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EXISTENCE OF POSITIVE SYMMETRIC SOLUTIONS FOR A CLASS OF SINGULAR SECOND ORDER THREE-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, by applying the Guo-Krasnosel'skii fixed point theorem and some fixed index theorem we study the existence of positive symmetric solutions for a class of singular second-order three-point boundary value problems. The nonlinear term may be singular at $t = 0, 1$ and can be allowed to change sign.

1. Introduction

In this paper, we will study the existence of positive symmetric solutions for the following boundary value problems(BVP):

$$\begin{cases} x''(t) + a(t)x'(t) + f(t, x(t)) = 0, & 0 < t < 1, \\ x(0) = x(1) = \alpha x(\eta), \end{cases} \tag{1.1}$$

where $\alpha \in [0, \frac{1}{2}]$, $\eta \in (0, 1)$ is a constant, $a(t) \in C[0, 1], f \in C((0, 1) \times [0, +\infty), R)$.

In recent years, the problems for the existence of positive solutions to second order multi-point BVP have received much attention, and many excellent

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results have been established, for instance, ([4]-[6], [8]). However, to the best of our knowledge, it seems there are few results about symmetric positive solutions to multi-point BVP in the relevant literatures. In [1] Avery and Henderson studied the existence of symmetric positive solutions for the two-point BVP

$$\begin{cases} u''(t) + f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

by applying a fixed point theorem due to Avery. In [10], Wang investigated three-point BVP

$$\begin{cases} u''(t) + h(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(t) = u(1-t), \ \alpha u'(0) - \beta u'(1) = \gamma u\left(\frac{1}{2}\right), \end{cases}$$

and obtained the existence of symmetric positive solutions by using the fixed index theorem in cones.

However, in the above literatures they assumed that the nonlinear term f was non-singular and non-negative. Motivated by the above excellent works, in this paper, we intend to study the existence of symmetric positive solutions to a class of BVP in which the nonlinear term f was allowed be singular at $t = 0, 1$ and be sign-changing by applying Guo-Krasnosel'skii fixed point theorem and some fixed point index theorem in cones.

For the convenience of readers several main assumptions are provided as the following:

(H1) for $a \in C[0,1]$ and $t \in [0,1]$, $a(t) = -a(1-t)$. In addition, $q(t) \in C(0,1)$ is nonnegative and $q(t) = q(1-t)$ for any $t \in (0,1)$.

(H2) $f(t,x) \in C((0,1) \times [0,+\infty), R)$ is symmetric on $(0,1)$ (i.e., $f(t,x) = f(1-t,x)$ for any $t \in (0,1)$). For any $t \in (0,1)$, $f(t,x)$ is nonincreasing about x on $(0,+\infty)$, and $f(t,x) \geq -q(t)x$ for any $(t,x) \in (0,1) \times [0,+\infty)$.

(H3) $|\max_{t\in[0,1]} \int_0^1 \gamma(t,s)p(s)f(s,k\phi_1(s)\phi_2(s))ds| < +\infty$ for any $k > 0$, $\max_{t\in[0,1]} \int_0^1 \gamma(t,s)p(s)q(s)ds < +\infty$ and there exists some $r > 0$ such that $\max_{t\in[0,1]} \int_0^1 \gamma(t,s)p(s)f(s,r\phi_1(s)\phi_2(s))ds > 0$, where $\gamma(t,s) = G(t,s) + \frac{\alpha(\phi_1(t)+\phi_2(t))}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))}G(\eta,s)$.

Throughout this paper, let $E = C[0,1]$ be a Banach space with the norm $\|x\| = \max_{0\leq t\leq 1} |x(t)|$ for any $x \in E$.

2. Preliminaries

In this section, we present some lemmas which are essential in the proof of our main results.

