Nonlinear Functional Analysis and Applications
Vol. 24, No. 2 (2019), pp. 221-239
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

FIXED AND COMMON FIXED POINT THEOREMS THROUGH MODIFIED $\omega-$DISTANCE MAPPINGS

Tariq Qawasmeh[1], Abdalla Tallafha[2] and Wasfi Shatanawi[3]

[1]Department of Mathematics, School of Science
University of Jordan, Amman, Jordan
e-mail: jorqaw@yahoo.com, tar9160251@fgs.ju.edu.jo

[2]Department of Mathematics, School of Science
University of Jordan, Amman, Jordan
e-mail: a.tallafha@ju.edu.jo

[3]Department of Mathematics and general courses
Prince Sultan University Riyadh, Saudi Arabia
e-mail: wshatanawi@psu.edu.sa

Department of Medical Research
China Medical University Hospital China Medical University, Taichung, Taiwan
e-mail: wshatanawi@yahoo.com

Department of Mathematics, Faculty of Science
Hashemite University, Zarqa, Jordan
e-mail: swasfi@hu.edu.jo

Abstract. In this paper, we introduce the notion of $(k, \varphi, L)-$mω contraction which based on the notion of ultra distance function. We employ our contraction to prove fixed and common fixed point theorems. Also, we introduce an example in order to support our work.

[0]Received September 19, 2018. Revised January 28, 2019.
[0]2010 Mathematics Subject Classification: 54H25, 47H10, 55M20.
[0]Keywords: Quasi metric space, nonlinear contraction, fixed point theorem, modified $\omega-$distance.
[0]Corresponding author: T. Qawasmeh(jorqaw@yahoo.com).

1. Introduction and preliminary

Banach contraction principle [7] is a classical and powerful tool in nonlinear analysis, more precisely a self-mapping Ton a complete metric space (X, d) such that

$$d\left(Tx, Ty\right) \leq kd\left(x, y\right) \text{ forall } x, y \in X, \text{where } k \in [0, 1)$$

has a unique fixed point.

Since then, the Banach contraction principle has been generalized and investigated in several direction.

In a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable map. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena.

Several generalization of metric spaces were proposed by several mathematicians such as 2-metric spaces, Gahler [10], D-metric spaces, Dhage [9], G-metric spaces [14], and Huaug and Zhang, b- metric spaces[11].

Recently, Tallafha and Khalil [22] defined a space which is a mixture of analysis and topology, namely semi-linear uniform space. Semi-linear uniform space is weaker than metric space and stronger than topological space since several authors studied the properties of semi-linear uniform spaces and fixed point in such spaces, see [5, 16, 17, 22, 23, 24, 25, 26].

A self-mapping T on a metric space (X, d) is called Kannan contraction if there is a $\alpha \in [0, 1\backslash 2)$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.$$

Kannan [13] proved that every Kannan contraction in a complete metric space has a uinque fixed point. It is worth mentioning that Kannan theorem is an important result since it characterizes the metric completeness.

The concept of quasi metric space was introduced by Wilson [27].

Definition 1.1. ([27]) Let X be a nonempty set and $q : X\times X \to [0, \infty)$ be a given function which satisfies:

(i) $q(x, y) = 0$ if and only if $x = y$.
(ii) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

Then q is called a quasi metric on X, and the pair (X, q) is called a quasi metric space.

It is obvious that every metric space is a quasi metric space, but the converse need not be true.

A quasi metric q induces a metric q_m as follows:

$$q_m(x, y) = \max\{q(x, y), q(y, x)\}.$$

The notion of convergence and completeness in quasi metric spaces are given as follows:

Definition 1.2. ([12]) Let (X, q) be a quasi metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. Then the sequence $\{x_n\}$ is convergent to x if $\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0$.

Definition 1.3. ([12]) Let (X, q) be a quasi metric space and $\{x_n\}$ be a sequence in X. Then

(i) We say that the sequence $\{x_n\}$ is left-Cauchy if and only if for every $\epsilon > 0$, there is a positive integer $N = N_\epsilon$ such that $q(x_n, x_m) < \epsilon$ for all $n \geq m > N$.

(ii) We say that the sequence $\{x_n\}$ is right-Cauchy if and only if for every $\epsilon > 0$, there is a positive integer $N = N_\epsilon$ such that $q(x_n, x_m) < \epsilon$ for all $m \geq n > N$.

Definition 1.4. ([12]) Let (X, q) be a quasi metric space and $\{x_n\}$ be a sequence in X. We say that the sequence $\{x_n\}$ is Cauchy if and only if for every $\epsilon > 0$, there is a positive integer $N = N_\epsilon$ such that $q(x_n, x_m) < \epsilon$ for all $n, m > N$.

It is obvious that a sequence $\{x_n\}$ in a quasi metric space (X, q) is Cauchy if and only if it is right-Cauchy and left-Cauchy.

Definition 1.5. ([12]) Let (X, q) be a quasi metric space. We say that

(i) (X, q) is left-complete if and only if every left-Cauchy sequence in X is convergent.

(ii) (X, q) is right-complete if and only if every right-Cauchy sequence in X is convergent.

(iii) (X, q) is complete if and only if every Cauchy sequence in X is convergent.

For some fixed point theorems in a quasi metric we refer the reader to [3, 4, 8, 15, 19, 20, 21, 27].

A modified $\omega-$distance mapping on quasi metric space was defined by Alegre and Marin [4] as follows:

Definition 1.6. ([4]) A modified $\omega-$distance (shortly m$\omega-$distance) on a quasi metric space (X, q) is a function $p : X\times X \to [0, \infty)$, which satisfies:

(W1) $p(x,y) \leq p(x,z) + p(z,y)$ for all $x,y,z \in X$.
(W2) $p(x,.) : X \to [0,\infty)$ is lower semi-continuous for all $x \in X$.
(mW3) for each $\epsilon > 0$ there exist $\delta > 0$ such that if $p(x,y) \leq \delta$ and $p(y,z) \leq \delta$, then $q(x,z) \leq \epsilon$ for all $x,y,z \in X$.

Definition 1.7. ([4]) A strong m$\omega-$distance on a quasi metric space (X,q) is an m$\omega-$distance $p : X \times X \to [0,\infty)$ and the following properties are satisfied:
(sW2) $p(.,x) : X \to [0,\infty)$ is lower semi -continuous for all $x \in X$.

Remark 1.8. ([4]) Every quasi metric q on X is an m$\omega-$distance on the quasi metric space (X,q).

