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HARMONIC MULTIVALENT FUNCTIONS ASSOCIATED WITH AN EXTENDED GENERALIZED LINEAR OPERATOR OF NOOR-TYPE

Hiba F. Al-Janaby[1], F. Ghanim[2] and Muhammad Zaini Ahmad[3]

[1]Department of Mathematics, College of Science
University of Baghdad, Baghdad, Iraq
e-mail: fawzihiba@yahoo.com

[2]Department of Mathematics, College of Science
University of Sharjah, Sharjah, United Arab Emirates
e-mail: fgahmed@sharjah.ac.ae

[3]Institute of Engineering Mathematics
Universiti Malaysia Perlis, 02600 Arau Perlis, Malaysia
e-mail: mzaini@unimap.edu.my

Abstract. This paper introduces a new extended generalized linear operator of Noor-type of harmonic multivalent functions correlated with Fox-Wright generalized hypergeometric functions (FWGH). Moreover, a certain subclass of harmonic multivalent functions, which include this new formulation of the operator, is posed. In this study, an attempt has also been made to investigate several geometric properties such as coefficient condition and by showing the significance of this condition for the negative coefficient, growth bounds, extreme points, convolution property, convex linear combination, and a class-preserving integral operator.

1. Introduction

Harmonic functions have widely known to have a plethora of applications in the seemingly diverse fields of medicine, engineering, electronics, physics, aerodynamics, operation research and other branches of applied mathematics.

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[0]Corresponding author: Hiba F. Al-Janaby(fawzihiba@yahoo.com).

From the perspective of geometric function theory (GFT), in 1984, Clunie and Sheil-Small [7] initiated the study of these functions by introducing class $\mathbb{S}_{\mathbb{H}}$ of normalized harmonic univalent functions defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In their studies, they managed to find its geometric properties such as the classical coefficient bounds, growth and distortion theorems and covering theorems. Actually, the study of harmonic univalent functions is a natural generalization of analytic univalent functions, which is opened up a new direction for numerous prominent complex analysts to investigate many other subclasses of harmonic univalent functions. Some of their main contributions, one may refer to Sheil-Small [41], Silverman [42], Jahangiri and Ahuja [25], Murugusundaramoorthy and Uma [29], Ahuja [2], Pathak et al. [35], Ponnusamy et al. [36], Nagpal and Ravichandran [30], Porwal [37], Ibrahim et al. ([22],[23]), Hussain et al. [21] and others.

Recall that the class $\mathbb{S}_{\mathbb{H}}$ of harmonic functions $\omega = \rho + \bar{\sigma}$ that are univalent, sense-preserving in the open unit disc $\mathbb{D}$, and normalized by the conditions $\omega(0) = \omega'(0) - 1 = 0$, where the analytic part ρ and the co-analytic part σ are given as follows [7]:

$$\rho(z) = z + \sum_{\kappa=2}^{\infty} \alpha_\kappa z^\kappa, \quad \sigma(z) = \sum_{\kappa=1}^{\infty} \beta_\kappa z^\kappa, \quad |\beta_1| < 1.$$

Note that $\mathbb{S}_{\mathbb{H}}$ reduces to the class $\mathbb{S}$ of normalized analytic univalent functions if the co-analytic part σ is zero. Consequently, the function $\omega(z)$ for this class can be expressed as

$$\omega(z) = z + \sum_{\kappa=2}^{\infty} \alpha_\kappa \ z^\kappa.$$

Encouraged by wide investigation in the study of harmonic univalent functions, many authors attempted to apply this technique to multivalent function theory (which is the natural generalization of univalent function theory) too and brought to daylight many new facets of this field. In 2001, Jahangiri and Ahuja [25] defined the class $\mathbb{S}_{\mathbb{H}(p)}$ of harmonic multivalent ($p-$valent) functions, $\omega = h + \bar{g}$ that are sense-preserving in $\mathbb{D}$, and ρ and σ are of the formula

$$\rho(z) = z^p + \sum_{\kappa=p+1}^{\infty} \alpha_\kappa z^\kappa, \;\; \sigma(z) = \sum_{k=p}^{\infty} \beta_\kappa z^\kappa, \;\; |\beta_p| < 1, \; p \in \mathbb{N} = \{1, 2, ...\}. \quad (1.1)$$

Note that $\mathbb{S}_{\mathbb{H}(p)}$ reduces to the class $\mathbb{M}_p$ of normalized analytic multivalent functions if the co-analytic part σ is zero. Consequently, the function $\omega(z)$ for this class can be expressed as

$$\omega(z) = z^p + \sum_{\kappa=p+1}^{\infty} \alpha_\kappa z^\kappa. \tag{1.2}$$

Also, denoted by $\mathbb{NS}_{\mathbb{H}(p)}$ the subclass of $\mathbb{S}_{\mathbb{H}(p)}$ consisting of functions $\omega = \rho + \bar{\sigma}$, such that functions ρ and σ are of the form

$$\rho(z) = z^p - \sum_{\kappa=p+1}^{\infty} |\alpha_\kappa| z^\kappa, \;\; \sigma(z) = -\sum_{\kappa=p}^{\infty} |\beta_\kappa \; z^\kappa, \;\; |\beta_p| < 1, \; p \in \mathbb{N}. \tag{1.3}$$

Since then, several interesting subclasses of harmonic multivalent functions have been successfully accomplished by some renowned mathematicians researchers. For instance, Ghanim et al. [20], El-Ashwah and Aouf [14], Yaşar and Yalçn [44], Ezhilarasi et al. [17], Seoudy [40], Al-Janaby and Ahmad [4] and others.

Convolution (Hadamard product) is a mathematical operation on two functions ω_1 and ω_2 to produce a third function. It is an important tool in GFT for defining new subclasses and operators. The following convolution is given by Hadamard in 1899 [9]:

For two functions $\omega_\imath \in \mathbb{S}$ is given by $\omega_\imath(z) = z + \sum_{\kappa=2}^{\infty} \alpha_{\kappa,\imath} z^\kappa, \; \imath = 1, 2, \; z \in \mathbb{D}$, the convolution is denoted by $\omega_1 * \omega_2$ and defined as

$$(\omega_1 * \omega_2)(z) = z + \sum_{\kappa=2}^{\infty} \alpha_{\kappa,1} \, \alpha_{\kappa,2} \; z^\kappa.$$

