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SUPERSTABILITY OF A GENERALIZED TRIGONOMETRIC FUNCTIONAL EQUATION

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Abstract. We will investigate the superstability of the trigonometric functional equation from the following Pexider type functional equation:

$$f(x+y) - f(x-y) = \lambda \cdot g(x)h(y), \quad \lambda \text{ is constant},$$

which is a trigonometric functional equation mixed by the sine and cosine function. Moreover, the equation can be considered by the mixed functional equation of the hyperbolic trigonometric functions, several exponential type functions, and Jensen type equation.

1. Introduction

In 1940, Ulam [24] conjectured the stability problem of the functional equation. Next year, Hyers [12] obtained partial answer for the case of additive mapping in this problem. Thereafter this problem was improved by Bourgin [8] in 1949, Aoki [2] in 1950, Rassias [23] in 1978, and Găvruta [11].

In 1979, Baker et al. [6] announced the new concept known as the superstability as follows: If f satisfies $|f(x+y) - f(x)f(y)| \leq \epsilon$ for some fixed $\epsilon > 0$, then either f is bounded or f satisfies the exponential functional equation $f(x+y) = f(x)f(y)$.

D'Alembert [1] in 1769(see, Kannappen book [14]) had introduced the cosine functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{C}$$

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which superstability is proved by Baker [5] in 1980.

Baker's result is generalized by Badora [3] in 1998 to a noncommutative group by using of the Kannappen condition [13]: $f(x+y+z) = f(x+z+y)$, and it again is improved by Badora and Ger [4] in 2002 under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$.

The d'Alembert equation (C) is generalized by Wilson as follows:

$$f(x+y)+f(x-y) = 2f(x)g(y) \tag{Wfg}$$

$$f(x+y)+f(x-y) = 2g(x)f(y), \tag{Wgf}$$

which is called as Wilson equation.

The superstability of the cosine type equations (C), (Wfg) and (Wgf) is founded in Badora, Ger, Kannappan, and Kim ([7], [15], [21], [22])

In 1983, Cholewa [9] investigated the superstability of the sine functional equation

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2. \tag{S}$$

His result was improved by Kim ([17], [19]) for the following generalized sine functional equation

$$g(x)h(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2. \tag{S_{gh}}$$

In [16], Kim had obtained the superstability of sine functional equation from the Pexider type functional equation

$$f(x+y)+g(x-y) = \lambda h(x)k(y). \tag{C_{fghk}}$$

Recently, Fassi, Kabbaj, and Kim [10] also obtained the superstability of cosine functional equation from (C_{fghk}).

The alternative difference type equation for the cosine functional equation (C) as follows:

$$f(x+y)-f(x-y) = 2f(x)f(y), \tag{T}$$

was investigated by Kim [18].

In 1769, d'Alembert [1] obtained solution the difference type equation $f(x+y)-f(x-y) = 2g(x)h(y)$, which stability is proved by Kim [18].

A trigonometric functional equation : $\cos(x+y)-\sin(x-y) = (\cos x - \sin x)(\cos y + \sin y)$ implies the functional equation $f(x+y)-g(x-y) = h(x)k(y)$.

The aim of this paper is to investigate the transferred superstability for the cosine and sine functional equation from the difference type trigonometric functional equation:

$$f(x+y)-f(x-y) = \lambda \cdot g(x)h(y), \quad \lambda \text{ is constant.} \tag{T^{λ}_{ffgh}}$$

under the conditions $\|f(x+y)-f(x-y)-\lambda g(x)h(y)\|_A \leq \varphi(x)$ or $\varphi(y)$ on a unital commutative normed algebra A.

The obtained superstability results for (T^{λ}_{ffgh}) can be applied to the hyperbolic trigonometric functions, several exponential functions, and Jensen equation as follows:

$$\begin{aligned}
\cosh(x+y) - \cosh(x-y) &= 2\sinh(x)\sinh(y)\\
\sinh(x+y) - \sinh(x-y) &= 2\cosh(x)\sinh(y)\\
\sinh^2\left(\tfrac{x+y}{2}\right) - \sinh^2\left(\tfrac{x-y}{2}\right) &= \sinh(x)\sinh(y)\\
e^{x+y} - e^{x-y} = 2\tfrac{e^x}{2}(e^y - e^{-y}) &= 2e^x\cosh(y)\\
n(x+y) - n(x-y) = 2ny &: \text{for } f(x) = nx.
\end{aligned}$$

Note that $(G,+)$ is a 2-divisible Abelian group, $\mathbb{C}$ is a set of complex numbers. V is a vector space, A is a unital commutative normed algebra with unit 1_A, and a^{-1} is an invertible element of $0 \neq a \in A$ (i.e., $a^{-1}a = aa^{-1} = 1_A$). Let $\varphi, \lambda > 0$ be a constant.