Lemma 2.1. [5] *Let* $a(t), b(t) \in C[0,1]$ *with* $b(t) > 0$ *for any* $t \in [0,1]$. *Assume that* $\phi_1(t), \phi_2(t)$ *are the solutions of the following BVP, respectively*

$$\begin{cases} \phi_1''(t) + a(t)\phi_1'(t) - b(t)\phi_1(t) = 0, \quad 0 < t < 1, \\ \phi_1(0) = 0, \quad \phi_1(1) = 1, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \phi_2''(t) + a(t)\phi_2'(t) - b(t)\phi_2(t) = 0, \quad 0 < t < 1, \\ \phi_2(0) = 1, \quad \phi_2(1) = 0, \end{cases} \tag{2.2}$$

then

(i) $\phi_1(t)$ *is strictly increasing on* $[0,1]$ *and* $\phi_1'(0) > 0$;
(ii) $\phi_2(t)$ *is strictly decreasing on* $[0,1]$.

Lemma 2.2. *The above two boundary value problems* (2.1) *and* (2.2) *have a unique solution.*

Proof. First, we show that BVP(2.1) has a unique solution.
In fact, if there exists $v_1(t)$ and $v_2(t)$ be two solutions of BVP(2.1), then $v_0(t) = v_1(t) - v_2(t)$ is also a solution of BVP(2.1). Moreover, $v_0(t)$ satisfies that

$$\begin{cases} v_0''(t) + a(t)v_0'(t) - b(t)v_0(t) = 0 \quad 0 < t < 1, \\ v_0(0) = 0, v_0(1) = 0. \end{cases} \tag{2.3}$$

Thus, by the maximum principle we obtain $v_0(t) \equiv 0,\ t \in [0,1]$. And so, BVP(2.1) has only one solution. Similarly, BVP(2.2) has also only one solution. □

Lemma 2.3. *Let* $a(t) = -a(1-t),\ t \in [0,1]$ *and* $a(t) \in C[0,1]$. *Then* $\phi_1(t) = \phi_2(1-t)$ *for any* $t \in [0,1]$

Proof. Following the above conditions, it is easy to prove $\phi_2(1-t)$ satisfying BVP(2.1). Further, by Lemma 2.2, we obtain $\phi_1(t) = \phi_2(1-t)$. The proof is complete. □

Lemma 2.4. *For any* $a(t),\ b(t) \in C[0,1]$ *with* $b(t) > 0,\ t \in [0,1],\ y(t) \in C[0,1]$, *the BVP*

$$\begin{cases} x''(t) + a(t)x'(t) - b(t)x(t) + y(t) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = \alpha x(\eta) \end{cases} \tag{2.4}$$

has a unique solution. Moreover, this solution can be expressed in the form

$$u(t) = \int_0^1 \gamma(t,s)p(s)y(s)ds, \tag{2.5}$$

where

$$\gamma(t,s) = G(t,s) + \frac{\alpha(\phi_1(t)+\phi_2(t))}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))}G(\eta,s),$$

$$G(t,s) = \frac{1}{\phi_1'(0)}\begin{cases}\phi_1(t)\phi_2(s), & t \le s,\\ \phi_1(s)\phi_2(t), & s \le t,\end{cases} \quad p(s) = \exp\left(\int_0^s a(\theta)d\theta\right).$$

Proof. It is not difficult to testify the above conclusions. □

Lemma 2.5. *For any $(t,s) \in [0,1]\times[0,1]$, we have $G(t,s) \ge 0$, $G(1-t,1-s) = G(t,s)$, $\phi_1(t)\phi_2(t)G(s,s) \le G(t,s) \le G(s,s)$, $\gamma(t,s) \le \tilde{\gamma}(s)$, where*

$$\tilde{\gamma}(s) = G(s,s) + \frac{2\alpha}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))}G(\eta,s).$$

Proof. Following the definition for Green's function $G(t,s)$, Lemma 2.1 and Lemma 2.2, we easily obtain the above conclusions. □

Now for any $h(t) \in C[0,1]$ with $h(t) \ge 0,\ t \in [0,1]$, we consider the following BVP

$$\begin{cases} x''(t) + a(t)x'(t) - q(t)x(t) + f(t,h(t)) + q(t)h(t) = 0, & 0 < t < 1,\\ x(0) = x(1) = \alpha x(\eta). \end{cases} \tag{2.6}$$