In the harmonic function case, the convolution of two functions belong to the class $\mathbb{S}_\mathbb{H}$ was initially studied by Clunie and Sheil-Small [7] as: for two functions $\omega_\imath \in \mathbb{S}_\mathbb{H}, \; \imath = 1, 2, \; z \in \mathbb{D}$ given by

$$\omega_\imath(z) = \rho_\imath(z) + \overline{\sigma_\imath(z)} = z + \sum_{\kappa=2}^{\infty} \alpha_{\kappa,\imath} \; z^\kappa + \overline{\sum_{\kappa=1}^{\infty} \beta_{\kappa,\imath} \; z^\kappa}, \;\; |\beta_{1,1}| < 1, \, |\beta_{1,2}| < 1,$$

the convolution is denoted by $\omega_1 * \omega_2$ and defined as

$$(\omega_1 * \omega_2)(z) = z + \sum_{\kappa=2}^{\infty} \alpha_{\kappa,1} \;\; \alpha_{k,2} \, z^\kappa + \overline{\sum_{\kappa=1}^{\infty} \beta_{\kappa,1} \;\; \beta_{\kappa,2} \; z^\kappa}.$$

More generally, the convolution of two functions $\omega_\imath \in \mathbb{S}_{\mathbb{H}(p)}, \; \imath = 1, 2, \; z \in \mathbb{D}$ given by (see, [28])

$$\omega_\imath(z) = \rho_\imath(z) + \overline{\sigma_\imath(z)} = z^p + \sum_{\kappa=p+1}^{\infty} \alpha_{\kappa,\imath} \; z^\kappa + \overline{\sum_{\kappa=p}^{\infty} \beta_{\kappa,\imath} \; z^\kappa}, \; |\beta_{p,1}| < 1, \; |\beta_{p,2}| < 1,$$

the convolution is denoted by $\omega_1 * \omega_2$ and defined as

$$(\omega_1 * \omega_2)(z) = z^p + \sum_{\kappa=p+1}^{\infty} \alpha_{\kappa,1}\, \alpha_{\kappa,2}\ z^{\kappa} + \overline{\sum_{\kappa=p}^{\infty} \beta_{\kappa,1}\, \beta_{\kappa,2}\ z^{\kappa}}. \tag{1.4}$$

The study of operators plays an important role in mathematics, especially in GFT. Indeed, operators are used to obtain new subclasses and their properties. Integral, differential and convolution are three typical types of operators. Since the beginning of the previous century, many prominent authors have employed various methods to study the different types of integral operators. The first integral operator defined on the class of analytic functions $\mathbb{A}$ was introduced by Alexander [3], in 1915. Since then, several types of the well-known classical integral operators have been introduced by notable complex analysts, such as Miller et al. [27], Pascu and Pescar [34], Ong et al. [33], Frasin and Breaz [18], El-Ashwah et al. [15], Deniz [10], Rahrovi [38], Al-Jnaaby and et al. [5] and others. The convolution technique has a significant part in the development of this area. Several differential and integral operators can be written in terms of the convolution of certain analytic functions. Recall that the Pochhammer symbol $(\gamma)_\kappa$ defined by

$$(\gamma)_\kappa := \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} = \begin{cases} 1, & (\kappa = 0), \\ \gamma(\gamma+1)(\gamma+2)...(\gamma+n-1), & (\kappa \in \mathbb{N}). \end{cases} \tag{1.5}$$

By utilizing the technique of convolution, in 1975, Ruscheweyh [39] imposed the differential operator $D^\gamma\omega(z)$ as: for $\omega \in \mathbb{A}$, $\gamma > -1$ and $D^\gamma : \mathbb{A} \to \mathbb{A}$, is given by

$$D^\gamma\omega(z) = \frac{z}{(1-z)^{\gamma+1}} * \omega(z) = z + \sum_{\kappa=2}^{\infty} \frac{(\gamma+1)_{\kappa-1}}{(\kappa-1)!}\ \alpha_\kappa\ z^\kappa, \tag{1.6}$$

such that $D^0\omega(z) = \phi(z)$ and $D^1\omega(z) = z\omega'(z)$.

Corresponds to the Ruscheweyh operator, in 1999, Noor [31] introduced an integral operator $I_\gamma\omega(z)$ called the Noor Integral of $\gamma-$th order as: for $\omega \in \mathbb{A}$, $\gamma \in \mathbb{N}_0$ and $I^\gamma : \mathbb{A} \to \mathbb{A}$, is defined by

$$\begin{aligned} I_\gamma\omega(z) &= \omega_\gamma^{(-1)}(z) * \omega(z) \\ &= \left[\frac{z}{(1-z)^{\gamma+1}}\right]^{-1} * \omega(z) \\ &= z + \sum_{\kappa=2}^{\infty} \frac{\kappa!}{(\gamma+1)_{\kappa-1}}\ \alpha_\kappa\ z^\kappa, \end{aligned} \tag{1.7}$$

such that

$$\omega_\gamma(z) * \omega_\gamma^{(-1)}(z) = \frac{z}{(1-z)^2}.$$

Notice that $I_0\omega(z) = z\omega'(z)$ and $I_1\omega(z) = \omega(z)$. This operator, which is a very useful tool in defining several subclasses of analytic functions, has attracted the attention of many renowned mathematicians globally. Recently, most of operators have been extended to harmonic functions. For example, El-Ashwah, and Aouf [14] Yaşar and Yalçn [44], Hussain et al. [21], Seoudy [40] and others.

The hypergeometric functions have been included in GFT. In 1984, De Branges [11] used these special functions in proofing the important problem called the 'Bieberbachs conjecture' which gave renewed stimulation authors to develop and study various special functions. Since then, the theory of hypergeometric functions has been utilized to study numerous linear and nonlinear operators in the field.

The incomplete beta function is given as: for ξ and ζ be real or complex numbers with ζ other than $0, -1, -2, ...$, and

$$\varphi(\xi;\zeta;z) = \sum_{\kappa=0}^{\infty} \frac{(\xi)_\kappa}{(\zeta)_\kappa}\, z^\kappa = 1 + \frac{\xi}{\zeta}z + \frac{\xi(\xi+1)}{\zeta(\zeta+1)}\frac{z^2}{2!} + \cdots . \tag{1.8}$$

This post provided by (1.8) has a generalized form, namely the Gauss hypergeometric function: for ξ, η and ζ be real or complex numbers with ζ other than $0, -1, -2, ...$, and

$$\mathcal{F}(\xi,\eta;\zeta;z) = \sum_{\kappa=0}^{\infty} \frac{(\xi)_\kappa(\eta)_\kappa}{(\zeta)_\kappa(1)_\kappa}\, z^\kappa = 1 + \frac{\xi\eta}{\zeta}z + \frac{\xi(\xi+1)\eta(\eta+1)}{\zeta(\zeta+1)}\frac{z^2}{2!} + \cdots . \tag{1.9}$$