2. Superstability of the Wilson type equation from (T^{λ}_{ffgh}).

We will investigate the superstability of the Wilson type equation from the mixed trigonometric functional equation (T^{λ}_{ffgh}).

Theorem 2.1. *Assume that $f, g, h : G \times G \to A$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)h(y)\| \le \varphi(x) \quad \forall x, y \in G. \tag{2.1}$$

Suppose that there exists a sequence $\{y_n\}$ in G such that

$$\lim_{n\to\infty} \|h(y_n)^{-1}\| = 0.$$

Then, we have the following statements:

(i) *g satisfies the Wilson type equation*

$$g(x+y) + g(x-y) = \lambda g(x)l_h(y) \tag{Wg_{l_h}}$$

for all $x, y \in G$, where $l_h : G \to A$ is an even function such that $l_h(0) = 2\lambda^{-1}$.

(ii) *In particular, if h satisfies the cosine type equation*

$$h(x+y) + h(x-y) = \lambda h(x)h(y), \tag{C^{λ}}$$

then g and h satisfy the Wilson type equation

$$g(x+y) + g(x-y) = \lambda \cdot g(x)h(y) \quad \forall x, y \in G. \tag{Wg_h}$$

Proof. (i) From assumption, we can choose $\{y_n\}$ such that $\|h(y_n)^{-1}\| \to 0$ as $n \to \infty$. Substituting $y = y_n$ (with $n \in \mathbb{N}$) in (2.1), we have

$$\|f(x+y_n) - f(x-y_n) - \lambda g(x)h(y_n)\| \le \varphi(x)$$

for all $x, y_n \in G$. Then

$$\begin{aligned}
&\|(f(x+y_n) - f(x-y_n))h(y_n)^{-1} - \lambda g(x)\| \\
&= \left\|[f(x+y_n) - f(x-y_n) - \lambda g(x)h(y_n)]h(y_n)^{-1}\right\| \\
&\le \|f(x+y_n) - f(x-y_n) - \lambda g(x)h(y_n)\| \, \|h(y_n)^{-1}\| \\
&\le \|h(y_n)^{-1}\|\varphi(x)
\end{aligned} \tag{2.2}$$

for all $y \in G$. As $n \to \infty$ in (2.2), we get

$$g(x) = \lambda^{-1} \lim_{n\to\infty} [f(x+y_n) - f(x-y_n)]h(y_n)^{-1} \tag{2.3}$$

for all $y \in G$. Replacing y by $y + y_n$ in (2.1),

$$\|f(x+(y+y_n)) - f(x-(y+y_n)) - \lambda g(x)h(y+y_n)\| \le \varphi(x) \tag{2.4}$$

for all $x, y, y_n \in V$. Substituting y by $-y + y_n$ in (2.1),

$$\|f(x+(-y+y_n)) - f(x-(-y+y_n)) - \lambda g(x)h(-y+y_n)\| \le \varphi(x) \tag{2.5}$$

for all $x, y \in G$. Then,

$$\begin{aligned}
&\|f(x+(y+y_n)) - f(x-(y+y_n)) - \lambda g(x)h(y+y_n) + f(x+(-y+y_n)) \\
&- f(x-(-y+y_n)) - \lambda g(x)h(-y+y_n)\| \le 2\varphi(x)
\end{aligned} \tag{2.6}$$

for all $x, y \in G$.

The inequality (2.6) implies that

$$\begin{aligned}
&\|[f((x+y)+y_n) - f((x+y)-y_n)]h(y_n)^{-1} \\
&\quad + [f((x-y)+y_n) - f((x-y)-y_n)]h(y_n)^{-1} \\
&\quad - \lambda g(x)[h(y+y_n) + h(-y+y_n)]h(y_n)^{-1}\| \\
&= \|[f((x+y)+y_n) - f((x+y)-y_n) - \lambda g(x)h(y+y_n) \\
&\quad + f((x-y)+y_n) - f((x-y)-y_n) - \lambda g(x)h(-y+y_n)]h(y_n)^{-1}\| \\
&\le \|h(y_n)^{-1}\| 2\varphi(x)
\end{aligned} \tag{2.7}$$

for all $x, y \in G$.

