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GENERALIZED COMMON FIXED POINT RESULTS FOR THREE SELF-MAPPINGS IN COMPLEX VALUED b-METRIC SPACES

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Abstract. In this paper, some generalized common fixed point results have been proved for three self maps in the context of complex valued b-metric spaces. Our results substantially improve and generalize a number of known results.

1. Introduction

The Banach contraction principle [3] is the first important result in metric fixed point theory. Over the years, it has been generalized in different directions and spaces by several authors. These generalization were made either by using the contractive condition or by imposing some additional conditions on an ambient space. There have been a number of generalizations of metric spaces such as, rectangular metric spaces, pseudo metric spaces, fuzzy metric

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spaces, quasi metric spaces, cone metric spaces, D-metric spaces (see [6], [7], [8], [9], [18]). Czerwik in [5] introduced the concept of a b-metric space. Recently, Azam et al. [1] introduced the notion of complex valued metric spaces and established common fixed point theorems. Subsequently, many authors have studied the existence and uniqueness of the fixed point and common fixed point of self-mappings in view of different type of contractive conditions in complex valued-metric spaces. Some of these results are described in [4, 12, 13, 14, 16, 17, 19]. In 2013, Rao et al. [15] introduced the concept of complex valued b-metric space which is a generalization of complex valued metric spaces. Since then, Mukheimer [10, 11] obtained some common fixed point theorems in complex valued b-metric spaces.

The aim of this paper is to obtain some generalized common fixed point results for three self-mappings in complex valued b-metric spaces. The obtained results are generalizations of recent results proved by Mukheimer [10, 11], Mohanta et al. [12] , Rouzkard et al. [16], Sintunavarat et al. [17, 19].

2. Basic facts and definitions

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. Thus $z_1 \precsim z_2$ if one of the following holds:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(C_3) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(C_4) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C_1), (C_2)$ and (C_3) is satisfied and we will write $z_1 \prec z_2$ if only (C_3) is satisfied.

Remark 2.1. We obtained that the following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \precsim bz$ for all $z \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 2.2. ([1]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(i) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 2.3. ([17]) Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|$, where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

Definition 2.4. ([15]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(*i*) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(*ii*) $d(x, y) = d(y, x)$;
(*iii*) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 2.5. ([15]) Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 2.6. ([15]) Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.
(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}, B(x, r) \bigcap (A - \{x\}) \neq \phi$.
(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of a set A.
(iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.
(v) A sub-basis for a Hausdorff topology τ on X is a family

$$F = \{B(x, r) : x \in X \quad \text{and} \quad 0 \prec r\}.$$

Definition 2.7. ([15]) Let (X, d) be a complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be complete.

Lemma 2.8. ([15]) *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.9. ([15]) *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if*

$$|d(x_n, x_{n+m})| \to 0$$

as $n \to \infty$, where $m \in \mathbb{N}$.

Lemma 2.10. ([2]) *Let X be a nonempty set and the mappings $S,T,f : X \to X$ have a unique point of coincidence $v \in X$. If (S,f) and (T,f) are weakly compatible then S,T and f have a unique common fixed point.*

Now we are ready to state and prove our main result.

3. Generalized common fixed point results

Theorem 3.1. *Let (X,d) be a complex valued b-metric space with the coefficient $s \geq 1$ and let $f,S,T : X \to X$. Suppose that there exist mappings $\lambda, \mu, \gamma : X \to [0,1)$ such that for all $x,y \in X$,*

(i) $\lambda(Sx) \leq \lambda(fx)$, $\mu(Sx) \leq \mu(fx)$ *and* $\gamma(Sx) \leq \gamma(fx)$,
(ii) $\lambda(Tx) \leq \lambda(fx)$, $\mu(Tx) \leq \mu(fx)$ *and* $\gamma(Tx) \leq \gamma(fx)$,
(iii) $s\lambda(fx) + \mu(fx) + \gamma(fx) < 1$,
(iv) $d(Sx,Ty) \precsim \lambda(fx)d(fx,fy) + \frac{\mu(fx)d(fx,Sx)d(fy,Ty)+\gamma(fx)d(fy,Sx)d(fx,Ty)}{1+d(fx,fy)}$.