The well-known generalized hypergeometric function named the Fox-Wright generalized hypergeometric (FWGH) function is given as follows:(see for example [19], and [43])

$$\begin{aligned} &\tau\mathcal{W}_\varsigma[(\mu_1,\mathcal{A}_1)\cdots(\mu_\tau,\mathcal{A}_\tau);(\nu_1,\mathcal{B}_1)\cdots(\nu_\varsigma,\mathcal{B}_\varsigma);z] \\ &= \tau\mathcal{W}_\varsigma[(\mu_l,\mathcal{A}_l)_{1,\tau};(\nu_l,\mathcal{B}_l)_{1,\varsigma};z] = \sum_{\kappa=0}^{\infty} \frac{\prod_{l=1}^{\tau}\Gamma(\mu_l+\kappa\ \mathcal{A}_l)}{\prod_{l=1}^{\varsigma}\Gamma(\nu_l+\kappa\ \mathcal{B}_l)}\ \frac{z^\kappa}{\kappa!}, \end{aligned} \tag{1.10}$$

where $\mathcal{A}_l > 0$ $(l = 1,2,...,\tau)$, $\mathcal{B}_l > 0$ $(l = 1,2,...,\varsigma)$, $1 + \sum_{l=1}^{\tau}\mathcal{A}_l - \sum_{l=1}^{\varsigma}\mathcal{B}_l \geq 0$, $\mu_l + \kappa\ \mathcal{A}_l \neq 0,-1,...$ $(l = 1,2,..\tau;\ \kappa = 0,1,2,...)$, $\nu_l + \kappa\ \mathcal{B}_l \neq 0,-1,...$ $(l = 1,2,..\varsigma;\ \kappa = 0,1,...)$ and $z \in \mathbb{C}$. The condition $1 + \sum_{l=1}^{\tau}\mathcal{A}_l - \sum_{l=1}^{\varsigma}\mathcal{B}_l \geq 0$ is essential so that the series in (1.10) is absolutely convergent for all $z \in \mathbb{C}$, and is an entire function of z (for details, see [26]).

Special case of FWGH defined in (1.10) which is given as: if $\mathcal{A}_l = 1,\ (l = 1, 2, ..., \tau)$, $\mathcal{B}_l = 1\ (l = 1, 2, ..., \varsigma),\ \tau \leq \varsigma + 1$ and

$$\Omega = \left(\prod_{l=1}^{\varsigma} \Gamma(\nu_l)\right) \left(\prod_{l=1}^{\tau} \Gamma(\mu_l)\right)^{-1}, \tag{1.11}$$

then

$$\Omega\ {}_\tau\mathcal{W}_\varsigma[(\mu_l, 1)_{1,\tau}; (\nu_l, 1)_{1,\varsigma}; z] = {}_\tau\mathcal{F}_\varsigma[(\mu_l, ...\mu_\tau; \nu_l, ...\nu_\varsigma; z],$$

where ${}_\tau\mathcal{F}_\varsigma[(\mu_l, ...\mu_\tau; \nu_l, ...\nu_\varsigma; z]$ is the generalized hypergeometric function, [12]. Other special cases of FWGH were introduced in [26]. Although many complex analysts have investigated the connections between the well-established theory of analytic univalent (or multivalent) functions and hypergeometric functions, the corresponding connections between the newly merging theory of harmonic univalent functions and hypergeometric functions have not been explored.

In 2004, Ahuja together with Silverman [1] discovered interesting corresponding studies on connections between hypergeometric functions and harmonic univalent functions. Recently, the connections between FWGH and harmonic functions have investigated and studied by several authors.

Some previous studies that deal with hypergeometric function and FWGH will be mentioned:

In 2004, Cho-Kwon-Srivastava [8], considered the operator $I_p^\gamma(\xi, \zeta)$ as:

$$\begin{aligned} I_p^\gamma(\xi, \zeta)\omega(z) &= \left[z\varphi_p^{-1}(\xi, \zeta; z)\right] * \omega(z) \\ &= z^p + \sum_{\kappa=p+1}^{\infty} \frac{(\zeta)_{\kappa-p}(\gamma+p)_{\kappa-p}}{(\xi)_{\kappa-p}(\kappa-p)!}\, \alpha_n z^\kappa, \end{aligned} \tag{1.12}$$

where $\xi, \zeta \in \mathbb{R}\backslash\mathbb{Z}_0^-,\ \gamma > -p,\ p \in \mathbb{N},\ \omega \in \mathbb{M}_p$ and $\varphi_p^{-1}(\xi, \zeta; z)$ is such that

$$\varphi_p^{-1}(\xi, \zeta; z) * \varphi_p(\xi, \zeta; z) = \frac{z^p}{(1-z)^{\gamma+p}}.$$

In 2004, by using FWGH , Dziok and Raina [13] considered the following linear operator $W[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}]\omega(z)$ on $\mathbb{S}$:

$$W(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}]\ \omega(z) = z + \sum_{\kappa=2}^{\infty} \Omega\left(\vartheta_\kappa\ [\mu_l, \nu_l]\right)\ \alpha_\kappa\, z^\kappa,$$

$$\vartheta_\kappa[\mu_l,\nu_l]=\frac{\Gamma(\mu_1+(\kappa-1)\mathcal{A}_1)\Gamma(\mu_2+(\kappa-1)\mathcal{A}_2)\cdots\Gamma(\mu_\tau+(\kappa-1)\mathcal{A}_\tau)}{\Gamma(\nu_1+(\kappa-1)\mathcal{B}_1)\Gamma(\nu_2+(\kappa-1)\mathcal{B}_2)\cdots\Gamma(\nu_\varsigma+(\kappa-1)\mathcal{B}_\varsigma)(\kappa-1)!},$$

and Ω is defined in (1.11). In 2016, the authors Hussain, Rasheed and Darus [21] consequently established a new subclass of harmonic functions by utilizing the above linear operator extended to harmonic functions. Moreover, they studied some properties as coefficient bounds, extreme points, and inclusion results and closed under an integral operator for this subclass.