By Lemma 2.4, if the BVP (2.6) has a solution, then the solution can be expressed by

$$u(t) = \int_0^1 \gamma(t,s)p(s)(f(s,h(s)) + q(s)h(s))ds.$$

Now for $h(t) \in C[0,1]$ with $h(t) \ge 0,\ t \in [0,1]$, define an operator

$$Ah(t) = \int_0^1 \gamma(t,s)p(s)(f(s,h(s)) + q(s)h(s))ds.$$

Let

$$P = \{x \in E \mid x(t) \ge 0, x(t) \text{ is symmetric on } [0,1], x(t) \ge \phi_1(t)\phi_2(t)\|x\|, \}.$$

It is easy to show that P is a cone in E.

Lemma 2.6. *Assume that (H_1), (H_2) and (H_3) hold. Then for any $0 < R_1 < R_2 < +\infty$, $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous, where $\Omega_i = \{x \in E \mid \|x\| < R_i\}, i = 1,2$.*

Proof. First, for any $x \in P$, by (H_1) and (H_2), $Ax(t) \geq 0$ for any $t \in [0,1]$.

$$
\begin{aligned}
&Ax(1-t)\\
&= \int_0^1 \gamma(1-t,s)p(s)(f(s,x(s)) + q(s)x(s))ds\\
&= \int_0^1 G(1-t,s)p(s)(f(s,x(s)) + x(s))ds\\
&\quad + \int_0^1 \frac{\alpha(\phi_1(1-t)+\phi_2(1-t))}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))} G(\eta,s)p(s)(f(s,x(s)) + q(s)x(s))ds\\
&= \int_0^1 G(1-t,1-s)p(1-s)(f(1-s,x(1-s)) + q(1-s)x(1-s))ds\\
&\quad + \int_0^1 \frac{\alpha(\phi_1(t)+\phi_2(t))}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))} G(\eta,s)p(s)(f(s,x(s)) + q(s)x(s))ds\\
&= \int_0^1 G(t,s)p(s)(f(s,x(s)) + q(s)x(s))ds\\
&\quad + \int_0^1 \frac{\alpha(\phi_1(t)+\phi_2(t))}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))} G(\eta,s)p(s)(f(s,x(s)) + q(s)x(s))ds\\
&= Ax(t).
\end{aligned}
$$

So, $Ax(t)$ is symmetric on [0,1].

Next, for any $x \in E, t \in [0,1]$, we get

$$
\begin{aligned}
Ax(t) &= \int_0^1 \gamma(t,s)p(s)(f(s,x(s)) + q(s)x(s))ds\\
&\leq \int_0^1 \left(G(s,s) + \frac{2\alpha}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))} G(\eta,s) \right)\\
&\quad \times p(s)(f(s,x(s)) + q(s)x(s))ds.
\end{aligned}
$$

Thus,

$$
\|Ax(t)\| \leq \int_0^1 \tilde{\gamma}(s)p(s)(f(s,x(s)) + q(s)x(s))ds.
$$

And also,

$$
\begin{aligned}
Ax(t) &\geq \int_0^1 \left(\phi_1(t)\phi_2(t)G(s,s) + \frac{\alpha(\phi_1(t)\phi_2(t)+\phi_2(t)\phi_1(t))}{1-\alpha(\phi_1(\eta)+\phi_2(\eta))} G(\eta,s) \right)\\
&\quad \times p(s)(f(s,x(s)) + q(s)x(s))ds\\
&= \phi_1(t)\phi_2(t) \int_0^1 \tilde{\gamma}(s)p(s)(f(s,x(s)) + q(s)x(s))ds\\
&\geq \phi_1(t)\phi_2(t)\|Ax\|.
\end{aligned}
$$