The right-hand side in (2.7) converges to zero as $n \to \infty$. Hence, a limit function $l_h : G \to F$ can be defined as follows:

$$l_h(y) := \lim_{n\to\infty} \frac{h(y+y_n) + h(-y+y_n)}{\lambda \cdot h(y_n)}, \tag{2.8}$$

for all $y \in G$, then $l_h(0) = 2\lambda^{-1}$, and l_h is an even function. Letting $n \to \infty$ in (2.7), we see from (2.3) that

$$g(x+y) + g(x-y) = \lambda g(x) l_h(y) \tag{2.9}$$

for all $x, y \in G$, as desired.

(ii) In the case h satisfies (C^λ), the limit l_h states nothing else but h from (2.8). Hence, g and h validate a required equation (Wg_h). □

Theorem 2.2. *Assume that $f, g, h : G \times G \to A$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)h(y)\| \le \varphi(y) \quad \forall x, y \in G. \tag{2.10}$$

Suppose that there exists a sequence $\{x_n\}$ in G such that

$$\lim_{n\to\infty} \|g(x_n)^{-1}\| = 0.$$

Then, we have the following statements:

(i) *h satisfies*

$$h(x+y) + h(x-y) = \lambda h(x) l_g(y) \tag{Wh_{l_g}}$$

for all $x, y \in G$, where $l_g : G \to A$ is an even function such that $l_g(0) = 2\lambda^{-1}$.

(ii) *In particular, if g satisfies (C^λ), then h and g satisfy*

$$h(x+y) + h(x-y) = \lambda \cdot h(x)g(y). \tag{Wh_g}$$

Proof. We choose $\{x_n\}$ in G such that $\lim_{n\to\infty} \|g(x_n))^{-1}\| = 0$.

Taking $x = x_n$ (with $n \in \mathbb{N}$) in (2.10), dividing both sides by $\|\lambda \cdot g(x_n)\|$, and passing to the limit as $n \to \infty$, then

$$h(y) = \lambda^{-1} \lim_{n\to\infty} [f(x_n + y) - f(x_n - y)] g(x_n)^{-1} \tag{2.11}$$

for all $x_n, y \in G$.

Replace (x, y) by $(x_n + y, x)$ and (x, y) by $(x_n - y, x)$ in (2.10). The same procedure as those of the equations (2.4) $\sim$ (2.7) of Theorem 2.1, is performed. That is, adding the above two inequalities obtained by replacing, and dividing by $\lambda \cdot g(x_n)$, as $n \to \infty$, then it imply the existence of a limit function

$$l_g(y) := \lim_{n\to\infty} \frac{g(x_n + y) + g(x_n - y)}{\lambda \cdot g(x_n)}, \tag{2.12}$$

where $l_g : G \to \mathbb{C}$ satisfies

$$h(x+y) + h(x-y) = \lambda h(x) l_g(y) \quad \forall x, y \in G$$

for all $x, y \in G$, as desired.

For remainder, let go through the same steps as Theorem 2.1, then we get the required results. □

The following corollary follows immediate from the Theorems 2.1 and 2.2.

Corollary 2.3. *Assume that $f, g, h : G \times G \to A$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)h(y)\| \le \begin{cases} \min\{\varphi(x), \varphi(y)\} & \text{or} \\ \varepsilon \end{cases} \quad \forall\, x, y \in G.$$

Then we have the following statements:

(i) *Suppose that there exists a sequence* $\{y_n\}$ *in* G *such that*

$$\lim_{n\to\infty} \|h(y_n)^{-1}\| = 0.$$

Then, g *satisfies* (Wg_{l_h}) *with the even function* l_h *s.t.* $l_h(0) = 2\lambda^{-1}$. *In particular, if* h *satisfies* (C^λ)*, then* g *and* h *satisfy* (Wg_h).

(ii) *Suppose that there exists a sequence* $\{x_n\}$ *in* G *such that*

$$\lim_{n\to\infty} \|g(x_n)^{-1}\| = 0.$$

Then, h *satisfies* (Wh_{l_g}) *with the even function* l_g *s.t.* $l_g(0) = 2\lambda^{-1}$. *In particular, if* g *satisfies* (C^λ)*, then* h *and* g *satisfy* (Wh_g).