In 2006, Noor [32] once again defined an integral operator by employing the Gauss hypergeometric function $\mathcal{F}(\xi,\eta;\zeta;z)$ given by (1.9) as:

$$\begin{aligned}I_\gamma(\xi,\eta;\zeta)\omega(z)&=[z\mathcal{F}(\xi,\eta;\zeta;z)]^{(-1)}*\omega(z)\\&=z+\sum_{\kappa=2}^{\infty}\frac{(\zeta)_{\kappa-1}(\gamma+1)_{\kappa-1}}{(\xi)_{\kappa-1}(\eta)_{\kappa-1}}\,\alpha_n z^\kappa,\end{aligned}\tag{1.13}$$

where

$$[z\mathcal{F}(\xi,\eta;\zeta;z)]*[z\mathcal{F}(\xi,\eta;\zeta;z)]^{(-1)}=\frac{z}{(1-z)^{\gamma+1}}.$$

Later, in 2008, Ibrahim and Darus [24] imposed a generalized integral operator $I_\gamma[(\mu_l,\mathcal{A}_l)_{1,\tau};(\nu_l,\mathcal{B}_l)_{1,\varsigma}]\omega(z)$ by using FWGH on $\mathbb{S}$ as:

$$I_\gamma[(\mu_l,\mathcal{A}_l)_{1,\tau};(\nu_l,\mathcal{B}_l)_{1,\varsigma}]\omega(z)=z+\sum_{\kappa=2}^{\infty}\frac{\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-1)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-1)\mathcal{A}_l)}(\gamma+1)_{\kappa-1}\,\alpha_\kappa\,z^\kappa\tag{1.14}$$

and

$$\frac{\Gamma(\nu_1)...\Gamma(\nu_\varsigma)}{\Gamma(\mu_1)...\Gamma(\mu_\tau)}=1.$$

In 2016, El-Ashwah and Hassan [16] introduced the following linear operator on $\mathbb{M}_p$ as:

$$\Theta[(\mu_l,\mathcal{A}_l)_{1,\tau};(\nu_l,\mathcal{B}_l)_{1,\varsigma}]\omega(z)=z^p+\sum_{\kappa=p+1}^{\infty}\Omega\left(\vartheta_\kappa[\mu_l,\nu_l]\right)\,\alpha_\kappa\,z^\kappa,$$

$$\vartheta[\mu_l,\nu_l]=\frac{\Gamma(\mu_1+(\kappa-p)\mathcal{A}_1)\Gamma(\mu_2+(\kappa-p)\mathcal{A}_2)\cdots\Gamma(\mu_\tau+(\kappa-p)\mathcal{A}_\tau)}{\Gamma(\nu_1+(\kappa-p)\mathcal{B}_1)\Gamma(\nu_2+(\kappa-p)\mathcal{B}_2)\cdots\Gamma(\nu_\varsigma+(\kappa-p)\mathcal{B}_\varsigma)(\kappa-p)!}$$

and Ω is defined in (1.11).

In an analogous manner, we define a linear operator by using FWGH extended to harmonic multivalent functions. Furthermore, we consider a certain subclass that involves this posed operator. For this subclass, we study several geometric properties, such as coefficient conditions, growth bounds, extreme points, convolution property, convex linear combination and a class-preserving integral operator.

2. Proposed operator $\mathcal{GN}_p[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}]\ \omega(z)$

This section introduces a new extended generalized linear operator of Noor-type $\mathcal{GN}_p[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}]\ \omega(z)$ for harmonic multivalent functions in the terms of FWGH.

By giving the extended of FWGH in (1.10)

$$z^p {}_\tau\mathcal{W}_\varsigma[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}; z] = \sum_{\kappa=p}^{\infty} \frac{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa - p)\ \mathcal{A}_l)}{\prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa - p)\ \mathcal{B}_l)} \frac{z^\kappa}{(\kappa - p)!}, \quad (2.1)$$

we consider a new function as:

$$z^p {}_\tau\mathcal{W}_\varsigma[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}; z]^{-1} = \frac{\prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa - p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa - p)\mathcal{A}_l)} (\gamma + p)_{\kappa - p}\, z^\kappa, \quad (2.2)$$

such that, for $\gamma > -p$

$$\begin{aligned}&\left(z^p {}_\tau\mathcal{W}_\varsigma[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}; z]\right) * \left(z^p {}_\tau\mathcal{W}_\varsigma[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}; z]\right)^{-1}\\ &= \frac{z^p}{(1-z)^{\gamma+p}} = \sum_{\kappa=p}^{\infty} \frac{(\gamma + p)_{\kappa-p}}{(\kappa - p)!}\ z^\kappa.\end{aligned}$$

Corresponding to (2.2), we impose the following linear operator:

$$\mathcal{GN}_p[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}]\ \omega(z) : \mathbb{S}_{\mathbb{H}(p)} \longrightarrow \mathbb{S}_{\mathbb{H}(p)}$$

defined by the convolution

$$\mathcal{GN}_p[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}]\omega(z) = \Delta\left(z^p {}_\tau\mathcal{W}_\varsigma[(\mu_l, \mathcal{A}_l)_{1,\tau}; (\nu_l, \mathcal{B}_l)_{1,\varsigma}; z]\right)^{-1} * \omega(z), \quad (2.3)$$

where

$$\Delta = \left(\prod_{l=1}^{\tau} \Gamma(\mu_l)\right) \left(\prod_{l=1}^{\varsigma} \Gamma(\nu_l)\right)^{-1}. \quad (2.4)$$

Therefore, the operator formula as:

$$\mathcal{GN}_p[(\mu_l,\mathcal{A}_l)_{1,\tau};(\nu_l,\mathcal{B}_l)_{1,\varsigma}]\ \omega(z)$$
$$= z^p + \sum_{\kappa=p+1}^{\infty} \frac{\Delta \prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)} (\gamma+p)_{\kappa-p}\ \alpha_\kappa z^\kappa. \tag{2.5}$$

Remark 2.1. For some suitably chosen parameters $p,\ \varsigma,\ \tau,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{B}_1,\ \mu_1,\ \mu_2$ and ν_1, the operator $\mathcal{GN}_p[(\mu_l,\mathcal{A}_l)_{1,\tau};(\nu_l,\mathcal{B}_l)_{1,\varsigma}]\omega(z)$ defined in (2.5) can be reduced to several operators mentioned above. The following are some special cases:

(1) For $p=1,\ \varsigma=1,\ \tau=2,\ \mathcal{A}_1=\mathcal{A}_2=\mathcal{B}_1=1$, and $\mu_1=\mu_2=\nu_1=1$ in (2.5), the Ruscheweyh's differential operator given in (1.6) is obtained.
(2) By taking $p=1,\ \varsigma=1,\ \tau=2,\ \mathcal{A}_1=\mathcal{A}_2=\mathcal{B}_1=1,\ \mu_1=\mu_2=1+\gamma$ and $\nu_1=2$ in (2.5), we obtain the Noor integral operator defined in (1.7).
(3) By taking $\varsigma=1,\ \tau=2,\ \mathcal{A}_1=\mathcal{A}_2=\mathcal{B}_1=1,\ \mu_1=\xi,\ \mu_2=1$ and $\nu_1=\zeta$ in (2.5), gives us the general linear operator defined by (1.12).
(4) For $p=1,\ \varsigma=1,\ \tau=2$, and $\mathcal{A}_1=\mathcal{A}_2=\mathcal{B}_1=1,\ \mu_1=\xi,\ \mu_2=\eta$ and $\nu_1=\zeta$, the operator (2.5) provides an integral operator given in (1.13).
(5) If $\Delta=1$, the operator (2.5) reduces to integral operator given by (1.14).