Definition 1.9. ([19]) The function $\varphi : [0,\infty) \to [0,\infty)$ is called ultra distance if the following properties are satisfied:
(i) $\varphi(t) = 0$ if and only if $t = 0$.
(ii) If $\{x_n\}$ is a sequence in $[0,\infty)$ such that $\lim\limits_{n\to\infty} \varphi(x_n) = 0$, then
$$\lim_{n\to+\infty} x_n = 0.$$

Here, we have an example of ultra distance function.

Example 1.10. ([19]) Define a function $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \sin(t)$ if $0 \leq t \leq 3\pi\backslash 4$ and $\varphi(t) = 1$ if $3\pi\backslash 4 \leq t < \infty$. Then it is clear that φ is an ultra distance function.

Lemma 1.11. ([15]) *Let (X,q) be a quasi metric space equipped with an m$\omega-$distance p. Let $\{x_n\}$ be a sequence in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero. Then we have the following statements:*
(i) *If $p(x_n, x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m \geq n$, then $\{x_n\}$ is a right Cauchy sequence in (X,q).*
(ii) *If $p(x_n, x_m) \leq \beta_m$ for any $n,m \in \mathbb{N}$ with $n \geq m$, then $\{x_n\}$ is a left Cauchy sequence in (X,q).*

Remark 1.12. ([15]) The above Lemma 1.11 implies that if $\lim\limits_{m,n\to\infty} p(x_n,x_m) = 0$, then $\{x_n\}$ is a Cauchy sequence in (X,q).

2. Main results

Definition 2.1. Let (X,q) be a quasi metric space equipped with an m$\omega-$distance mapping p and $F,G : X \to X$ be two self mappings. Then the pair (F,G) is called a $(k,\varphi,L)-$ mω contraction if there exist an ultra distance function φ and a constant $k \in [0,1)$ such that for all $x,y \in X$, we have
$$\varphi\left(p\left(Fx,Gy\right)\right) \leq k \max\left\{\varphi\left(p\left(x,Fx\right)\right), \varphi\left(p\left(y,Gy\right)\right)\right\} + L \min\left\{q\left(x,Gy\right), q\left(y,Fx\right), q\left(x,Fx\right)\right\}$$

and

$$\varphi\left(p\left(Gx,Fy\right)\right)\leq k\max\left\{\varphi\left(p\left(x,Gx\right)\right),\varphi\left(p\left(y,Fy\right)\right)\right\} + L\min\left\{q\left(x,Fy\right),q\left(y,Gx\right),q\left(x,Gx\right)\right\}.$$

Theorem 2.2. *Let (X,q) be a complete quasi metric space equipped with an $m\omega-$distance mapping p and $F,G:X\to X$ be two self mappings such that the pair (F,G) is a $(k,\varphi,L)-$ $m\omega$ contraction. Also, assume that one of the following conditions is satisfied:*

(i) *F and G are continuous.*
(ii) *If F or G is continuous and for each $u\in X$, $Fu\neq Gu$ then*

$$\inf\left\{p\left(Fx,u\right)+p\left(Gy,u\right):x,y\in X\right\}>0. \tag{2.1}$$

Then F and G have a unique common fixed point in X.

Proof. Let $x_0\in X$. Define a sequence $\{x_n\}$ in X such that $x_{2n+1}=Fx_{2n}$, $x_{2n+2}=Gx_{2n+1}$ for all $n\geq 0$. Now we want to show that if there exists $k\in\mathbb{N}\cup\{0\}$ such that $p\left(x_k,x_{k+1}\right)=0$ or $p\left(x_{k+1},x_k\right)=0$, then x_k is a common fixed point for F and G.
Case I: If $p\left(x_k,x_{k+1}\right)=0$ for some $k\in\mathbb{N}\cup\{0\}$.

Let k be even. Then $k=2t$ for some $t\in\mathbb{N}\cup\{0\}$. So, $p\left(x_{2t},x_{2t+1}\right)=0$, and so by definition of φ, we have $\varphi\left(p\left(x_{2t},x_{2t+1}\right)\right)=0$. Since the pair (F,G) is a $(k,\varphi,L)-$ mω contraction, we have

$$\begin{aligned}
&\varphi\left(p\left(x_{2t+1},x_{2t+2}\right)\right)\\
&=\varphi\left(p\left(Fx_{2t},Gx_{2t+1}\right)\right)\\
&\leq k\max\left\{\varphi\left(p\left(x_{2t},Fx_{2t}\right)\right),\varphi\left(p\left(x_{2t+1},Gx_{2t+1}\right)\right)\right\}\\
&\quad+L\min\left\{q\left(x_{2t},Gx_{2t+1}\right),q\left(x_{2t+1},Fx_{2t}\right),q\left(x_{2t},Fx_{2t}\right)\right\}\\
&=k\max\left\{\varphi\left(p\left(x_{2t},x_{2t+1}\right)\right),\varphi\left(p\left(x_{2t+1},x_{2t+2}\right)\right)\right\}\\
&\quad+L\min\left\{q\left(x_{2t},x_{2t+2}\right),q\left(x_{2t+1},x_{2t+1}\right),q\left(x_{2t},x_{2t+1}\right)\right\}\\
&=k\max\left\{\varphi\left(p\left(x_{2t},x_{2t+1}\right)\right),\varphi\left(p\left(x_{2t+1},x_{2t+2}\right)\right)\right\}\\
&=k\varphi\left(p\left(x_{2t+1},x_{2t+2}\right)\right).
\end{aligned}$$

Since $k<1$, we get $\varphi\left(p\left(x_{2t+1},x_{2t+2}\right)\right)=0$. Hence we have

$$p\left(x_{2t+1},x_{2t+2}\right)=0. \tag{2.2}$$

Now, also we have

$$
\begin{aligned}
&\varphi\left(p\left(x_{2t+2}, x_{2t+1}\right)\right) \\
&= \varphi\left(p\left(G x_{2t+1}, F x_{2t}\right)\right) \\
&\leq k \max\left\{\varphi\left(p\left(x_{2t+1}, G x_{2t+1}\right)\right), \varphi\left(p\left(x_{2t}, F x_{2t}\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{2t+1}, F x_{2t}\right), q\left(x_{2t}, G x_{2t+1}\right), q\left(x_{2t+1}, G x_{2t+1}\right)\right\} \\
&= k \max\left\{\varphi\left(p\left(x_{2t+1}, x_{2t+2}\right)\right), \varphi\left(p\left(x_{2t}, x_{2t+1}\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{2t+1}, x_{2t+1}\right), q\left(x_{2t}, x_{2t+2}\right), q\left(x_{2t+1}, x_{2t+2}\right)\right\} \\
&= k \max\left\{\varphi\left(p\left(x_{2t+1}, x_{2t+2}\right)\right), \varphi\left(p\left(x_{2t}, x_{2t+1}\right)\right)\right\} \\
&= 0.
\end{aligned}
$$

