. \tag{3.24}$$

Therefore

$$\limsup_{k\to\infty}\langle g_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle=\lim_{\ell\to\infty}\langle g_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle\le 0. \tag{3.25}$$

Similarly, we can also prove that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle g_2(x_3^*)-x_2^*,\ x_2^{n_k+1}-x_2^*\rangle\le 0,\\
&\qquad\vdots\\
&\limsup_{k\to\infty}\langle g_N(x_1^*)-x_N^*,\ x_N^{n_k+1}-x_N^*\rangle\le 0.
\end{aligned}\tag{3.26}$$

Hence, we have the desired inequalities.

(iv) Finally, we prove that the sequences $\{x_1^n\}, \cdots, \{x_N^n\}$ generated by (3.8) converge to $x_1^*, \cdots, x_N^*$, respectively. It is clear that

$$\begin{aligned} &\|x_1^{n+1} - x_1^*\|\|x_2^n - x_2^*\| + \|x_2^{n+1} - x_2^*\|\|x_3^n - x_3^*\| \\ &\quad + \cdots + \|x_N^{n+1} - x_N^*\|\|x_1^n - x_1^*\| \\ &\le \sqrt{\|x_1^n - x_1^*\|^2 + \cdots + \|x_N^n - x_N^*\|^2} \\ &\quad \times \sqrt{\|x_1^{n+1} - x_1^*\|^2 + \cdots + \|x_N^{n+1} - x_N^*\|^2}. \end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.13), we have

$$\begin{aligned} &\|x_1^{n+1} - x_1^*\|^2 + \|x_2^{n+1} - x_2^*\|^2 + \cdots + \|x_N^{n+1} - x_N^*\|^2 \\ &\le (1-\alpha_n)^2(\|x_1^n - x_1^*\|^2 + \cdots + \|x_N^n - x_N^*\|^2) \\ &\quad + 2\alpha_n\tau\Big\{\sqrt{\|x_1^n - x_1^*\|^2 + \cdots + \|x_N^n - x_N^*\|^2} \\ &\quad \times \sqrt{\|x_1^{n+1} - x_1^*\|^2 + \cdots + \|x_N^{n+1} - x_N^*\|^2}\Big\} \\ &\quad + 2\alpha_n(\langle g_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle + \langle g_2(x_3^*) - x_2^*,\ x_2^{n+1} - x_2^*\rangle \\ &\quad + \cdots + \langle g_N(x_1^*) - x_N^*,\ x_N^{n+1} - x_N^*\rangle). \end{aligned} \tag{3.28}$$

Let

$$\begin{aligned} a_n &= \|x_1^n - x_1^*\|^2 + \|x_2^n - x_2^*\|^2 + \cdots + \|x_N^n - x_N^*\|^2, \\ b_n &= 2(\langle g_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle + \cdots + \langle g_N(x_1^*) - x_N^*,\ x_N^{n+1} - x_N^*\rangle). \end{aligned} \tag{3.29}$$

Then, we have the following statements:

(1) From (i), $\{a_n\}$ is bounded sequence.
(2) From (3.28) $a_{n+1} \le (1-\alpha_n)^2 a_n + 2\alpha_n\tau\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n,\ \forall n \ge 1.$
(3) From (iii), if $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying

$$\lim_{k\to\infty} \inf(a_{n_k+1} - a_{n_k}) \ge 0, \tag{3.30}$$

then we have

$$\lim_{k\to\infty} \sup b_{n_k} \le 0.$$

Therefore, it follows from Lemma 2.6 that

$$\lim_{n\to\infty}(\|x_1^n - x_1^*\|^2 + \cdots + \|x_N^n - x_N^*\|^2) = 0. \tag{3.31}$$

Hence, we obtain that

$$\begin{aligned}\lim_{n\to\infty}\|x_1^n - x_1^*\| &= \lim_{n\to\infty}\|x_2^n - x_2^*\| \\ &\ \ \vdots \\ &= \lim_{n\to\infty}\|x_N^n - x_N^*\| \\ &= 0.\end{aligned} \tag{3.32}$$

This completes the proof. □

Theorem 3.3. *Let $A_i, \Omega_i, \mathcal{K}_i$ and $\mathcal{K}_{i,\beta}$ $(i = 1, 2, \cdots, N)$ satisfying the conditions $(C1)$-$(C2)$ and $G : \mathcal{H} \to \mathcal{H}$ be an μ-Lipschitz continuous and r-strongly monotone mapping. Let $\{x_1^n\}, \cdots, \{x_N^n\}$ be the sequences defined by*

$$\begin{cases} x_1^0, \cdots, x_N^0 \in \mathcal{H}, \\ x_1^{n+1} = (1-\alpha_n)\mathcal{K}_{1,\beta}x_1^n + \alpha_n g_1(\mathcal{K}_{2,\beta}(x_2^n)), \\ x_2^{n+1} = (1-\alpha_n)\mathcal{K}_{2,\beta}x_2^n + \alpha_n g_2(\mathcal{K}_{3,\beta}(x_3^n)), \\ \quad\vdots \\ x_N^{n+1} = (1-\alpha_n)\mathcal{K}_{N,\beta}x_N^n + \alpha_n g_N(\mathcal{K}_{1,\beta}(x_1^n)), \end{cases} \tag{3.33}$$

for $n = 0, 1, 2, \cdots$, where $g_1 = I - \eta_1 G,\ g_2 = I - \eta_2 G, \cdots, g_N = I - \eta_N G$ with $\eta_1, \cdots, \eta_N \in (0, \frac{2r}{\mu})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequences $\{x_1^n\}, \{x_2^n\} \cdots, \{x_N^n\}$ converge to $x_1^*, x_2^*, \cdots, x_N^*$, respectively, where $(x_1^*, \cdots, x_N^*)$ is the unique element in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ such that (2.3) is satisfied.