For brevity, (2.5) is written as

$$\mathcal{GN}_p[\mu_l]\ \omega(z) = \mathcal{GN}_p[(\mu_l,\mathcal{A}_l)_{1,\tau};(\nu_l,\mathcal{B}_l)_{1,\varsigma}]\ \omega(z),$$

where $\mathcal{GN}_p[\mu_l]\ \omega(z)$ satisfy the recurrence relation

$$z\Big(\mathcal{GN}_p^{\tau,\varsigma}[\mu_1+l]\omega(z)\Big)' = \frac{\mu_1}{\mathcal{A}_l}\Big(\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z)\Big) - \frac{\mu_1-p\mathcal{A}_l}{\mathcal{A}_l}\Big(\mathcal{GN}_p^{\tau,\varsigma}[\mu_1+l]\omega(z)\Big). \tag{2.6}$$

The operator $\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z)$ when extended to harmonic multivalent function $\omega=\rho+\bar{\sigma}$ is defined by

$$\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z) = \mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\rho(z) + \overline{\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\sigma(z)}, \tag{2.7}$$

where

$$\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z) = z^p + \sum_{\kappa=p+1}^{\infty} \frac{\Delta \prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)} (\gamma+p)_{\kappa-p}\ \alpha_\kappa z^\kappa$$

and

$$\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\sigma(z) = z^p + \sum_{\kappa=p}^{\infty} \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa - p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa - p)\mathcal{A}_l)} (\gamma + p)_{\kappa-p}\ \beta_\kappa z^\kappa.$$

3. Geometric outcomes

In this section, by invoking the operator $\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z)$ given in (2.7), we proceed to introduce a certain geometric subclass of $\mathbb{S}_{\mathbb{H}}$. Then, some properties are acquired by including coefficient bounds, growth formula, extreme points, convolution, and convex combinations as well as discuss a class-preserving integral operator.

Definition 3.1. A function $f \in \mathbb{S}_{\mathbb{H}}$ is said to be in the subclass $\mathcal{H}_{p,b}(\tau,\varsigma)$ if it satisfies the following inequality:

$$\Re\left\{(1-a)\ \frac{\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\ \omega(z)}{z^p} + a\ \frac{\left[\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\ \omega(z)\right]'}{pz^{p-1}}\right\} \geq \frac{b}{p} \tag{3.1}$$

where $\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z)$ is given by (2.7), $0 < a \leq 1,\ 0 \leq b < p$.

Also let $\mathcal{NH}_{p,b}(\tau,\varsigma) = \mathcal{H}_{p,b}(\tau,\varsigma) \bigcap \mathbb{NS}_{\mathbb{H}(p)}$.

The first theorem provides a sufficient coefficient condition for function belong to the class $\mathcal{H}_{p,b}(\tau,\varsigma)$.

Theorem 3.2. *Let $\omega = \rho + \overline{\sigma}$ be of the form* (1.1)*. If*

$$\sum_{\kappa=p+1}^{\infty} [(\kappa-p)a + p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa - p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa - p)\mathcal{A}_l)} (\gamma + p)_{\kappa-p} |\alpha_\kappa|$$

$$+ \sum_{\kappa=p}^{\infty} [(\kappa-p)a + p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa - p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa - p)\mathcal{A}_l)} (\gamma + p)_{\kappa-p} |\beta_\kappa| \leq p - b, \tag{3.2}$$

where $0 < a \leq 1,\ 0 \leq b < p$, then $\omega \in \mathcal{H}_{p,b}(\tau,\varsigma)$.

Proof. Let

$$\omega(z) = (1-a)\frac{\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z)}{z^p} + a\frac{\left[\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\omega(z)\right]'}{pz^{p-1}}.$$

To prove $\Re\,\omega(z) > \frac{b}{p}$, it suffices to show that $|p-b+p\omega(z)| \geq |p+b-p\omega(z)|$. Substituting for $\omega(z)$ and making use of (2.7), we find that

$$\begin{aligned}
&|p-b+p\;\omega(z)| \\
&\geq 2p-b \\
&\quad - \sum_{\kappa=p+1}^{\infty} [(\kappa-p)a+p]\;\frac{\Delta \prod\limits_{l=1}^{\varsigma} \Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau} \Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\;(\gamma+p)_{\kappa-p}\;|\alpha_\kappa|\;|z|^{\kappa-p} \\
&\quad - \sum_{\kappa=p}^{\infty} [(\kappa-p)a+p]\;\frac{\Delta \prod\limits_{l=1}^{\varsigma} \Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau} \Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\;(\gamma+p)_{\kappa-p}\;|\beta_\kappa|\;|z|^{\kappa-p}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&|p+b-p\omega(z)| \\
&\leq b \\
&\quad + \sum_{\kappa=p+1}^{\infty} [(\kappa-p)a+p]\;\frac{\Delta \prod\limits_{l=1}^{\varsigma} \Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau} \Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\;(\gamma+p)_{\kappa-p}\,|\alpha_\kappa|\;|z|^{\kappa-p} \\
&\quad + \sum_{\kappa=p}^{\infty} [(\kappa-p)a+p]\;\frac{\Delta \prod\limits_{l=1}^{\varsigma} \Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau} \Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\;(\gamma+p)_{\kappa-p}\,|\beta_\kappa|\;|z|^{\kappa-p}.
\end{aligned} \tag{3.4}$$

Evidently, the inequalities (3.3) and (3.4) in conjunction with (3.2) yield

$$\begin{aligned}
&|p-b+p\omega(z)| \geq |p+b-p\omega(z)| \\
&\geq 2\Bigg[(p-b) - \sum_{\kappa=p+1}^{\infty} [(\kappa-p)a+p]\frac{\Delta \prod\limits_{l=1}^{\varsigma} \Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau} \Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\;(\gamma+p)_{\kappa-p}\;|\alpha_\kappa| \\
&\quad - \sum_{\kappa=p}^{\infty} [(\kappa-p)a+p]\frac{\Delta \prod\limits_{l=1}^{\varsigma} \Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau} \Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\;(\gamma+p)_{\kappa-p}\;|\beta_\kappa|\Bigg] \geq 0.
\end{aligned} \tag{3.5}$$

The harmonic function

$$\begin{aligned}\omega(z) = & z^p + \sum_{\kappa=p+1}^{\infty} \frac{x_\kappa}{[(\kappa-p)a+p]\ \frac{\Delta \prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\ (\gamma+p)_{\kappa-p}\ |\alpha_\kappa|} z^\kappa \\ & + \sum_{\kappa=p}^{\infty} \frac{\overline{y}_\kappa}{[(\kappa-p)a+p]\ \frac{\Delta \prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\ (\gamma+p)_{\kappa-p}\ |\beta_\kappa|} \overline{z}^\kappa, \end{aligned} \tag{3.6}$$

where $\sum_{\kappa=p+1}^{\infty}|x_\kappa|+\sum_{\kappa=p}^{\infty}|y_\kappa| = p-b$ shows that the coefficient bounds given by (3.2) are sharp.