Hence, $\varphi\left(p\left(x_{2t+2}, x_{2t+1}\right)\right) = 0$. It implies that

$$p\left(x_{2t+2}, x_{2t+1}\right) = 0. \tag{2.3}$$

Therefore, by (W1) of the definition of p we have

$$
\begin{aligned}
p\left(x_{2t}, x_{2t+2}\right) &\leq p\left(x_{2t}, x_{2t+1}\right) + p\left(x_{2t+1}, x_{2t+2}\right) \\
&= 0.
\end{aligned}
$$

Thus,

$$p\left(x_{2t}, x_{2t+2}\right) = 0. \tag{2.4}$$

Using (2.3), (2.4) and (mW3), we get

$$q\left(x_{2t}, x_{2t+1}\right) = 0. \tag{2.5}$$

Therefore $q\left(x_k, x_{k+1}\right) = 0$ which implies that $x_k = x_{k+1}$. Hence, x_k is a fixed point for F.

Also, we have

$$
\begin{aligned}
p\left(x_{2t}, x_{2t+1}\right) &\leq p\left(x_{2t}, x_{2t+2}\right) + p\left(x_{2t+2}, x_{2t+1}\right) \\
&= 0.
\end{aligned} \tag{2.6}
$$

Using (2.2), (2.6) and (mW3), we get that

$$q\left(x_{2t}, x_{2t+2}\right) = 0. \tag{2.7}$$

So, we have

$$
\begin{aligned}
q\left(x_{2t+1}, x_{2t+2}\right) &\leq q\left(x_{2t+1}, x_{2t}\right) + q\left(x_{2t}, x_{2t+2}\right) \\
&= 0.
\end{aligned}
$$

Thus, $x_{2t} = x_{2t+1} = x_{2t+2}$ and so $x_k = x_{k+1} = x_{k+2}$. Therefore, x_k is a common fixed point of F and G.

Case II: If k is odd, then $k = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$, so,

$$p\left(x_{2t+1}, x_{2t+2}\right) = 0. \tag{2.8}$$

By the definition of φ, we get

$$\varphi\left(p\left(x_{2t+1}, x_{2t+2}\right)\right) = 0.$$

Since the pair (F, G) is a $(k, \varphi, L)-$ mω contraction, we have

$$\begin{aligned}
&\varphi\left(p\left(x_{2t+2}, x_{2t+3}\right)\right) \\
&= \varphi\left(p\left(Gx_{2t+1}, Fx_{2t+2}\right)\right) \\
&\leq k \max\left\{\varphi\left(p\left(x_{2t+1}, Gx_{2t+1}\right)\right), \varphi\left(p\left(x_{2t+2}, Fx_{2t+2}\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{2t+1}, Fx_{2t+2}\right), q\left(x_{2t+2}, Gx_{2t+1}\right), q\left(x_{2t+1}, Gx_{2t+1}\right)\right\} \\
&= k \max\left\{\varphi\left(p\left(x_{2t+1}, x_{2t+2}\right)\right), \varphi\left(p\left(x_{2t+2}, x_{2t+3}\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{2t+1}, x_{2t+3}\right), q\left(x_{2t+2}, x_{2t+2}\right), q\left(x_{2t+1}, x_{2t+2}\right)\right\}.
\end{aligned}$$

Therefore, $\varphi\left(p\left(x_{2t+2}, x_{2t+3}\right)\right) \leq k\varphi\left(p\left(x_{2t+2}, x_{2t+3}\right)\right)$. So we have

$$\varphi\left(p\left(x_{2t+2}, x_{2t+3}\right)\right) = 0.$$

It implies that

$$p\left(x_{2t+2}, x_{2t+3}\right) = 0. \tag{2.9}$$

Now,

$$\begin{aligned}
&\varphi\left(p\left(x_{2t+3}, x_{2t+2}\right)\right) \\
&= \varphi\left(p\left(Fx_{2t+2}, Gx_{2t+1}\right)\right) \\
&\leq k \max\left\{\varphi\left(p\left(x_{2t+2}, Fx_{2t+2}\right)\right), \varphi\left(p\left(x_{2t+1}, Gx_{2t+1}\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{2t+2}, Gx_{2t+1}\right), q\left(x_{2t+1}, Fx_{2t+2}\right), q\left(x_{2t+2}, Fx_{2t+2}\right)\right\} \\
&= k \max\left\{\varphi\left(p\left(x_{2t+2}, x_{2t+3}\right)\right), \varphi\left(p\left(x_{2t+1}, x_{2t+1}\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{2t+2}, x_{2t+2}\right), q\left(x_{2t+1}, x_{2t+3}\right), q\left(x_{2t+2}, x_{2t+3}\right)\right\}.
\end{aligned}$$

Therefore, $\varphi\left(p\left(x_{2t+3}, x_{2t+2}\right)\right) = 0$ and hence

$$p\left(x_{2t+3}, x_{2t+2}\right) = 0. \tag{2.10}$$

By (W1) we have

$$\begin{aligned}
p\left(x_{2t+1}, x_{2t+3}\right) &\leq p\left(x_{2t+1}, x_{2t+2}\right) + p\left(x_{2t+2}, x_{2t+3}\right) \\
&= 0.
\end{aligned}$$

Hence we have

$$p\left(x_{2t+1}, x_{2t+3}\right) = 0. \tag{2.11}$$

Using (2.10), (2.11) and (mW3), we get that

$$q\left(x_{2t+1}, x_{2t+2}\right) = 0. \tag{2.12}$$

Using (2.8), (2.9) and (mW3), we get

$$q\left(x_{2t+1}, x_{2t+3}\right) = 0. \tag{2.13}$$

Thus, $x_{2t} = x_{2t+1} = x_{2t+2}$ and so $x_k = x_{k+1} = x_{k+2}$. Hence, x_k is a common fixed point of F and G.

Similarly, we can prove that if $p\left(x_{k+1}, x_k\right) = 0$, then x_k is a common fixed point of F and G.