Finally, we will show that $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous. Suppose $\{x_n\}_{n\geq 1} \subseteq P\cap(\bar{\Omega}_2\setminus\Omega_1)$ and $x_0 \in P\cap(\bar{\Omega}_2\setminus\Omega_1)$ with $\lim_{n\to\infty} x_n = x_0$. Then we obtain $R_1\phi_1(t)\phi_2(t) \leq x_n(t) \leq R_2$ for any $t \in [0,1]$ and $n \geq 0$.
By virtue of

$$\begin{aligned}\max_{t\in[0,1]} Ax_n(t) &= \max_{t\in[0,1]} \int_0^1 \gamma(t,s)p(s)(f(s,x_n(s)) + q(s)x_n(s))ds \\ &\leq \max_{t\in[0,1]} \int_0^1 \gamma(t,s)p(s)f(s,R_1\phi_1(s)\phi_2(s))ds \\ &\quad + R_2 \max_{t\in[0,1]} \int_0^1 \gamma(t,s)p(s)q(s)ds < +\infty,\end{aligned}$$

we have from the Lebesgue dominated convergence theorem that $\|Ax_n - Ax_0\| \to 0, \quad (n\to\infty)$. Thus, $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ is continuous.

Next, we will show that $A(P \cap (\bar{\Omega}_2 \setminus \Omega_1))$ is relatively compact.
For any $x \in P\cap(\bar{\Omega}_2\setminus\Omega_1)$, we have $R_1\phi_1(t)\phi_2(t) \leq x(t) \leq R_2$ for any $t \in [0,1]$.

$$\begin{aligned}Ax(t) &\leq \max t \in [0,1] \int_0^1 \gamma(t,s)p(s)f(s,R_1\phi_1(s)\phi_2(s))ds \\ &\quad + R_2 \max_{t\in[0,1]} \int_0^1 \gamma(t,s)p(s)q(s)ds < +\infty, \qquad t\in[0,1],\end{aligned}$$

which means that $A(P \cap (\bar{\Omega}_2 \setminus \Omega_1))$ is bounded.

Now, for any $t_1, t_2 \in [0,1]$ and $x \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ we get

$$\begin{aligned}&|Ax(t_1) - Ax(t_2)| \\ &\leq \int_0^1 |\gamma(t_1,s) - \gamma(t_2,s)|p(s)(f(s,x(s)) + q(s)x(s))ds \\ &\leq \int_0^1 |\gamma(t_1,s) - \gamma(t_2,s)|p(s)(f(s,R_1\phi_1(s)\phi_2(s)) + R_2q(s))ds.\end{aligned}$$

Thus, we have that $A(P \cap (\bar{\Omega}_2 \setminus \Omega_1))$ is equicontinuous on [0, 1]. So, by the Arzela-Ascoli theorem we obtain that $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous. □

Lemma 2.7. [2] *Let Ω_1, Ω_2 be bounded open subset in a real Banach space E, P be a cone of E, $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|,\ u \in P \cap \partial\Omega_1,$ *and* $\|Au\| \geq \|u\|,\ u \in P\cap\partial\Omega_2$;

or

(ii) $\|Au\| \geq \|u\|,\ u \in P \cap \partial\Omega_1,$ *and* $\|Au\| \leq \|u\|,\ u \in P\cap\partial\Omega_2$.

Then A has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.8. [2] *Let X be a Banach space, and $P \subset X$ be a cone in E. Assume Ω is a bounded open subset of E with $\theta \in \Omega$, and let $A : P \cap \bar{\Omega} \to P$ be a completely continuous operator with $Ax \neq x$ for any $x \in P \cap \partial\Omega$.*

(i) *If $\|x\| \leq \|Ax\|,\ x \in P \cap \partial\Omega$, then $i(A, P \cap \Omega, P) = 0$;*

(ii) *If $Tu \neq \mu u$ for any $u \in P \cap \partial\Omega,\ \mu > 1$, then $i(A, P \cap \Omega, P) = 1$.*

Lemma 2.9. [7] *Let P be a cone of a real Banach space E and $B : P \to P$ be a completely continuous operator. Assume that B is order-preserving and positively homogeneous of degree 1 and that there exist $v \in P \setminus \{\theta\}, \lambda > 0$ such that $Bv \geq \lambda v$. Then $r(B) \geq \lambda$, where $r(B)$ denotes the spectral radius of B.*

3. Main results

In this section, we will present several main results.