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is complete, then f,S and T have a unique point of coincidence. Moreover, if (S,f) and (T,f) are weakly compatible, then f,S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Choose a point $x_1 \in X$ such that $fx_1 = Sx_0$ which is possible since $S(X) \subseteq f(X)$. Also, we may choose a point $x_2 \in X$ satisfying $fx_2 = Tx_1$ since $T(X) \subseteq f(X)$. Continuing in this way, we can construct a sequence $\{x_n\}$ in $f(X)$ such that $fx_{2n+1} = Sx_{2n}$ and $fx_{2n+2} = Tx_{2n+1}$, for $n \geq 0$.

Now, we show that the sequence $\{fx_n\}$ is Cauchy. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (iv), we get

$$\begin{aligned} d(fx_{2n+1}, fx_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \lambda(fx_{2n})d(fx_{2n}, fx_{2n+1}) \\ &\quad + \frac{\mu(fx_{2n})d(fx_{2n}, Sx_{2n})d(fx_{2n+1}, Tx_{2n+1})}{1 + d(fx_{2n}, fx_{2n+1})} \\ &\quad + \frac{\gamma(fx_{2n})d(fx_{2n+1}, Sx_{2n})d(fx_{2n}, Tx_{2n+1})}{1 + d(fx_{2n}, fx_{2n+1})} \end{aligned}$$

$$
\begin{aligned}
&= \lambda(Tx_{2n-1})d(fx_{2n}, fx_{2n+1}) \\
&\quad + \frac{\mu(Tx_{2n-1})d(fx_{2n}, fx_{2n+1})d(fx_{2n+1}, fx_{2n+2})}{1 + d(fx_{2n}, fx_{2n+1})} \\
&\quad + \frac{\gamma(Tx_{2n-1})d(fx_{2n+1}, fx_{2n+1})d(fx_{2n}, fx_{2n+2})}{1 + d(fx_{2n}, fx_{2n+1})} \\
&\precsim \lambda(fx_{2n-1})d(fx_{2n}, fx_{2n+1}) \\
&\quad + \frac{\mu(fx_{2n-1})d(fx_{2n}, fx_{2n+1})d(fx_{2n+1}, fx_{2n+2})}{1 + d(fx_{2n}, fx_{2n+1})} \\
&= \lambda(Sx_{2n-2})d(fx_{2n}, fx_{2n+1}) \\
&\quad + \frac{\mu(Sx_{2n-2})d(fx_{2n}, fx_{2n+1})d(fx_{2n+1}, fx_{2n+2})}{1 + d(fx_{2n}, fx_{2n+1})} \\
&\precsim \lambda(fx_{2n-2})d(fx_{2n}, fx_{2n+1}) \\
&\quad + \frac{\mu(fx_{2n-2})d(fx_{2n}, fx_{2n+1})d(fx_{2n+1}, fx_{2n+2})}{1 + d(fx_{2n}, fx_{2n+1})} \\
&\vdots \\
&\precsim \lambda(fx_0)d(fx_{2n}, fx_{2n+1}) \\
&\quad + \frac{\mu(fx_0)d(fx_{2n}, fx_{2n+1})d(fx_{2n+1}, fx_{2n+2})}{1 + d(fx_{2n}, fx_{2n+1})}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\mid d(fx_{2n+1}, fx_{2n+2}) \mid &\leq \lambda(fx_0) \mid d(fx_{2n}, fx_{2n+1}) \mid \\
&\quad + \mu(fx_0) \mid d(fx_{2n+1}, fx_{2n+2}) \mid\mid \left(\frac{d(fx_{2n}, fx_{2n+1})}{1 + d(fx_{2n}, fx_{2n+1})} \right) \mid \\
&\leq \lambda(fx_0) \mid d(fx_{2n}, fx_{2n+1}) \mid \\
&\quad + \mu(fx_0) \mid d(fx_{2n+1}, fx_{2n+2}) \mid
\end{aligned}
$$

which implies that

$$
|d(fx_{2n+1}, fx_{2n+2})| \leq \frac{\lambda(fx_0)}{1 - \mu(fx_0)} |d(fx_{2n}, fx_{2n+1})|. \tag{3.1}
$$