Now assume that $p\left(x_n, x_{n+1}\right) \neq 0$ and $p\left(x_{n+1}, x_n\right) \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since the pair (F, G) is a $(k, \varphi, L)-$ mω contraction, we have

$$\begin{aligned}\varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right) &= \varphi\left(p\left(Fx_{2n}, Gx_{2n+1}\right)\right)\\ &\leq k \max\left\{\varphi\left(p\left(x_{2n}, x_{2n+1}\right)\right), \varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right)\right\}\\ &\quad + L \min\left\{q\left(x_{2n}, x_{2n+2}\right), q\left(x_{2n+1}, x_{2n+1}\right), q\left(x_{2n}, x_{2n+1}\right)\right\},\end{aligned}$$

$$\varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right) \leq k \max\left\{\varphi\left(p\left(x_{2n}, x_{2n+1}\right)\right), \varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right)\right\}.$$

If $\max\left\{\varphi\left(p\left(x_{2n}, x_{2n+1}\right)\right), \varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right)\right\} = \varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right)$, then

$$\varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right) = 0$$

which is a contraction. So,

$$\max\left\{\varphi\left(p\left(x_{2n}, x_{2n+1}\right)\right), \varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right)\right\} = \varphi\left(p\left(x_{2n}, x_{2n+1}\right)\right).$$

Thus,

$$\varphi\left(p\left(x_{2n+1}, x_{2n+2}\right)\right) \leq k\varphi\left(p\left(x_{2n}, x_{2n+1}\right)\right).$$

By the same process, we can show that:

$$\varphi\left(p\left(x_{2n}, x_{2n+1}\right)\right) \leq k\varphi\left(p\left(x_{2n-1}, x_{2n}\right)\right).$$

Hence,

$$\varphi\left(p\left(x_n, x_{n+1}\right)\right) \leq k\varphi\left(p\left(x_{n-1}, x_n\right)\right) \ \ \forall n \in \mathbb{N} \cup \{0\}. \tag{2.14}$$

Now,

$$\begin{aligned}\varphi\left(p\left(x_{n+1}, x_n\right)\right) &= \varphi\left(p\left(Fx_n, Gx_{n-1}\right)\right)\\ &\leq k \max\{\varphi\left(p\left(x_n, x_{n+1}\right), \varphi\left(p\left(x_{n-1}, x_n\right)\right)\right)\}\\ &\quad + L \min\{q(x_n, x_n), q(x_{n-1}, x_{n+1}), q(x_n, x_{n+1})\}.\end{aligned}$$

Using (2.14) we get

$$\varphi(p\left(x_{n+1}, x_n\right)) \leq k\varphi\left(p\left(x_{n-1}, x_n\right)\right).$$

Repeating this process n-times, we get that

$$\varphi p\left(x_n, x_{n+1}\right) \leq k^n \varphi\left(p\left(x_\circ, x_1\right)\right) \ \ \forall n \in \mathbb{N} \cup \{0\} \tag{2.15}$$

and

$$\varphi\left(p\left(x_{n+1}, x_n\right)\right) \leq k^n \varphi\left(p\left(x_\circ, x_1\right)\right) \ \ \forall n \in \mathbb{N} \cup \{0\}. \tag{2.16}$$

Therefore,

$$\max\left\{\varphi\left(p\left(x_n,x_{n+1}\right)\right),\varphi\left(p\left(x_{n+1},x_n\right)\right)\right\}\leq k^n\varphi\left(p\left(x_1,x_\circ\right)\right). \tag{2.17}$$

Letting $n\to\infty$, we get that

$$\lim_{n\to\infty}\varphi\left(p(x_n,x_{n+1})\right)=0$$

and

$$\lim_{n\to\infty}\varphi\left(p(x_{n+1},x_n)\right)=0.$$

Since φ is a ultra distance function, we have

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0$$

and

$$\lim_{n\to\infty}p(x_{n+1},x_n)=0.$$

To prove that $\{x_n\}$ is a Cauchy sequence, first we show that $\lim\limits_{r,s\to\infty}p(x_r,x_s)=0$ for each $r,s\in\mathbb{N}$ with r odd and s even or r even and s odd.
Case(1): If r is odd and s is even with $r<s$, then we have

$$\begin{aligned}
\varphi\left(p\left(x_r,x_s\right)\right)&=\varphi\left(p\left(Fx_{r-1},Gx_{s-1}\right)\right)\\
&\leq k\max\left\{\varphi\left(p\left(x_{r-1},Fx_{r-1}\right)\right),\varphi\left(p\left(x_{s-1},Gx_{s-1}\right)\right)\right\}\\
&\quad+L\min\left\{q\left(x_{r-1},Gx_{s-1}\right),q\left(x_{s-1},Fx_{r-1}\right),q\left(x_{r-1},Fx_{r-1}\right)\right\}\\
&=k\max\left\{\varphi\left(p\left(x_{r-1},x_r\right)\right),\varphi\left(p\left(x_{s-1},x_s\right)\right)\right\}\\
&\quad+L\min\left\{q\left(x_{r-1},x_s\right),q\left(x_{s-1},x_r\right),q\left(x_{r-1},x_r\right)\right\}\\
&\leq k\max\left\{\varphi\left(p\left(x_{r-1},x_r\right)\right),\varphi\left(p\left(x_{s-1},x_s\right)\right)\right\}+Lq\left(x_{r-1},x_r\right)\\
&=k\varphi\left(p\left(x_{r-1},x_r\right)\right)+Lq\left(x_{r-1},x_r\right).
\end{aligned}$$

Thus, we get that

$$\varphi\left(p\left(x_r,x_s\right)\right)\leq k^r\varphi\left(p\left(x_\circ,x_1\right)\right)+Lq\left(x_{r-1},x_r\right).$$