By replacing h by f, g by f, h by g in Theorem 2.1 and Theorem 2.2, we obtain the following corollaries.

Corollary 2.4. *Assume that* $f, g : G \times G \to A$ *satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)f(y)\| \le \varphi(x), \qquad \forall x, y \in G.$$

Suppose that there exists a sequence $\{y_n\}$ *in* G *such that*

$$\lim_{n\to\infty} \|f(y_n)^{-1}\| = 0.$$

Then, we have the following statements:

(i) g *satisfies*

$$g(x+y) + g(x-y) = \lambda g(x)l_f(y), \qquad \forall x, y \in G,$$

where $l_f : G \to A$ *is an even function such that* $l_f(0) = 2\lambda^{-1}$.

(ii) *In particular, if* f *satisfies* (C^λ)*, then* g *and* f *satisfy*

$$g(x+y) + g(x-y) = \lambda \cdot g(x)f(y).$$

Corollary 2.5. *Assume that* $f, g : G \times G \to A$ *satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)f(y)\| \le \varphi(y), \qquad \forall x, y \in G.$$

Suppose that there exists a sequence $\{x_n\}$ *in* G *such that*

$$\lim_{n\to\infty} \|g(x_n)^{-1}\| = 0.$$

Then, we have the following statements:

(i) h *satisfies*

$$f(x+y) + f(x-y) = \lambda f(x)l_g(y), \qquad \forall x, y \in G,$$

where $l_g : G \to A$ *is an even function such that* $l_g(0) = 2\lambda^{-1}$.

(ii) *In particular, if* g *satisfies* (C^λ)*, then* f *and* g *satisfy*

$$f(x+y) + f(x-y) = \lambda \cdot f(x)g(y).$$

Corollary 2.6. *Assume that $f, h : G \times G \to A$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda f(x)h(y)\| \le \varphi(x), \qquad \forall x, y \in G.$$

Suppose that there exists a sequence $\{y_n\}$ in G such that

$$\lim_{n\to\infty} \|h(y_n)^{-1}\| = 0.$$

Then, we have the following statements:

(i) *f satisfies*

$$f(x+y) + f(x-y) = \lambda f(x)l_h(y), \qquad \forall x, y \in G,$$

where $l_h : G \to A$ is an even function such that $l_h(0) = 2\lambda^{-1}$.

(ii) *In particular, if h satisfies (C^λ), then f and h satisfy*

$$f(x+y) + f(x-y) = \lambda \cdot f(x)h(y).$$

Corollary 2.7. *Assume that $f, h : G \times G \to A$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda f(x)h(y)\| \le \varphi(y), \qquad \forall x, y \in G.$$

Suppose that there exists a sequence $\{x_n\}$ in G such that

$$\lim_{n\to\infty} \|f(x_n)^{-1}\| = 0.$$

Then, we have the following statements:

(i) *h satisfies*

$$h(x+y) + h(x-y) = \lambda h(x)l_f(y), \qquad \forall x, y \in G.$$

where $l_f : G \to A$ is an even function such that $l_f(0) = 2\lambda^{-1}$.

(ii) *In particular, if f satisfies (C^λ), then h and f satisfy*

$$h(x+y) + h(x-y) = \lambda \cdot h(x)f(y).$$

Corollary 2.8. *Assume that $f, g : G \times G \to A$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)g(y)\| \le \varphi(x) \ \text{ or } \ \varphi(y), \qquad \forall x, y \in G.$$

Suppose that there exists sequences $\{x_n\}$ or $\{y_n\}$ in G such that

$$\lim_{n\to\infty} \|g(x_n)^{-1}\| = 0$$

or

$$\lim_{n\to\infty} \|g(y_n)^{-1}\| = 0,$$

respectively. Then, g satisfies

$$g(x+y) + g(x-y) = \lambda g(x)l_g(y)$$

where $l_g : G \to A$ is an even function such that $l_g(0) = 2\lambda^{-1}$.