Proof. It is easy to see that $g_1, g_2, \cdots, g_N$ are contraction mappings and all the conditions in Theorem 3.2 are satisfied. From Theorem 3.2, we has the sequence $(\{x_1^n\}, \cdots, \{x_N^n\}\ )$ which converges to $(x_1^*, \cdots, x_N^*) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ such that the following inequalities are satisfied.

$$\begin{cases} \langle x_1^* - g_1(x_2^*),\ x_1 - x_1^*\rangle \ge 0,\ \forall x_1 \in \Omega_1, \\ \langle x_2^* - g_2(x_3^*),\ x_2 - x_2^*\rangle \ge 0,\ \forall x_2 \in \Omega_2, \\ \quad\vdots \\ \langle x_{N-1}^* - g_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^*\rangle \ge 0,\ \forall x_{N-1} \in \Omega_{N-1}, \\ \langle x_N^* - g_N(x_1^*),\ x_N - x_N^*\rangle \ge 0,\ \forall x_N \in \Omega_N. \end{cases} \tag{3.34}$$

Substituting $g_1 = I - \eta_1 G,\ g_2 = I - \eta_2 G, \cdots, g_N = I - \eta_N G$ in (3.34), we obtain that the sequence $(\{x_1^n\}, \cdots, \{x_N^n\})$ converges to $(x_1^*, \cdots, x_N^*) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ such that (2.3) is satisfied. This completes the proof. □

Acknowledgments: This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

References

[1] M. K. Ahmad and Salahuddin, *Perturbed three step approximation process with errors for a general implicit nonlinear variational inequalities*, Int. J. Math. Math. Sci., Article ID 43818, (2006), 1-14.

[2] M. K. Ahmad and Salahuddin, *A stable perturbed algorithms for a new class of generalized nonlinear implicit quasi variational inclusions in Banach spaces*, Advances in Pure Mathematics, **2**(2) (2012), 139-148.

[3] J. Bracken and J.M. McGill, *Mathematical programs with optimization problems in the constraints*, Oper. Res., **21** (1973), 37-44.

[4] J. Bracken and J.M. McGill, *A method for solving mathematical programs with nonlinear programs in the constraints*, Oper. Res., **22** (1974), 1097-1101.

[5] S. S. Chang, J. K. Kim, H. W. Lee and C. K. Chun, *On the hierarchical variational inclusion problems in Hilbert spaces*, Fixed Point Theory App., 2013:179 (2013).

[6] F. Cianciaruso, V. Calao, L. Muglia and H. K. Xu, *On an implicit hierarchical fixed point approach to variational inequalities*, Bull. Austral. Math. Soc., **80**(1) (2009), 117-124.

[7] R. Kraikaew and S. Saejung, *On Mainge's approach for hierarchical optimization problems*, J. Optim. Theory Appl., **154**(1) (2012), 71-87.

[8] J. K. Kim, Salahuddin and A. Farajzadeh, *A new system of extended nonlinear regularized nonconvex set valued variational inequalities*, Commun. Appl. Nonlinear Anal., **21**(3) (2014), 21-28.

[9] K. S. Kim, *Extended general variational inequality for relaxed cocoercive mappings in Hilbert spaces*, Nonlinear Funct. Anal. Appl., **23**(4) (2018), 767-777.

[10] B. S. Lee and Salahuddin, *A general system of regularized nonconvex variational inequalities*, Appl. Comput. Math., **3**(4) (2014), 1-7.

[11] P. E. Mainge, *New approach to solving a system of variational inequalities and hierarchical problems*, J. Optim. Theory Appl., **138**(3) (2008), 459-477.

[12] P. E. Mainge, *The viscosity approximation process for quasi nonexpansive mappings in Hilbert spaces*, Comput. Math. Appl., **59**(1) (2010), 74-79.

[13] P. E. Mainge and A. Moudafi, *Strong convergence of an iterative method for hierarchical fixed point problems*, Pacific J. Optim., **3**(3) (2007), 529-538.

[14] G. Marino, V. Colao, L. Muglia and Y. Yao, *Krasnoselski-Mann iteration for hierarchical fixed points and equilibrium problems*, Bulletin Austral. Math. Soc., **79**(2) (2009), 187-200.

[15] A. Moudafi and P. Mainge, *Towards viscosity approximations of hierarchical fixed point problems*, Fixed Point Theory App., 2006, Article 95453 (2006), 10 pages.

[16] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yakohama Publishers, Yakohama, Japan, 2009.

[17] Y. K. Tang, S. S. Chang and Salahuddin, *A System of Nonlinear Set Valued Variational Inclusions*, Springer Plus, **3**(2014) , 318, Doi:10.1186/2193-180-3-318.

[18] N. Wairojjana and P. Kumam, *Existence and algorithm for the systems of Hierarchical variational inclusion problems*, Abst. Appl. Anal., 2014, ID 589679 (2014), 10 pages.

[19] H. K. Xu, *Viscosity method for hierarchical fixed point approach to variational inequalities*, Taiwanese J. Math., **14**(2) (2010), 463-478.

[20] Y. Yao, J. C. Cho and Y. C. Liou, *Iterative algorithms for hierarchical fixed points problems and variational inequalities*, Math. Comput. Modelling, **52**(9-10) (2010), 1697-1705.

[21] S. Zhang, J. H. W. Lee and C. K. Chan, *Algorithms of common solutions to quasi variational inclusions and fixed point problems*, Appl. Math. Mech., **29**(5) (2008), 571-581.