Letting $s,r\to\infty$, we have $\lim\limits_{s,r\to\infty}\varphi\left(p(x_r,x_s)\right)=0$. Since φ is an ultra distance function, we have

$$\lim_{s,r\to\infty}p(x_r,x_s)=0 \text{ with } r<s. \tag{2.18}$$

Case(2): If $s, r \in \mathbb{N}$ such that r is odd and s is even with $r > s$, then we have

$$\begin{aligned}
\varphi\left(p\left(x_r, x_s\right)\right) &= \varphi\left(p\left(Fx_{r-1}, Gx_{s-1}\right)\right) \\
&\leq k \max\left\{\varphi\left(p\left(x_{r-1}, Fx_{r-1}\right)\right), \varphi\left(p\left(x_{s-1}, Gx_{s-1}\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{r-1}, Gx_{s-1}\right), q\left(x_{s-1}, Fx_{r-1}\right), q\left(x_{r-1}, Gx_{r-1}\right)\right\} \\
&= k \max\left\{\varphi\left(p\left(x_{r-1}, x_r\right)\right), \varphi\left(p\left(x_{s-1}, x_s\right)\right)\right\} \\
&\quad + L \min\left\{q\left(x_{r-1}, x_s\right), q\left(x_{s-1}, x_r\right), q\left(x_{r-1}, x_r\right)\right\} \\
&\leq k\varphi\left(p\left(x_{s-1}, x_s\right)\right) + Lq\left(x_{r-1}, x_r\right) \\
&\leq k^s\varphi\left(p\left(x_\circ, x_1\right)\right) + Lq\left(x_{r-1}, x_r\right).
\end{aligned}$$

Thus, we have

$$\varphi\left(p\left(x_r, x_s\right)\right) \leq k^s\varphi\left(p\left(x_\circ, x_1\right)\right) + Lq\left(x_{r-1}, x_r\right).$$

Letting $s, r \to \infty$, we get $\lim_{r,s\to\infty} \varphi\left(p(x_r, x_s)\right) = 0$ and so

$$\lim_{r,s\to\infty} p(x_r, x_s) = 0 \text{ with } r > s. \tag{2.19}$$

Thus, for all $r, s \in \mathbb{N}$ with r odd and s even we have

$$\lim_{s,r\to\infty} p(x_r, x_s) = 0 \text{ with } r > s.$$

By the same argument, we can show that for all $r, s \in \mathbb{N}$ with r even and s odd, then

$$\lim_{s,r\to\infty} p(x_r, x_s) = 0 \text{ with } r < s. \tag{2.20}$$

Now, we show that $\{x_n\}$ is a right Cauchy sequence. To prove that $\{x_n\}$ is a right Cauchy sequence we have the following cases:

Case(i): If $n, m \in \mathbb{N}$ such that n is odd and m is even with $m > n$, then by (2.18) we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Case (ii): If $n, m \in \mathbb{N}$ such that n is even and m is odd with $m > n$, then by (2.20) we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Case (iii): If $n, m \in \mathbb{N}$ such that n m are both even with $m > n$, then we have

$$p\left(x_n, x_m\right) \leq p\left(x_n, x_{n+1}\right) + p\left(x_{n+1}, x_m\right).$$

Therefore,

$$\lim_{m,n\to\infty} p(x_n, x_m) = 0, \;\; m > n.$$

Case (iv): If $n, m \in \mathbb{N}$ such that n m are both odd with $m > n$, then we have

$$p\left(x_n, x_m\right) \leq p\left(x_n, x_{m+1}\right) + p\left(x_{m+1}, x_m\right).$$

Therefore,

$$\lim_{m,n\to\infty} p(x_n, x_m) = 0,\ m > n.$$

By the same argument, we can show that $\{x_n\}$ is a left Cauchy sequence. By Lemma 1.11, (x_n) is a Cauchy sequence.

Since (X, q) is a complete quasi metric space, there exists $u \in X$ such that $x_{2n} \to u$. Thus,

$$\lim_{n\to\infty} q(x_{2n}, u) = \lim_{n\to\infty} q(u, x_{2n}) = 0.$$

Now, suppose that F and G are continuous mappings. Then

$$\lim_{n\to\infty} q(Fx_{2n}, Fu) = \lim_{n\to\infty} q(Fu, Fx_{2n}) = 0.$$

To show that $u = Fu$, we have

$$\lim_{n\to\infty} q(x_{2n+1}, Fu) = \lim_{n\to\infty} q(Fu, x_{2n+1}) = 0.$$

So, $x_{2n+1} \to Fu$ and hence $u = Fu$.

Similarly, we can prove that $u = Gu$. Hence F and G have a common fixed point.

Now, suppose that F or G is continuous. Without loss of generality, we may assume that F is continuous. As above argument, we figure out u is a fixed point of F.

Now, we show that u is a fixed point of G.

Since $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, for given $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $p(x_n, x_m) \leq \frac{\epsilon}{2}$ for all $n, m \geq k$. By the lower semi continuity of p, we have

$$p(x_n, u) \leq \lim_{l\to\infty} \inf p(x_n, x_l) \leq \frac{\epsilon}{2}, \text{ for all } n \geq k.$$

Assume that $Fu \neq Gu$. Then by (2.1) we have

$$\begin{aligned}
\inf\{p(Fx, u) + p(Gx, u) : x \in X\} &\leq \inf\{p(Fx_n, u) + p(Gx_{n+1}, u) : n \in \mathbb{N}\} \\
&= \inf\{p(x_{n+1}, u) + p(x_{n+2}, u) : n \in \mathbb{N}\} \\
&\leq \epsilon,
\end{aligned}$$

for all $\epsilon > 0$. This is a contradiction. Therefore $u = Fu = Gu$. Hence, u is a common fixed point for F and G.

To prove the uniqueness of u, first we show that if $z \in X$ is a common fixed point of F and G, then $p(z, z) = 0$. We note that

$$\begin{aligned}\varphi(p(z,z)) &= \varphi(p(Fz,Gz)) \\ &\leq k\max\{\varphi(p(z,Fz)),\varphi(p(z,Gz))\} \\ &\quad + L\min\{q(z,Gz),q(z,Fz),q(z,Fz)\} \\ &= k\varphi(p(z,z)).\end{aligned}$$

Hence, $\varphi(p(z,z)) = 0$. It implies that $p(z,z) = 0$.

Now, assume that there exists $v \in X$ such that $Fv = Gv = v$. Then

$$\begin{aligned}\varphi(p(u,v)) &= \varphi(p(Fu,Gv)) \\ &\leq k\max\{\varphi(p(u,Fu)),\varphi(p(v,Gv))\} \\ &\quad + L\min\{q(u,Gv),q(v,Fu),q(u,Fu)\} \\ &= k\max\{\varphi(p(u,u)),\varphi(p(v,v))\} \\ &\quad + L\min\{q(u,v),q(v,u),q(u,u)\}.\end{aligned}$$

Thus, $\varphi(p(u,v)) = 0$ and hence $p(u,v) = 0$. Since $p(u,u) = 0$, we get $q(u,v) = 0$ and so $u = v$. This completes the proof. □

Putting $L = 0$ in Theorem 2.2, we get the following result:

Corollary 2.3. *Let (X,q) be a complete quasi metric space equipped with an $m\omega-$distance mapping p and $F, G : X \to X$ be two self mappings. Assume the following hypotheses:*

(1) *If there exists a ultra distance function φ and $k \in [0,1)$ such that for all $x, y \in X$, we have*

$$\varphi(p(Fx,Gy)) \leq k\max\{\varphi(p(x,Fx)),\varphi(p(y,Gy))\}$$

and

$$\varphi(p(Gx,Fy)) \leq k\max\{\varphi(p(x,Gx)),\varphi(p(y,Fy))\}.$$

(2) *If one of the following condition is satisfied:*
(i) F and G are continuous.
(ii) If F or G is continuous and for each $u \in X$ if $Fu \neq Gu$ then

$$\inf\{p(Fx,u) + p(Gy,u) : x, y \in X\} > 0.$$

Then F and G have a unique common fixed point in X.