Corollary 2.9. *Assume that $f : G \times G \to A$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda f(x)f(y)\| \leq \varphi(x) \quad or \quad \varphi(y), \qquad \forall x, y \in G.$$

Suppose that there exists sequences $\{x_n\}$ or $\{y_n\}$ in G such that

$$\lim_{n\to\infty} \|g(x_n)^{-1}\| = 0$$

or

$$\lim_{n\to\infty} \|g(y_n)^{-1}\| = 0,$$

respectively. Then, f is bounded.

Proof. Assume that f is not bounded. Then, by applying $g = f$ in Corollary 2.4 and Corollary 2.5, f satisfies (C^λ). By the way, that f satisfies (C^λ) implies f satisfies (T^λ) by Theorem 1 in [18]. Hence, f satisfies simultaneously (C^λ) and (T^λ). This forces that f is a zero function. But we know that there exists the cosine function which satisfies (C) as a non-zero. Hence the result is obtained by a contradiction. □

Remark 2.10. In all results in this section, by applying $\varphi(x) = \varphi(y) = \varepsilon$, we can obtain the same results as those.

3. Applications of the case $\widetilde{f}(x) := f(x)f(0)^{-1}$ in equation (T^λ_{ffgh})

Note that $\widetilde{f}(x) := f(x)f(0)^{-1}$. The following lemmas which is easy to verify shows that the similar argument holds without assuming the continuity. To ease the presentation, we continue using this notation $\widetilde{f}$ and note that it is legel only when $f(0) \neq 0$.

We consider the following type functional equation:

$$f(x+y) - f(x-y) = \lambda f(x)f(y). \tag{T^λ}$$

Lemma 3.1. ([10]) *Let $f, g : V \to A$ be functions satisfying*

$$f(x+y) + f(x-y) = \lambda f(x)g(y), \text{ for all } x, y \in V.$$

If f is an even function, then either $f \neq 0$ or $\widetilde{f}$ satisfies (C).

Lemma 3.2. *Let $f, g : V \to A$ be functions satisfying*

$$f(x+y) - f(x-y) = \lambda f(x)g(y), \text{ for all } x, y \in V. \tag{$T^\lambda fg$}$$

If f is odd function, then either $f \neq 0$ or $\widetilde{f}$ satisfies (T).

Proof. Since f is odd, let $x = 0$ in $(T^\lambda fg)$. Then $f(y) - f(-y) = \lambda f(0)g(y)$. Thus $2f(y) = \lambda f(0)g(y)$. □

Theorem 3.3. *Let $\varphi : V \to [0, +\infty)$ be a function, and $f, g, h : V \to A$ be functions satisfying*

$$\|f(x+y) - f(x-y) - \lambda g(x)h(y)\| \leq \varphi(x) \tag{3.1}$$

for all $x, y \in V$ with $\lambda > 0$. Suppose that there exists a sequence $\{y_n\}$ in V such that

$$\lim_{n\to\infty} \|h(y_n)^{-1}\| = 0.$$

Then, we have the following statements:

(i) *$g \neq 0$ is an even function, then $\widetilde{g}$ satisfies* (C).
(ii) *$g \neq 0$ is odd function, then $\widetilde{g}$ satisfies (T).*

Proof. Let $f, g, h : V \to A$ be functions satisfying (3.1) for all $x, y \in V$ with $\lambda > 0$. Assume that there exists a sequence $\{y_n\}$ in V such that $\lim_{n\to\infty} \|h(y_n)^{-1}\| = 0$. Then, by Theorem 2.1, there exists an even function $l_h : V \to A$ with $l_h(0) = 2\lambda^{-1}$ and g satisfies (Wg_{l_h}). Then, (i) holds by the Lemma 3.1. By Lemma 3.2, (ii) holds, that is, $\widetilde{g}$ satisfies (T). □

Corollary 3.4. *Let $\varphi : V \to [0, +\infty)$ be a function and $f, h, k : V \to \mathbb{C}$ be functions satisfying*

$$|f(x+y) - f(x-y) - \lambda g(x)h(y)| \leq \varphi(x)$$

for all $x, y \in V$ with $\lambda > 0$. Suppose that $g \neq 0$ and h is unbounded, Then, we have the following statements:

(i) *if g is even, then $\widetilde{g}$ satisfies* (C).
(ii) *if g is odd, then $\widetilde{g}$ satisfies (T).*

Corollary 3.5. *Let $\varphi : V \to [0, +\infty)$ be a function and $f, g : V \to A$ be functions satisfying*

$$\|f(x+y) - f(x-y) - \lambda g(x)f(y)\| \leq \varphi(x)$$

for all $x, y \in V$ with $\lambda > 0$. If $f \neq 0$ is odd and there exists a sequence $\{y_n\}$ in V such that $\lim_{n\to\infty} \|f(y_n)^{-1}\| = 0$, then $\widetilde{g} = (\lambda/2) \cdot g$ satisfies (C).