Similarly, we obtain

$$
|d(fx_{2n+2}, fx_{2n+3})| \leq \frac{\lambda(fx_0)}{1 - \mu(fx_0)} |d(fx_{2n+1}, fx_{2n+2})|. \tag{3.2}
$$

Putting $h := \frac{\lambda(fx_0)}{1-\mu(fx_0)}$ for all $n \geq 0$, then we have

$$\begin{aligned}|d(fx_{2n+1}, fx_{2n+2})| &\leq h|d(fx_{2n}, fx_{2n+1})| \\ &\leq h^2|d(fx_{2n-1}, fx_{2n})| \\ &\vdots \\ &\leq h^{2n+1}|d(fx_0, fx_1)|.\end{aligned}$$

That is,

$$|d(fx_{n+1}, fx_{n+2})| \leq h^{n+1}|d(fx_0, fx_1)|$$

or

$$|d(fx_n, fx_{n+1})| \leq h^n|d(fx_0, fx_1)|. \tag{3.3}$$

Thus for any $m > n$ and $m, n \in \mathbb{N}$, since $sh = \frac{s\lambda(fx_0)}{1-\mu(fx_0)} < 1$, we have

$$\begin{aligned}d(fx_n, fx_m) &\precsim sd(fx_n, fx_{n+1}) + sd(fx_{n+1}, fx_m) \\ &\precsim sd(fx_n, fx_{n+1}) + s^2d(fx_{n+1}, fx_{n+2}) + s^2d(fx_{n+2}, fx_m) \\ &\vdots \\ &\precsim sd(fx_n, fx_{n+1}) + s^2d(fx_{n+1}, fx_{n+2}) \\ &\quad + \cdots + s^{m-n-1}d(fx_{m-2}, fx_{m-1}) + s^{m-n}d(fx_{m-1}, fx_m).\end{aligned}$$

By using (3.3), we get

$$\begin{aligned}d(fx_n, fx_m) &\precsim sh^nd(fx_0, fx_1) + s^2h^{n+1}d(fx_0, fx_1) \\ &\quad + \cdots + s^{m-n}h^{m-1}d(fx_0, fx_1) \\ &= \sum_{i=1}^{m-n} s^ih^{i+n-1}d(fx_0, fx_1).\end{aligned}$$

Therefore, we have

$$\begin{aligned}d(fx_n, fx_m) &\precsim \sum_{i=1}^{m-n} s^{i+n-1}h^{i+n-1}d(fx_0, fx_1) \\ &= \sum_{t=n}^{m-1} s^th^td(fx_0, fx_1) \\ &\precsim \sum_{t=n}^{\infty}(sh)^td(fx_0, fx_1) \\ &= \frac{(sh)^n}{1-sh}d(fx_0, fx_1).\end{aligned} \tag{3.4}$$

Hence,

$$|d(fx_n, fx_m)| \le \frac{(sh)^n}{1-sh}|d(fx_0, fx_1)| \to 0$$

as $m, n \to \infty$. This implies that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. By completeness of $f(X)$, there exists $u, v \in X$ such that $fx_n \to v = fu$. Now

$$\begin{aligned}
d(fu, Tu) &\precsim sd(fu, fx_{2n+1}) + sd(fx_{2n+1}, Tu) \\
&= sd(fu, fx_{2n+1}) + sd(Sx_{2n}, Tu) \\
&\precsim sd(fu, fx_{2n+1}) + s\lambda(fx_{2n})d(fx_{2n}, fu) \\
&\quad + \frac{s\mu(fx_{2n})d(fx_{2n}, Sx_{2n})d(fu, Tu)}{1 + d(fx_{2n}, fu)} \\
&\quad + \frac{s\gamma(fx_{2n})d(fu, Sx_{2n})d(fx_{2n}, Tu)}{1 + d(fx_{2n}, fu)} \\
&= sd(fu, fx_{2n+1}) + s\lambda(fx_{2n})d(fx_{2n}, fu) \\
&\quad + \frac{s\mu(fx_{2n})d(fx_{2n}, fx_{2n+1})d(fu, Tu)}{1 + d(fx_{2n}, fu)} \\
&\quad + \frac{s\gamma(fx_{2n})d(fu, fx_{2n+1})d(fx_{2n}, Tu)}{1 + d(fx_{2n}, fu)}
\end{aligned}$$

which implies that

$$\begin{aligned}
|d(fu, Tu)| &\le s|d(fu, fx_{2n+1})| + s\lambda(fx_0)|d(fx_{2n}, fu)| \\
&\quad + \frac{s\mu(fx_0)|d(fx_{2n}, fx_{2n+1})||d(fu, Tu)|}{|1 + d(fx_{2n}, fu)|} \\
&\quad + \frac{s\gamma(fx_0)|d(fu, fx_{2n+1})||d(fx_{2n}, Tu)|}{|1 + d(fx_{2n}, fu)|}.
\end{aligned}$$