Corollary 2.4. *Let (X,q) be a complete quasi metric space equipped with an $m\omega-$distance mapping p and $F, G : X \to X$ be two self mappings. Assume the following hypotheses:*

(1) *If there exists an ultra distance function φ, and two positive numbers α, β with $\alpha+\beta<1$, and $L \geq 0$ such that for all $x, y \in X$, we have*

$$\varphi(p(Fx, Gy)) \leq \alpha\varphi(p(x, Fx)) + \beta\varphi(p(y, Gy)) + L\min\{q(x, Gy), q(y, Fx), q(x, Fx)\}$$

and

$$\varphi(p(Gx, Fy)) \leq \alpha\varphi(p(x, Gx)) + \beta\varphi(p(y, Fy)) + L\min\{q(x, Fy), q(y, Gx), q(x, Gx)\}.$$

(2) *If one of the following condition is satisfied:*
(i) F and G are continuous.
(ii) If F or G is continuous and for each $u \in X$ if $Fu \neq Gu$ then

$$\inf\{p(Fx, u) + p(Gy, u) : x, y \in X\} > 0.$$

Then F and G have a unique common fixed point in X.

Proof. Since,
$\alpha\varphi(p(x, Fx)) + \beta\varphi(p(y, Gy)) \leq (\alpha+\beta)\max\{\varphi(p(x, Fx)), \varphi(p(y, Fy))\}$, we have desired result. □

By taking $G = I$ in Theorem 2.2, we have the following result:

Corollary 2.5. *Let (X, q) be a complete quasi metric space equipped with an $m\omega-$distance mapping p and $F : X \to X$ be a self mapping. Assume the following hypotheses:*

(1) *If there exists an ultra distance function φ, $k \in [0, 1)$, and $L \geq 0$ such that for all $x, y \in X$, we have*

$$\varphi(p(Fx, y))\} \leq k\max\{\varphi(p(x, Fx)), \varphi(p(y, y))\} + L\min\{p(x, y), p(y, Fx), p(x, Fx)\}$$

and

$$\varphi(p(x, Fy)) \leq k\max\{\varphi(p(x, x)), \varphi(p(y, Fy))\}.$$

(2) *If one of the following condition is satisfied:*
(i) F is continuous.
(ii) If F is any mapping and for each $u \in X$ if $u \neq Fu$ then

$$\inf\{p(x, u) + p(Fx, u) : x \in X\} > 0.$$

Then F has a unique fixed point in X.

By taking $F = G$ in Theorem 2.2 we get the following result:

Corollary 2.6. *Let (X, q) be a complete quasi metric space equipped with an $m\omega-$distance mapping p and $F : X \to X$ be a self mapping. Assume the following hypotheses:*

(1) *If there exists an ultra distance function* φ *and* $k \in [0,1)$, $L \geq 0$ *such that for all* $x, y \in X$, *we have*

$$\varphi(p(Fx, Fy)) \leq k \max\{\varphi(p(x, Fx)), \varphi(p(y, Fy))\} + L \min\{q(x, Fy), q(y, Fx), q(x, Fx)\}$$

(2) *If one of the following condition is satisfied:*
(i) F *is continuous.*
(ii) If F *is any mapping and for each* $u \in X$ *if* $u \neq Fu$ *then*

$$\inf\{p(Fx, u) : x \in X\} > 0.$$

Then F *has a unique fixed point in* X.

Corollary 2.7. *Let* (X, q) *be a complete quasi metric space equipped with an* $m\omega-$*distance mapping* p *and let* $F : X \to X$ *be a self mapping such that the following hypotheses hold true*

(1) *If there exists an ultra distance function* φ *and and* $\alpha + \beta < 1$ *with* $\alpha, \beta \geq 0$, *such that for all* $x, y \in X$, *we have*

$$\varphi(p(Fx, Fy)) \leq \alpha\varphi(p(x, Fx)) + \beta\varphi(p(y, Fy)).$$

(2) *If one of the following condition is satisfied:*
(i) F *is continuous.*
(ii) *for each* $u \in X$ *if* $u \neq Fu$ *then*

$$\inf\{p(x, u) + p(Fx, u) : x \in X\} > 0.$$

Then F *has a unique fixed point in* X.

Corollary 2.8. *Let* (X, q) *be a complete quasi metric space equipped with an* $m\omega-$*distance mapping* p *and* $F : X \to X$ *be a self continuous mapping. Assume that there exists* $\alpha \in [0, \frac{1}{2})$ *such that for all* $x, y \in X$, *we have*

$$p(Fx, Fy) \leq \alpha[\varphi(p(x, Fx)) + \varphi(p(y, Fy))].$$

Then F *has a unique fixed point in* X.

Now, we give an example to show the useability of our results.

Example 2.9. Let $X = \mathbb{N} \cup \{0\}$. Define $q : X \times X \to [0, \infty)$ as follow:

$$q(x, y) = \begin{cases} 0 & \text{if } x = y, \\ x + 2y & \text{if } x \neq y. \end{cases}$$

Define $p : X \times X \to X$ by $p(x, y) = \frac{1}{3}(x + 2y)$. And define the mappings F,$G : X \to X$ by

$$Fx = \begin{cases} 0, & x = 0, 1 \\ 1, & x \geq 2. \end{cases} \quad , \quad Gy = \begin{cases} 0, & y = 0 \\ 1, & y \geq 1. \end{cases}$$

Also, define $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(t) = \begin{cases} e^t - 1 & \text{if } t \in [0,1) \\ e^t & \text{if } t \geq 1. \end{cases}$$

Then, we know that the following statements are hold:

(1) φ is an ultra distance function.
(2) (X,q) is a complete quasi metric space.
(3) (X,p) is an m$\omega-$distance on q.
(4) The pair (F,G) is $(e^{-\frac{1}{3}},\varphi,0)-m\omega$ contraction i.e., $\forall x,y\in X$ we have

$$\varphi\left(p\left(Fx,Gy\right)\right) \leq e^{-\frac{1}{3}} \max\left\{\varphi\left(p\left(x,Fx\right)\right),\varphi\left(p\left(y,Gy\right)\right)\right\}$$

and

$$\varphi\left(p\left(Gx,Fy\right)\right) \leq e^{-\frac{1}{3}} \max\left\{\varphi\left(p\left(x,Gx\right)\right),\varphi\left(p\left(y,Fy\right)\right)\right\}.$$

(5) F and G are continuous functions.