Proof. Replace h by f in Theorem 3.3. If $f \neq 0$ is odd, then, by (2.3), it implies that g is even with $g(0) = 2\lambda^{-1}$. Hence the required result holds from (i) in Theorem 3.3. □

4. Representation of solution for the Wilson type equations.

The solution of the Wilson equation(Wfg) was investigated in ([1], [14]).

In the following Lemma 4.1, a solution for the Wilson type Eqs.(Wf_g) and (Wg_f) will be investigated. Hence, its explicit solutions for results obtained in Sections 2 can be represented immediately from Lemma 4.1.

In this section, let $\mathbb{C}$ be a set of all complex numbers, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Whenever we only deal with $\mathbb{C}$, $(G, +)$ needs the Abelian which is not λ-divisible.

It is easy to verify shows that the following lemma holds.

Lemma 4.1. *Let $f, g : G \times G \to \mathbb{C}^*$ satisfy the Wilson type equation:*

$$f(x+y) + f(x-y) = \lambda f(x)g(y), \qquad (Wf_g)$$

$$f(x+y) + f(x-y) = \lambda g(x)f(y). \qquad (Wg_f)$$

Then, g satisfies (C^λ), and g, f are given by

$$g(x) = \frac{E(x) + E(-x)}{\lambda}, \quad f(x) = c(E(x) - E(-x)) + \frac{d}{\lambda}(E(x) + E(-x)),$$

where $c, d \in \mathbb{C}$, and $E : G \to \mathbb{C}^*$ is a homomorphism.

Corollary 4.2. *Assume that $f, g, h : G \times G \to \mathbb{C}^*$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)h(y)\| \leq \varphi(x), \quad \forall\, x, y \in G.$$

If h fails to be bounded, then g and l_h satisfy (Wg_{l_h}). In particular, if h satisfies (C), then g and h satisfy (Wg_h) and g, h are given by

$$h(x) = \frac{E(x) + E(-x)}{\lambda}, \quad g(x) = c(E(x) - E(-x)) + \frac{d}{\lambda}(E(x) + E(-x)),$$

where $c, d \in \mathbb{C}$, and $E : G \to \mathbb{C}^*$ is a homomorphism.

Corollary 4.3. *Assume that $f, g, h : G \times G \to \mathbb{C}^*$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - \lambda g(x)h(y)\| \leq \varphi(y), \quad \forall\, x, y \in G.$$

If g fails to be bounded, then h and l_g satisfy (Wh_{l_g}). In particular, if g satisfies (C), then h and g satisfies (Wh_g), and h, g are given by

$$g(x) = \frac{E(x) + E(-x)}{\lambda}, \quad h(x) = c(E(x) - E(-x)) + \frac{d}{\lambda}(E(x) + E(-x)),$$

where $c, d \in \mathbb{C}$, and $E : G \to \mathbb{C}^*$ is a homomorphism.

Corollary 4.4. *Suppose that $f, g, h : G \to \mathbb{C}^*$ satisfy the inequality*

$$|f(x+y) - f(x-y) - \lambda g(x)h(y)| \leq \begin{cases} \min\{\varphi(x), \varphi(y)\} & or \\ \varepsilon \end{cases} \quad \forall\, x, y \in G. \tag{4.1}$$

Then, we have the following statements:

(i) *If h fails to be bounded, then g and l_h satisfy (Wg_{l_h}). In particular, if h satisfies (C), then g and h satisfy (Wg_h) and g, h are given by*

$$h(x) = \frac{E(x) + E(-x)}{\lambda}$$

and

$$g(x) = c(E(x) - E(-x)) + \frac{d}{\lambda}(E(x) + E(-x)),$$

where $c, d \in \mathbb{C}$, and $E : G \to \mathbb{C}^*$ is a homomorphism.