Let

$$Bh(t) = \int_0^1 \gamma(t, s)p(s)q(s)h(s)ds.$$

Obviously, $B : P \to P$ is a completely continuous operator.

Theorem 3.1. *Assume that (H_1), (H_2) and (H_3) hold, in addition, suppose that*

(H4) *There is a constant $\rho_1 > 0,\ 0 < \alpha < \beta < 1$, such that*

$$\min_{\sigma\rho_1 \leq h(t) \leq \rho_1,\ \alpha \leq t \leq \beta} (f(t, h(t)) + q(t)h(t)) \geq \wedge_1 \rho_1,$$

where $\wedge_1 = \max_{\alpha \leq t \leq \beta} \left(\int_\alpha^\beta \gamma(t, s)p(s)ds \right)^{-1}$, $\sigma = \min_{t \in [\alpha,\beta]} \phi_1(t)\phi_2(t)$.

If $\|B\| < 1$, then the BVP (1.1) has at least one positive symmetric solution.

Proof. By the assumption (H_4), let

$$\Omega_1 = \{h \in P \mid \|h\| < \rho_1\}.$$

If $h \in \Omega_1$, then $\rho_1\phi_1(t)\phi_2(t) \leq h(t) \leq \rho_1$. Obviously, for any $\alpha \leq t \leq \beta$, we have $\sigma\rho_1 \leq h(t) \leq \rho_1$ and

$$\begin{aligned} \|Ah\| \geq Ah(t) &= \int_0^1 \gamma(t, s)p(s)(f(s, h(s)) + q(s)h(s))ds \\ &\geq \int_\alpha^\beta \gamma(t, s)p(s)(f(s, h(s)) + q(s)h(s))ds \\ &\geq \int_\alpha^\beta \gamma(t, s)p(s)ds \cdot \wedge_1 \rho_1 \geq \rho_1 = \|h\|. \end{aligned}$$

Let $\rho_2 > \max\left\{r, \rho_1, \frac{1}{1-\|B\|}\max_{t\in[0,1]}\int_0^1 \gamma(t,s)p(s)f(s, r\phi_1(s)\phi_2(s))ds\right\}$ and $\Omega_2 = \{h \in P \mid \|h\| < \rho_2\}$. Then for any $t \in [0,1]$, we get

$$\begin{aligned} Ah(t) &= \int_0^1 \gamma(t,s)p(s)f(s,h(s))ds + Bh(t) \\ &\leq \int_0^1 \gamma(t,s)p(s)f(s, r\phi_1(s)\phi_2(s))ds + Bh(t). \end{aligned}$$

Thus, for any $t \in [0,1]$,

$$\begin{aligned} |Ah(t)| &\leq \max_{t\in[0,1]}\int_0^1 \gamma(t,s)p(s)f(s, r\phi_1(s)\phi_2(s))ds + \|B\| \cdot \rho_2 \\ &\leq \rho_2(1-\|B\|) + \|B\|\cdot\rho_2 = \rho_2 = \|h\|, \end{aligned}$$

that is, $\|Ah\| \leq \|h\|$. So, by Lemma 2.7, the operator A has a fixed point $h^* \in \bar{\Omega}_2 \setminus \Omega_1$, further, $h^*(t)$ is one positive symmetric solution of the BVP(1.1). □

Corollary 3.2. *Assume that* (H_1), (H_2) *and* (H_3) *are satisfied, and there exists* $0 < \alpha < \beta < 1$, *such that* $\liminf_{h\to 0^+}\min_{\alpha\leq\beta}(f(t,h) + q(t)h) > 0$. *If* $\|B\| < 1$, *then the BVP (1.1) has at least one positive symmetric solution.*

Proof. Obviously, the condition $\liminf_{h\to 0^+}\min_{\alpha\leq\beta}(f(t,h)+q(t)h) > 0$ can implies that the assumption (H_4) holds, hence this conclusion holds. □