Since $1 \le |1 + d(fx_{2n}, fu)|$, we get that

$$\begin{aligned}
|d(fu, Tu)| &\le s|d(fu, fx_{2n+1})| + s\lambda(fx_0)|d(fx_{2n}, fu)| \\
&\quad + s\mu(fx_0)|d(fx_{2n}, fx_{2n+1})||d(fu, Tu)| \\
&\quad + s\gamma(fx_0)|d(fu, fx_{2n+1})||d(fx_{2n}, Tu)|.
\end{aligned} \tag{3.5}$$

Taking the limit of (3.5) as $n \to \infty$, we get that $|d(fu, Tu)| = 0$, and hence $d(fu, Tu) = 0$. Therefore $fu = Tu = v$. Similarly, we can show that $fu = Su = v$. Thus, $fu = Su = Tu = v$ and so v becomes a common point of coincidence of f, S and T.

For uniqueness, let there exists another point $w(\neq v) \in X$ such that $fx = Sx = Tx = w$ for some $x \in X$. Then

$$\begin{aligned} d(v,w) &= d(Su,Tx) \\ &\precsim \lambda(fu)d(fu,fx) + \frac{\mu(fu)d(fu,Su)d(fx,Tx) + \gamma(fu)d(fx,Su)d(fu,Tx)}{1+d(fu,fx)} \\ &= \lambda(v)d(v,w) + \frac{\mu(v)d(v,v)d(w,w) + \gamma(v)d(w,v)d(v,w)}{1+d(v,w)}. \end{aligned}$$

Since $(1 + d(v,w)) \succsim d(v,w)$ therefore $d(v,w) \precsim \lambda(v)d(v,w) + \gamma(v)d(v,w)$ which implies that $|d(v,w)| \leq (\lambda+\gamma)(v)|d(v,w)|$. Since $0 \leq (\lambda+\gamma)(v) < 1$, it follows that $|d(v,w)| = 0$ and so $v = w$. If (S,f) and (T,f) are weakly compatible then by Lemma (2.10), f, S and T have a unique common fixed point in X. This completes the proof. □

As an application of Theorem 3.1, we have the following results.

Corollary 3.2. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S,T : X \to X$. If there exist mappings $\lambda, \mu, \gamma : X \to [0,1)$ such that for all $x,y \in X$,*

(i) $\lambda(Sx) \leq \lambda(x)$, $\mu(Sx) \leq \mu(x)$ *and* $\gamma(Sx) \leq \gamma(x)$;
(ii) $\lambda(Tx) \leq \lambda(x)$, $\mu(Tx) \leq \mu(x)$ *and* $\gamma(Tx) \leq \gamma(x)$;
(iii) $s\lambda(x) + \mu(x) + \gamma(x) < 1$;
(iv) $d(Sx,Ty) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,Sx)d(y,Ty)+\gamma(x)d(y,Sx)d(x,Ty)}{1+d(x,y)}$.

Then S and T have a unique common fixed point in X.

Proof. The result follows from Theorem 3.1 by taking $f = I$, the identity mapping. □

Corollary 3.3. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S,T : X \to X$ be mappings satisfying*

$$d(Sx,Ty) \precsim \lambda' d(x,y) + \frac{\mu' d(x,Sx)d(y,Ty) + \gamma' d(y,Sx)d(x,Ty)}{1+d(x,y)}$$

for all $x,y \in X$, where λ', μ', γ' are nonnegative real numbers with $s\lambda' + \mu' + \gamma' < 1$. Then S and T have a unique common fixed point.