The functions of the from (3.2) are in $\mathcal{H}_{p,b}(\tau,\varsigma)$ because

$$\begin{aligned} & \sum_{\kappa=p+1}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)} (\gamma+p)_{\kappa-p}\ |\alpha_\kappa| \\ & + \sum_{\kappa=p}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)} (\gamma+p)_{\kappa-p}\ |\beta_\kappa| \\ & = \sum_{\kappa=p+1}^{\infty} |x_\kappa| + \sum_{\kappa=p}^{\infty} |y_\kappa| = p-b. \end{aligned} \tag{3.7}$$

This completes the proof of Theorem 3.2. □

The next theorem gives a sufficient coefficient condition for function to be in $\mathcal{H}_{p,b}(\tau,\varsigma)$.

Theorem 3.3. *Let* $\omega = \rho + \overline{\sigma}$ *be of the form* (1.3). *Then* $\omega \in \mathcal{H}_{p,b}(\tau,\varsigma)$ *if and only if the condition* (3.2) *is as:*

$$\sum_{\kappa=p+1}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)} (\gamma+p)_{\kappa-p}\,|\alpha_\kappa|$$

$$+\sum_{\kappa=p}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\,|\beta_\kappa|\leq p-b,$$

where $0<a\leq 1,\ 0\leq b<p.$

Proof. As $\omega\in\mathcal{NH}_{p,b}(\tau,\varsigma)\subset\mathcal{NH}_{p,b}(\tau,\varsigma)$, we only need to prove the "only if" part of this theorem. To this end, for functions ω of the form (1.3), condition (3.1) is as follows:

$$\Re\Big\{(1-a)\ \frac{\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\ \omega(z)}{z^p}+a\ \frac{\big[\mathcal{GN}_p^{\tau,\varsigma}[\mu_1]\ \omega(z)\big]'}{pz^{p-1}}\Big\}\geq\frac{b}{p},$$

which implies that

$$\Re\Big\{(p-b)-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\ |\alpha_\kappa|z^{\kappa-p}$$
$$-\sum_{\kappa=p}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\ (\gamma+p)_{\kappa-p}\ |\beta_\kappa|\overline{z}^{\kappa-p}\Big\}\geq 0.$$

The above-mentioned required condition must hold for all values of z in $\mathbb{D}$. Upon choosing the values of z on the positive real axis where $0<|z|=r<1$, we must have

$$(p-b)-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}|\alpha_\kappa|\ r^{\kappa-p}$$
$$-\sum_{\kappa=p}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}\ (\gamma+p)_{\kappa-p}\ |\beta_\kappa|\ r^{\kappa-p}\geq 0.$$

Letting $r\to -1$ through real values, it follows that

$$\begin{aligned}&(p-b)-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\ |\alpha_\kappa|\\&-\sum_{\kappa=p}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\ |\beta_\kappa|\geq 0.\end{aligned}\tag{3.8}$$

Therefore, (3.8) gives (3.2). This completes the proof. □

The following theorem considers the growth relation of the function $\omega \in \mathcal{NH}_{p,b}(\tau,\varsigma)$.

Theorem 3.4. *Let $\omega \in \mathcal{NH}_{p,b}(\tau,\varsigma)$. Then for $r = |z| < 1$,*

$$|\omega(z)| \leq r^p\ (1+|\beta_p|) + \frac{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)\ [p(1-|\beta_p|)-b]}{[a+p]\ \Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}\ r^{p+1}$$

and

$$|\omega(z)| \geq r^p\ (1+|\beta_p|) - \frac{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)\ [p(1-|\beta_p|)-b]}{[a+p]\ \Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}\ r^{p+1}.$$

Proof. Let $\omega \in \mathcal{NH}_{p,b}(\tau,\varsigma)$. By taking the modulus value of ω and using Theorem 3.3, we have

$$\begin{aligned}|\omega(z)| &\leq (1+|\beta_p|)r^p + \sum_{\kappa=p+1}^{\infty}(|\alpha_\kappa|+|\beta_\kappa|)\, r^\kappa\\&\leq (1+|\beta_p|)r^p + r^{p+1}\sum_{\kappa=p+1}^{\infty}(|\alpha_\kappa|+|\beta_\kappa|)\end{aligned}$$

$$\leq (1+|\beta_p|)r^p + \frac{r^{p+1}\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)}{[a+p]\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}$$

$$\times\left(\sum_{\kappa=p+1}^{\infty}[a+p]\frac{\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)}(\gamma+p)_1\left(|\alpha_\kappa|+|\beta_\kappa|\right)\right)$$

$$\leq (1+|\beta_p|)r^p + \frac{r^{p+1}\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)}{[a+p]\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}$$

$$\times\left(\sum_{\kappa=p+1}^{\infty}[(k-p)a+p]\frac{\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+(k-p)\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+(k-p)\mathcal{A}_l)}(\gamma+p)_{k-p}\left(|\alpha_\kappa|+|\beta_\kappa|\right)\right)$$

$$\leq (1+|\beta_p|)r^p + \frac{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)\ [p(1-|\beta_p|)-b]}{[a+p]\ \Delta\prod\limits_{l=1}^{\varsigma}\ \Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}\ r^{p+1}.$$