Theorem 3.3. *Assume that* (H_1), (H_2) *and* (H_3) *hold. If there exists* $\rho_2 > 0$, $0 < \alpha < \beta < 1$, *such that*

$$\min_{\sigma\rho_1 \leq h(t)\leq \rho_1,\ \alpha\leq t\leq\beta}(f(t,h(t)) + q(t)h(t)) > \wedge_1\rho_2,$$

where $\wedge_1 = \max_{\alpha\leq t\leq\beta}\left(\int_\alpha^\beta \gamma(t,s)p(s)ds\right)^{-1}$, $\sigma = \min_{t\in[\alpha,\beta]}\phi_1(t)\phi_2(t)$, *then as* $r(B) < 1$ *and there exists* $R > 0$ *such that*

$$\max_{t\in[0,1]}\int_0^1 \gamma(t,s)p(s)f(s, R\phi_1(s)\phi_2(s))ds \leq 0,$$

the BVP(1.1) has at least one positive symmetric solution.

Proof. Let $\Omega_1 = \{h \in P \mid \|h\| < \rho_2\}$. Then for any $\rho > \rho_2$ and $\Omega_\rho = \{h \in P \mid \|h\| < \rho\}$, by Lemma 2.6 we obtain $A : \bar{\Omega}_\rho \setminus \Omega_1 \to P$ is completely continuous. By the extended theorem of completely continuous operators, there exists a completely continuous function $\tilde{A} : \bar{\Omega}_\rho \to P$ such that $\tilde{A}h = Ah$ for any $h \in \bar{\Omega}_\rho \setminus \Omega_1$.

Next, we show $\tilde{A}h \neq h$ for any $h \in \partial\Omega_1$.

In fact, if not, then there exists some $h \in \partial\Omega_1$ such that $\tilde{A}h = h$. For any

$t \in [\alpha, \beta]$, we get

$$\|h\| \geq h(t) = \tilde{A}h(t) = Ah(t) \geq \int_\alpha^\beta \gamma(t,s)p(s)(f(s,h(s)) + q(s)h(s))ds$$
$$> \int_\alpha^\beta \gamma(t,s)p(s)ds \cdot \wedge_1 \rho_2 \geq \rho_2 = \|h\|,$$

which gets a contradiction. Modeling the proof of the Theorem 3.1, we obtain $\|\tilde{A}h\| \geq \|h\|$ for any $h \in \partial\Omega_1$. Hence, by Lemma 2.8 $i(\tilde{A}, \Omega_1, P) = 0$.

Let $\rho_3 > \max\{\rho_2, R\}$ and $\Omega_2 = \{h \in P \mid \|h\| < \rho_3\}$.

Finally, we show $\tilde{A}h = \mu h, h \in \partial\Omega_2 \Rightarrow \mu < 1$.

If not, there exists $\mu_0 \geq 1$, $h_0 \in \partial\Omega_2$ such that $\tilde{A}h_0 = \mu_0 h_0$, then for any $t \in [0,1]$ we obtain

$$h_0(t) = \frac{1}{\mu_0}\tilde{A}h_0 = \frac{1}{\mu_0}Ah_0 \leq \int_0^1 \gamma(t,s)p(s)f(s,h_0(s))ds + Bh_0(t)$$
$$\leq \int_0^1 \gamma(t,s)p(s)f(s,R\phi_1(s)\phi_2(s))ds + Bh_0(t)$$
$$\leq Bh_0(t).$$

Thus, by Lemma 2.9 $r(B) \geq 1$ which is a contradiction for $r(B) < 1$.

So, by Lemma 2.8 $i(\tilde{A}, \Omega_2, P) = 1$. Hence the operator A has a fixed point on $\bar{\Omega}_2 \setminus \Omega_1$, further, the BVP (1.1) has at least one positive symmetric solution. □

Remark 3.4. *Even if $a(t) \equiv 0$ for any $t \in [0,1]$, the results obtained in this paper are also new.*

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