Proof. We can prove the result by applying Theorem 3.1 with $\lambda(x) = \lambda'$, $\mu(x) = \mu'$, $\gamma(x) = \gamma'$ and $f = I$. □

Corollary 3.4. *Let (X,d) be a complex valued b-metric space with the coefficient $s \geq 1$ and let $f,T : X \to X$ be such that $T(X) \subseteq f(X)$ and $f(X)$ is complete. If there exist mappings $\lambda, \mu, \gamma : X \to [0,1)$ such that for all $x,y \in X$,*

(i) $\lambda(Tx) \leq \lambda(fx)$, $\mu(Tx) \leq \mu(fx)$ *and* $\gamma(Tx) \leq \gamma(fx)$;

(ii) $s\lambda(fx) + \mu(fx) + \gamma(fx) < 1$;

(iii) $d(Tx,Ty) \precsim \lambda(fx)d(fx,fy) + \frac{\mu(fx)d(fx,Tx)d(fy,Ty)+\gamma(fx)d(fy,Tx)d(fx,Ty)}{1+d(fx,fy)}$.

Then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. The desired result can be obtained from Theorem 3.1 by taking $S = T$. □

Corollary 3.5. *Let (X, d) be a complex valued b-metric space with the coefficient $s \geq 1$ and let $f, T : X \to X$ satisfy*

$$d(Tx,Ty) \precsim \lambda^{'} d(fx,fy) + \frac{\mu' d(fx,Tx)d(fy,Ty) + \gamma^{'} d(fy,Tx)d(fx,Ty)}{1 + d(fx,fy)}$$

for all $x, y \in X$, where $\lambda^{'}, \mu^{'}, \gamma^{'}$ are nonnegative real numbers with $s\lambda^{'} + \mu^{'} + \gamma^{'} < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is complete, then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed points in X.

Proof. We can prove this result by applying Theorem (3.1) with $S = T$, $\lambda(x) = \lambda^{'}, \mu(x) = \mu^{'}, \gamma(x) = \gamma^{'}$. □

Corollary 3.6. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exist mappings $\lambda, \mu, \gamma : X \to [0, 1)$ such that for all $x, y \in X$,*

(i) $\lambda(Tx) \leq \lambda(x), \mu(Tx) \leq \mu(x)$ *and* $\gamma(Tx) \leq \gamma(x)$;

(ii) $s\lambda(x) + \mu(x) + \gamma(x) < 1$;

(iii) $d(Tx,Ty) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,Tx)d(y,Ty)+\gamma(x)d(y,Tx)d(x,Ty)}{1+d(x,y)}$.

Then T has a unique fixed point in X.

Proof. The proof of the Corollary follows from Theorem (3.1) by considering $S = T$ and $f = I$. □

Corollary 3.7. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exist mappings $\lambda, \mu, \gamma : X \to [0, 1)$ such that for all $x, y \in X$ and for some $n \in \mathbb{N}$,*

(i) $\lambda(T^n x) \leq \lambda(x), \mu(T^n x) \leq \mu(x)$ *and* $\gamma(T^n x) \leq \gamma(x)$;

(ii) $s\lambda(x) + \mu(x) + \gamma(x) < 1$;

(iii) $d(T^n x, T^n y) \precsim \lambda(x)d(x,y) + \frac{\mu(x)d(x,T^n x)d(y,T^n y)+\gamma(x)d(y,T^n x)d(x,T^n y)}{1+d(x,y)}$.

Then T has a unique fixed point.

Proof. From Corollary 3.6, we get T^n has a unique fixed point u. It follows from $T^n(Tu) = T(T^n u) = Tu$ that is, Tu is a fixed point of T^n. Therefore $Tu = u$ by the uniqueness of a fixed point of T^n and then u is also a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. □

Theorem 3.8. *Let (X, d) be a complex valued b-metric space with the coefficient $s \geq 1$ and let $f, S, T : X \to X$. If there exist mappings $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$,*

(i) $\lambda(Sx) \leq \lambda(fx)$ *and* $\mu(Sx) \leq \mu(fx)$*;*
(ii) $\lambda(Tx) \leq \lambda(fx)$ *and* $\mu(Tx) \leq \mu(fx)$*;*
(iii) $s\lambda(fx) + \mu(fx) < 1$*;*
(iv) $d(Sx, Ty) \precsim \lambda(fx)d(fx, fy) + \frac{\mu(fx)d(fx,Sx)d(fy,Ty)}{1+d(fx,fy)}$

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is complete, then f, S and T have a unique point of coincidence. Moreover, if (S, f) and (T, f) are weakly compatible, then f, S and T have a unique common fixed point in X.