(ii) *If g fails to be bounded, then h and l_g satisfy (Wh_{l_g}). In particular, if g satisfies (C), then h and g satisfies (Wh_g), and h, g are given by*

$$g(x) = \frac{E(x) + E(-x)}{\lambda}$$

and

$$h(x) = c(E(x) - E(-x)) + \frac{d}{\lambda}(E(x) + E(-x)),$$

where $c, d \in \mathbb{C}$, and $E : G \to \mathbb{C}^*$ is a homomorphism.

5. Extension to the Banach algebra

In all the results presented in Sections 2 and 3, the range of functions on the Abelian group can be extended to the semisimple commutative Banach algebra. We will represent just for the main equation (T^{λ}_{ffgh}).

Theorem 5.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \to E$ satisfy one of each inequalities*

$$\|f(x+y) - f(x-y) - \lambda \cdot g(x)h(y)\| \leq \begin{cases} (i)\ \varphi(x) \\ (ii)\ \varphi(y) \end{cases} \tag{5.1}$$

for all $x, y \in G$ with $\lambda > 0$ is a constant. For an arbitrary linear multiplicative functional $x^* \in E^*$, then we have the following statements:

(i) *If $x^* \circ h$ fails to be bounded, then g satisfies (Wg_{l_h}): $g(x+y) + g(x-y) = \lambda g(x)l_h(y)$, where $l_h : G \to A$ is an even function such that $l_h(0) = 2\lambda^{-1}$. In particular, if h satisfies (C^{λ}), then g and h satisfy the (Wg_h).*

(ii) *If $x^* \circ g$ fails to be bounded, then g satisfies (Wh_{l_g}): $h(x+y)+h(x-y) = \lambda h(x)l_g(y)$, where $l_g : G \to A$ is an even function such that $l_g(0) = 2\lambda^{-1}$. In particular, if g satisfies (C^λ), then h and g satisfies the (Wh_g).*

Proof. (i) Assume that (5.1) holds and arbitrarily fixes a linear multiplicative functional $x^* \in E^*$. As is well known, given that $\|x^*\| = 1$, for every $x, y \in G$,

$$\begin{aligned}\varphi(x) &\geq \|f(x+y) - f(x-y) - \lambda \cdot g(x)h(y)\| \\ &= \sup_{\|y^*\|=1} \left|y^*\big(f(x+y) - f(x-y) - \lambda \cdot g(x)h(y)\big)\right| \\ &\geq \left|x^*\big(f(x+y)\big) - x^*\big(f(x-y)\big) - \lambda \cdot x^*\big(g(x)\big)x^*\big(h(y)\big)\right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$, and $x^* \circ h$ yield a solution of inequality (2.1) in Theorem 2.1. Given the assumption, the superposition $x^* \circ h$ is unbounded, an appeal to (i) of Theorem 2.1 shows that the two results hold.

First, the superposition $x^* \circ g$ solves (Wg_{l_h}) with $x^* \circ l_h$, that is,

$$(x^* \circ g)\big(x+y\big) + (x^* \circ g)\big(x-y\big) = \lambda(x^* \circ g)(x)(x^* \circ l_h)(y).$$

Since x^* is a linear multiplicative functional, we have

$$x^*(g(x+y) + g(x-y) - \lambda \cdot g(x)l_h(y)) = 0.$$

Hence, an unrestricted choice of x^* implies that

$$g(x+y) + g(x-y) - \lambda \cdot g(x)l_h(y) \in \bigcap\{\ker x^* : x^* \in E^*\}.$$

Since the space E is semisimple, $\bigcap\{\ker x^* : x^* \in E^*\} = 0$, which means that g satisfies the claimed equation (Wg_{l_h}) with $l_h(0) = 2\lambda^{-1}$

In particular, if h satisfies (C^λ), then l_h implies h. Hence, (Wg_h) holds.

(ii) The second cases are also the same because each case of Theorem 2.2 is similar to case (i). □

Remark 5.2. In all results, we can obtain more corollaries from setting as the following.

(i) Choosing f, g, h in three places, $\{\varphi(x), \varphi(y), \min\{\varphi(x), \varphi(y)\}, \varepsilon$: constant$\}$, and $\lambda = 2$, from the such setting, we can obtain more the stability results for the following number of equations: $3! \times 4 \times 2$.

(ii) The case $\lambda = 2$ can be applied well known stability results for the cosine type "+"(Wilson) equation, which are found in papers (Badora [3], Ger [4], Baker [3], Kannappan and Kim [15], Kim ([16] $\sim$ [22]).

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