In addition,

$$|\omega(z)| \geq (1+|\beta_p|)r^p - \sum_{\kappa=p+1}^{\infty}\left(|\alpha_\kappa|+|\beta_\kappa|\right)r^\kappa$$

$$\geq (1+|\beta_p|)r^p - r^{p+1}\sum_{\kappa=p+1}^{\infty}\ \left(|\alpha_\kappa|+|\beta_\kappa|\right)$$

$$\geq (1+|\beta_p|)r^p - \frac{r^{p+1}\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)}{[a+p]\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}$$

$$\times\left(\sum_{\kappa=p+1}^{\infty}[a+p]\frac{\Delta\prod\limits_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)}{\prod\limits_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)}(\gamma+p)_1\left(|\alpha_\kappa|+|\beta_\kappa|\right)\right)$$

$$\geq (1+|\beta_p|)r^p - \frac{r^{p+1}\prod_{l=1}^{\tau}\Gamma(\mu_l+\mathcal{A}_l)}{[a+p]\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}$$

$$\times\left(\sum_{\kappa=p+1}^{\infty}[(k-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(k-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(k-p)\mathcal{A}_l)}(\gamma+p)_{k-p}\left(|\alpha_\kappa|+|\beta_\kappa|\right)\right)$$

$$\geq (1+|\beta_p|)r^p - \frac{\prod_{l=1}^{\tau} r^{p+1}\Gamma(\mu_l+\mathcal{A}_l)[p(1-|\beta_p|)-b]}{[a+p]\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+\mathcal{B}_l)\ (\gamma+p)_1}.$$

This completes the proof. □

The following theorem determines the extreme points of closed convex hulls of $\mathcal{NH}_{p,b}(\tau,\varsigma)$ denoted by $\overline{co}\mathcal{NH}_{p,b}(\tau,\varsigma)$.

Theorem 3.5. *A function $\omega\in\overline{co}\mathcal{NH}_{p,b}(\tau,\varsigma)$ if and only if*

$$\omega(z)=\sum_{\kappa=p}^{\infty}\left(X_\kappa h_\kappa(z)+Y_\kappa g_\kappa(z)\right), \tag{3.9}$$

where

$$h_p(z)=z^p,$$

$$h_\kappa(z)=z^p-\frac{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)\ (\gamma+p)_{\kappa-p}}z^\kappa,$$

$$(\kappa=p+1,p+2,...),$$

and

$$g_\kappa(z)=z^p-\frac{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)\,(\gamma+p)_{\kappa-p}}\overline{z}^\kappa,$$

$$(\kappa=p,p+1,...),$$

and $\sum_{\kappa=p}^{\infty}(X_\kappa+Y_\kappa)=1$, $X_\kappa\geq 0$ *and* $Y_\kappa\geq 0$.

Proof. For a function ω of the form (3.9), we have

$$\begin{aligned}
\omega(z) &= \sum_{\kappa=p}^{\infty} \left(X_\kappa h_\kappa(z) + Y_\kappa g_\kappa(z)\right) \\
&= \sum_{\kappa=p}^{\infty} \left(X_\kappa + Y_\kappa\right) z^p \\
&\quad - \sum_{\kappa=p+1}^{\infty} \frac{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)\ (\gamma+p)_{\kappa-p}} X_\kappa z^\kappa \\
&\quad - \sum_{\kappa=p}^{\infty} \frac{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)\ (\gamma+p)_{\kappa-p}} Y_\kappa \overline{z}^\kappa \\
&= z^p - \sum_{\kappa=p+1}^{\infty} \frac{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)\ (\gamma+p)_{\kappa-p}} X_\kappa z^\kappa \\
&\quad - \sum_{\kappa=p}^{\infty} \frac{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)\ (\gamma+p)_{\kappa-p}} Y_\kappa \overline{z}^\kappa.
\end{aligned}$$

Therefore, in view of Theorem 3.3, we obtain

$$\begin{aligned}
&\sum_{\kappa=p}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)}\ (\gamma+p)_{\kappa-p} \\
&\quad \times \left[\frac{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)\ (\gamma+p)_{\kappa-p}} X_\kappa \right] \\
&\quad + \sum_{\kappa=p}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)}\ (\gamma+p)_{\kappa-p}
\end{aligned}$$

$$\times\left[\frac{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)(p-b)}{[(\kappa-p)a+p]\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)\ (\gamma+p)_{\kappa-p}}Y_\kappa\right]$$
$$\leq (p-b)\left(\sum_{\kappa=p}^{\infty}(X_\kappa+Y_\kappa)-X_p\right)$$
$$=(p-b)\left(1-X_p\right)$$
$$\leq p-b.$$

Therefore, $\omega\in\overline{co}\mathcal{NH}_{p,b}(\tau,\varsigma)$.

Conversely, suppose that $\omega\in\overline{co}\mathcal{NH}_{p,b}(\tau,\varsigma)$. Set

$$X_\kappa=[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)(p-b)}\ (\gamma+p)_{\kappa-p}|\alpha_\kappa|,$$
$$(\kappa=p+1,p+2,...),$$

and

$$Y_\kappa=[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)(p-b)}(\gamma+p)_{\kappa-p}|\beta_\kappa|,$$
$$(\kappa=p,p+1,p+2,...).$$

On the basis of Theorem 3.3, we note that $0\leq X_\kappa\leq 1,\ (\kappa=p+1,p+2,...)$ and $0\leq Y_\kappa\leq 1,\ (\kappa=p,p+1,p+2,...)$. Let $X_p=1-\sum_{\kappa=p+1}^{\infty}X_\kappa+\sum_{\kappa=p}^{\infty}Y_\kappa$ and note that by Theorem 3.3, $X_p\geq 0$. Consequently, $\omega(z)=\sum_{\kappa=p}^{\infty}\left(X_\kappa h_\kappa(z)+Y_\kappa g_\kappa(z)\right)$ is obtained as required. □

Subclass $\mathcal{NH}_p(\tau,\varsigma)$ is closed under convolution and will be shown in the next theorem.

Theorem 3.6. $0\leq c\leq b<1$, *let* $\omega\in\mathcal{NH}_{p,c}(\tau,\varsigma)$ *and* $\mathcal{F}(z)\in\mathcal{NH}_{p,b}(\tau,\varsigma)$. *Then* $(\omega*\mathcal{F})\in\mathcal{NH}_{p,c}(\tau,\varsigma)\subset\mathcal{NH}_{p,b}(\tau,\varsigma)$.