Proof. We can prove this result by applying Theorem 3.1 with $\gamma = 0$. □

Corollary 3.9. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$. If there exist mappings $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$:*

(i) $\lambda(Sx) \leq \lambda(x)$ *and* $\mu(Sx) \leq \mu(x)$*;*
(ii) $\lambda(Tx) \leq \lambda(x)$ *and* $\mu(Tx) \leq \mu(x)$*;*
(iii) $s\lambda(x) + \mu(x) < 1$*;*
(iv) $d(Sx, Ty) \precsim \lambda(x)d(x, y) + \frac{\mu(x)d(x,Sx)d(y,Ty)}{1+d(x,y)}$.

Then S and T have a unique common fixed point in X.

Proof. The result follows from Theorem 3.8 by taking $f = I$, the identity mapping. □

Corollary 3.10. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ be mappings satisfying*

$$d(Sx, Ty) \precsim \lambda' d(x, y) + \frac{\mu' d(x, Sx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ', μ' are nonnegative real numbers with $s\lambda' + \mu' < 1$. Then S and T have a unique common fixed point.

Proof. . We can prove this result by applying Theorem 3.8 with $\lambda(x) = \lambda'; \mu(x) = \mu'$ and $f = I$. □

Corollary 3.11. *Let (X, d) be a complex valued b-metric space with the coefficient $s \geq 1$ and let $f, T : X \to X$ be such that $T(X) \subseteq f(X)$ and $f(X)$ is complete. If there exist mappings $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$,*

(i) $\lambda(Tx) \leq \lambda(fx)$ *and* $\mu(Tx) \leq \mu(fx)$;
(ii) $s\lambda(fx) + \mu(fx) < 1$;
(iii) $d(Tx, Ty) \precsim \lambda(fx)d(fx, fy) + \frac{\mu(fx)d(fx,Tx)d(fy,Ty)}{1+d(fx,fy)}$.

Then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. The result can be obtained from Theorem 3.8 by taking $S = T$. □

Corollary 3.12. *Let (X, d) be a complex valued b-metric space with the coefficient $s \geq 1$ and let $f, T : X \to X$ satisfy*

$$d(Tx, Ty) \precsim \lambda^{'} d(fx, fy) + \frac{\mu^{'} d(fx, Tx)d(fy, Ty)}{1 + d(fx, fy)}$$

for all $x, y \in X$, where $\lambda^{'}, \mu^{'}$ are nonnegative real numbers with $s\lambda^{'} + \mu^{'} < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is complete, then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof. We can prove this result by applying Theorem 3.8 with $S = T$, $\lambda(fx) = \lambda^{'}$ and $\mu(fx) = \mu^{'}$. □

Corollary 3.13. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exist mappings $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$,*

(i) $\lambda(Tx) \leq \lambda(x)$ *and* $\mu(Tx) \leq \mu(x)$;
(ii) $s\lambda(x) + \mu(x) < 1$;
(iii) $d(Tx, Ty) \precsim \lambda(x)d(x, y) + \frac{\mu(x)d(x,Tx)d(y,Ty)}{1+d(x,y)}$.

Then T has a unique fixed point in X.

Proof. The proof of the Corollary follows from Theorem 3.8 by putting $S = T$ and $f = I$. □

Corollary 3.14. *Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exist mappings $\lambda, \mu : X \to [0, 1)$ such that for all $x, y \in X$ and for some $n \in \mathbb{N}$,*

(i) $\lambda(T^n x) \leq \lambda(x)$ *and* $\mu(T^n x) \leq \mu(x)$;
(ii) $s\lambda(x) + \mu(x) < 1$;
(iii) $d(T^n x, T^n y) \precsim \lambda(x)d(x, y) + \frac{\mu(x)d(x,T^n x)d(y,T^n y)}{1+d(x,y)}$.

Then T has a unique fixed point.

Proof. The proof of the Corollary is similar as Corollary 3.7. □

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