The statement (1) is clear. Also it is easy matter to figure out that (X,q) is a quasi metric space and (X,p) is an m$\omega-$distance. To show that q is complete, let $\{x_n\}$ be a Cauchy sequence in X. Then for each $n,m\in\mathbb{N}$ we have

$$\lim_{m,n\to\infty} q(x_n,x_m) = 0.$$

So, we deduce that $x_n = x_m$ for all $n,m\in\{0,1,2,\ldots\}$ but possible for finitely many. Thus, $\{x_n\}$ is a convergent sequence in X. Hence (X,q) is a complete quasi metric space. To prove (4), given $x,y\in X$. We divide our proof into the following cases:

Case I: $x = 0,1,\ y = 0$.

$$\varphi\left(p\left(Fx,Gy\right)\right) = 0 \leq e^{-\frac{1}{3}} \max\left\{\varphi\left(p\left(x,Fx\right)\right),\varphi\left(p\left(y,Gy\right)\right)\right\}$$

and

$$\varphi\left(p\left(Gx,Fy\right)\right) = 0 \leq e^{-\frac{1}{3}} \max\left\{\varphi\left(p\left(x,Gx\right)\right),\varphi\left(p\left(y,Fy\right)\right)\right\}.$$

Case II: $x \geq 2,\ y = 0$.

$$\varphi\left(p\left(Fx,Gy\right)\right) = \varphi\left(p\left(1,0\right)\right) = \varphi(\frac{1}{3}) = e^{\frac{1}{3}} - 1,$$

$$\varphi\left(p\left(x,Fx\right)\right) = \varphi\left(p\left(x,1\right)\right) = \varphi\left(\frac{x+2}{3}\right) = e^{\left(\frac{x+2}{3}\right)} \geq e^{\frac{4}{3}},$$

$$\varphi\left(p\left(y,Gy\right)\right) = \varphi\left(0\right) = 0.$$

Hence, we have

$$\begin{aligned}\varphi\left(p\left(Fx,Gy\right)\right) &= e^{\frac{1}{3}}-1\\ &\leq e^{-\frac{1}{3}}\max\left\{\varphi\left(p\left(x,Fx\right)\right),\varphi\left(p\left(y,Gy\right)\right)\right\}.\end{aligned}$$

Also we have

$$\begin{aligned}\varphi\left(p\left(Gx,Fy\right)\right) &= e^{\frac{1}{3}}-1\\ &\leq e^{-\frac{1}{3}}\max\left\{\varphi\left(p\left(x,Gx\right)\right),\varphi\left(p\left(y,Fy\right)\right)\right\}.\end{aligned}$$

Case III: $x=0,1$, $y\geq 1$. We have the following subcases:
Subcase (1): $x=0$, $y=1$.

$$\varphi\left(p\left(Fx,Gy\right)\right)=\varphi\left(p\left(0,1\right)\right)=\varphi(\frac{2}{3})=e^{\frac{2}{3}}-1,$$

$$\varphi\left(p\left(x,Fx\right)\right)=0,$$

$$p\left(y,Gy\right)=p(1,1)=1$$

and

$$\varphi\left(p\left(y,Gy\right)\right)=e.$$

Thus,

$$\begin{aligned}\varphi\left(p\left(Fx,Gy\right)\right) &= e^{\frac{2}{3}}-1\\ &\leq e^{-\frac{1}{3}}\max\left\{\varphi\left(p\left(x,Fx\right)\right),\varphi\left(p\left(y,Gy\right)\right)\right\}.\end{aligned}$$

Also, $p\left(Gx,Fy\right)=p(0,0)=0$ and hence,

$$\begin{aligned}\varphi\left(p\left(Gx,Fy\right)\right) &= 0\\ &\leq e^{-\frac{1}{3}}\max\left\{\varphi\left(p\left(x,Gx\right)\right),\varphi\left(p\left(y,Fy\right)\right)\right\}.\end{aligned}$$

Subcase(2): $x=0$, $y\geq 2$.

$$\begin{aligned}\varphi\left(p\left(Fx,Gy\right)\right) &= e^{\frac{2}{3}}-1\\ &\leq e^{-\frac{1}{3}}\max\left\{\varphi\left(p\left(x,Fx\right)\right),\varphi\left(p\left(y,Gy\right)\right)\right\}\\ &= e^{-\frac{1}{3}}\times e^{\frac{y+2}{3}}\end{aligned}$$

and

$$\varphi(p\left(Gx,Fy\right))=\varphi(p(0,1))=e^{\frac{2}{3}}-1\leq e^{-\frac{1}{3}}\times e^{\frac{y+2}{3}}.$$

Subcase(3): $x = 1$, $y = 1$.

$$\begin{aligned}\varphi\left(p\left(Fx, Gy\right)\right) &= e^{\frac{2}{3}} - 1 \leq e^{-\frac{1}{3}} \times e \\ &= e^{\frac{2}{3}}.\end{aligned}$$

and

$$\varphi(p\left(Gx, Fy\right)) = e^{\frac{1}{3}} - 1 \leq e^{-\frac{1}{3}} \times e = e^{\frac{2}{3}}.$$

Subcase(4): $x = 1$, $y \geq 2$.

$$\varphi\left(p\left(Fx, Gy\right)\right) = e^{\frac{2}{3}} - 1 \leq e^{-\frac{1}{3}} \times e^{\frac{y+2}{3}}$$

and

$$\varphi(p\left(Gx, Fy\right)) = e \leq e^{-\frac{1}{3}} \times e^{\frac{y+2}{3}}.$$

Case IV: $x \geq 2$, $y \geq 1$. Then we have two subcases:
Subcase(1): $x \geq 2$, $y = 1$.

$$\varphi\left(p\left(Fx, Gy\right)\right) = \varphi\left(p\left(1,1\right)\right) = e,$$

$$\varphi\left(p\left(x, Fx\right)\right) = \varphi\left(p\left(x, 1\right)\right) = \varphi(\frac{x+2}{3}) = e^{\left(\frac{x+2}{3}\right)} \geq e^{\frac{4}{3}},$$

$$\varphi\left(p\left(y, Gy\right)\right) = \varphi\left(p\left(y, 1\right)\right) = \varphi\left(1\right) = e,$$

and

$$\begin{aligned}\varphi\left(p\left(Fx, Gy\right)\right) &= e \\ &\leq e^{-\frac{1}{3}} \max\left\{\varphi\left(p\left(x, Fx\right)\right), \varphi\left(p\left(y, Gy\right)\right)\right\} \\ &= e^{-\frac{1}{3}} \times e^{\left(\frac{x+2}{3}\right)}.\end{aligned}$$

Also, $\varphi(p\left(Gx, Fy\right)) = e^{\frac{1}{3}} - 1 \leq e^{-\frac{1}{3}} \times e^{\left(\frac{x+2}{3}\right)}$.
Subcase(2): $x \geq 2$, $y \geq 2$.

$$\varphi\left(p\left(Fx, Gy\right)\right) = e \leq e^{-\frac{1}{3}} \times e^{\left(\frac{x+2}{3}\right)}$$

and

$$\varphi(p\left(Gx, Fy\right)) = e \leq e^{-\frac{1}{3}} \times e^{\left(\frac{x+2}{3}\right)}.$$

Hence, by Theorem 2.1, F and G have a unique common fixed point. Here 0 is the unique common fixed point of F and G.

References

[1] K. Abodayeh, W. Shatanawi, A. Bataihah and A. H. Ansari, *Some fixed point and common fixed point results through $\Omega-$distance under nonlinear contraction*, Gazi University Journal of Science, **30** (2017), 293-302.

[2] K. Abodayeh, Anwar Batiahah and W. Shatanawi, *Generalized $\Omega-$distance mappings and some fixed point theorems*, Politehn Univ. Bucharest Sci. Bull. Ser. A, Appl. Math. Phys., **79** (2017), 223-232.

[3] K. Abodayeh, W. Shatanawi and D. Turkoglu, *Some fixed point theorems in quasi-metric spaces under quasi weak conractions*, Glob. J. Pure Appl. Math., **12**(6) (2016), 4771-4780.

[4] C. Alegre and J. Marin, *Modified ω-distance on quasi metric spaces and fixed point theorems on complete quasi metric spaces,* Topol. its Appl., **203** (2016), 120-129.

[5] S. A. Alhihi and M. A. Al-Fayyad, *Topological properties of semi-linear uniform spaces,* Glob. J. Pure and Appl. Math., **12**(6) (2016), 5329–5341.

[6] S. A**.** Alhihi, *More properties of semi linear uniform spaces*, Appl. Math., **6** (2015), 1069-1075.

[7] S. Banach, Sur les, *Opération dans les ensembles abstraits et leur application aux equations integral*, Fundam. Math., **3** (1922), 133-181.

[8] N. Bilgili, E. Karapmar and B. Samet, *Generalized quasi metric $\alpha-\psi$ contractive mapping in quasi metric spaces and related fixed point theorems*, J. Inequ. Appl., 2014:36, (2014).

[9] B. C. Dhage, *Generalized metric spaces and mapping with fixed point* **,** Bull. Calcutta Math. Soc., **84**(4) (1992), 329-338.

[10] S. Gahler, *2-metriscue Raume undihve topologische strulcture,* Math. Nachvichten, **26** (1963), 115-148.

[11] L-G. Huaug and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings***,** J. Math. Anal. Appl. **332** (2007), 1468-1476.

[12] M. Jleli and B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appl., 2012: 210, (2012).

[13] R. Kannan, *Some results on fixed point*, Bull. Calcutta Math. Soc., **60** (1968), 71-76.

[14] l Z. Mustafa and B. Sims, *Some remavles concerning D-metric spaces,* in proceeding of the international Conference on Fixed point theory and Applications,Vallencia, Spain, (2004), 189-198.

[15] I. Nuseir, W. Shatanawi, Issam Abu-Irwaq and Anwar Batiahah, *Nonlinear contraction and fixed point theorems with modified ω-distance mappings in complete quasi metric spaces*, J. Nonlinear Sci. Appl., **10** (2017), 5342-5350.

[16] A. Rawashdeh and A. Tallafha, *Fixed Point in Modified Semi-linear Uniform Spaces,* Global J. Pure and Appl. Math., **13**(2) (2017), 4461-4476.

[17] l A. Rawashdeh and A. Tallafha, *Fixed point in semi-linear uniform spaces and convex metric spaces*, Submitted.

[18] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer.Math. Soc., **136** (2008), 1861-1869.

[19] W. Shatanawi, *Fixed and common fixed point theoerms in frame of quasi metric spaces based on ultra distance functions*, Nonlinear Analysis: Modelling and Control, **23**(5) (2018), 724–748.

[20] W. Shatanawi, M. S. Noorani, H. Alsamir and A. Bataihah, *Fixed and common fixed point theorems in partially ordered quasi metric spaces*, J. Math. Compu. Sci., **16** (2012), 516-528.

[21] W. Shatanawi and A. Pitea, *Some coupled fixed point theorems in quasi-partial metric spaces*, Fixed Point Theory Appl., 2013:153, (2013).

[22] A. Tallafha and R. Khalil, *Best Approximation in Uniformity type spaces*, Euro. J. Pure and Appl. Math. **2**(2) (2009), 231-238.

[23] A. Tallafha, *Some properties of semi-linear uniform spaces*, Boletin da Sociedade Paranaense Dematematica, **29**(2) (2011), 9-14.

[24] A. Tallafha, *Open Problems in Semi-Linear Uniform Spaces*, J. Appl. Funct. Anal., **8**(2) (2013), 223-228.

[25] A. Tallafha, *Fixed point in Semi-Linear Uniform Spaces*, Euro. J. Math. Sci.. **12**(3) (2013), 334-340.

[26] A. Tallafha and S. Alhihi, *Fixed point in a non-metrizable spaces* Accepted for publication as a part of the proceedings volume " Computational Analysis: Contributions from AMAT 2015" in Springer- New York.

[27] W. A. Wilson, *On quasi-metric spaces*, Amer. J. Math., **53** (1931), 675-684.