Proof. Using convolution concept, let the harmonic function $\omega(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\alpha_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\beta_\kappa|\overline{z}^\kappa$ and $\mathcal{F}(z)=z^p-\sum_{\kappa=p+1}^{\infty}|A_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|B_\kappa|\overline{z}^\kappa$. Then, the convolution of ω and $\mathcal{F}$ is

$$\left(\omega*\mathcal{F}\right)(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\alpha_\kappa A_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\beta_\kappa B_\kappa|\overline{z}^\kappa.$$

In Theorem 3.3, because $\mathcal{F}(z) \in \mathcal{NH}_{p,b}(\tau,\varsigma)$, we conclude that $|A_\kappa| \le 1$ and $|B_\kappa| \le 1$. However, $\omega \in \mathcal{NH}_{p,c}(\tau,\varsigma)$. We then have

$$\begin{aligned}
&\sum_{\kappa=p+1}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\ \mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)\,(p-c)}\ (\gamma+p)_{\kappa-p}\ |\alpha_\kappa| \\
&\quad + \sum_{\kappa=p}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)\,(p-c)}\ (\gamma+p)_{\kappa-p}\ |\beta_\kappa| \\
&\le \sum_{\kappa=p+1}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)\,(p-b)}\ (\gamma+p)_{\kappa-p}\ |\alpha_\kappa| \\
&\quad + \sum_{\kappa=p}^{\infty} [(\kappa-p)a+p] \frac{\Delta \prod_{l=1}^{\varsigma} \Gamma(\nu_l + (\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau} \Gamma(\mu_l + (\kappa-p)\mathcal{A}_l)\,(p-b)}\ (\gamma+p)_{\kappa-p}\ |\beta_\kappa| \\
&\le 1.
\end{aligned}$$

Thus $(\omega * \mathcal{F}) \in \mathcal{NH}_{p,c}(\tau,\varsigma) \subset \mathcal{NH}_{p,b}(\tau,\varsigma)$. □

In the following result, we show that the convex combination of subclass $\mathcal{NH}_{p,b}(\tau,\varsigma)$. Let the functions $\omega_j(z)$ be defined, for $j = 1, 2, ...$, by

$$\omega_j(z) = z^p + \sum_{\kappa=p+1}^{\infty} |\alpha_{\kappa,j}| z^\kappa - \sum_{\kappa=p}^{\infty} |\beta_{\kappa,j}| \overline{z}^\kappa. \tag{3.10}$$

Theorem 3.7. *Let the functions $\omega_j(z)$ defined by (3.10) be in $\mathcal{NH}_{p,b}(\tau,\varsigma)$ for every $j = 1, 2, \ldots$. Then, the function $\theta(z)$ defined by*

$$\theta(z) = \sum_{j=1}^{\infty} c_j \omega_j(z) \qquad (0 \le c_j \le 1), \tag{3.11}$$

is also in $\mathcal{NH}_{p,b}(\tau,\varsigma)$, where $\sum_{j=1}^{\infty} c_j = 1$.

Proof. According to the definition of θ, we can write

$$\theta(z) = z^p + \sum_{\kappa=p+1}^{\infty} \left(\sum_{j=1}^{\infty} c_j |\alpha_{\kappa,j}| \right) z^\kappa - \sum_{\kappa=p}^{\infty} \left(\sum_{j=1}^{\infty} c_j |\beta_{\kappa,j}| \right) \overline{z}^\kappa.$$

Then, by Theorem 3.3, we have

$$\sum_{\kappa=p+1}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\left(\sum_{j=1}^{\infty}c_j|\alpha_{\kappa,j}|\right)$$
$$+\sum_{\kappa=p}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\left(\sum_{j=1}^{\infty}c_j|\beta_{\kappa,j}|\right)$$
$$=\sum_{j=1}^{\infty}c_j\left(\sum_{\kappa=p+1}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\,|\alpha_{\kappa,j}|\right.$$
$$\left.+\sum_{\kappa=p}^{\infty}[(\kappa-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(\kappa-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(\kappa-p)\mathcal{A}_l)}(\gamma+p)_{\kappa-p}\,|\beta_{\kappa,j}|\right)$$
$$\leq\sum_{j=1}^{\infty}c_j=1.$$

This completes the proof. □

In this last result, a closure property of subclass $\mathcal{NH}_{p,b}(\tau,\varsigma)$ is examined under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{F}(z)$ defined as: (see [6])

$$\mathcal{F}(z)=\frac{(\mu+p)}{z^{\mu}}\int_0^z t^{\mu-1}\,\omega(t)dt,\quad(\mu>-p).$$

Theorem 3.8. *Let* $\omega\in\mathcal{NH}_{p,b}(\tau,\varsigma)$*. Then* $\mathcal{F}\in\mathcal{NH}_{p,b}(\tau,\varsigma)$*.*

Proof. Let

$$\omega(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\alpha_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\beta_\kappa|\overline{z}^\kappa.$$

From the representation of $\mathcal{F}(z)$, it follows that

$$\begin{aligned}\mathcal{F}(z) &= \frac{(\mu+p)}{z^{\mu}}\int_0^z t^{\mu-1}\left\{\rho(z)+\overline{\sigma(z)}\right\}dt\\ &= \frac{(\mu+p)}{z^{\mu}}\left\{\int_0^z t^{\mu-1}\left(t^p-\sum_{\kappa=p+1}^{\infty}|\alpha_\kappa|t^\kappa\right)dt-\overline{\int_0^z t^{\mu-1}\left(\sum_{\kappa=p}^{\infty}|\beta_\kappa|\, t^\kappa\right)dt}\right\}\\ &= z^p-\sum_{\kappa=p+1}^{\infty}A_\kappa z^\kappa-\sum_{\kappa=p}^{\infty}B_\kappa\overline{z}^\kappa,\end{aligned}$$

where

$$A_\kappa=\left(\frac{\mu+p}{\mu+\kappa}\right)|\alpha_\kappa| \quad \text{and} \quad B_\kappa=\left(\frac{\mu+p}{\mu+\kappa}\right)|\beta_\kappa|.$$

Thus, because $\omega\in\mathcal{NH}_{p,b}(\tau,\varsigma)$,

$$\begin{aligned}&\sum_{k=p+1}^{\infty}[(k-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(k-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(k-p)\mathcal{A}_l)}(\gamma+p)_{k-p}\left(\frac{\mu+p}{\mu+k}\right)|\alpha_k|\\ &\quad+\sum_{k=p}^{\infty}[(k-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(k-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(k-p)\mathcal{A}_l)}(\gamma+p)_{k-p}\left(\frac{\mu+p}{\mu+k}\right)|\beta_k|\\ &\le\sum_{k=p+1}^{\infty}[(k-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(k-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(k-p)\mathcal{A}_l)}(\gamma+p)_{k-p}\,|\alpha_k|\\ &\quad+\sum_{k=p}^{\infty}[(k-p)a+p]\frac{\Delta\prod_{l=1}^{\varsigma}\Gamma(\nu_l+(k-p)\mathcal{B}_l)}{\prod_{l=1}^{\tau}\Gamma(\mu_l+(k-p)\mathcal{A}_l)}(\gamma+p)_{k-p}\,|\beta_k|\\ &\le p-b.\end{aligned}$$

By considering Theorem 3.3, we have $\mathcal{F}(z)\in\mathcal{NH}_{p,b}(\tau,\varsigma)$. □

4. Conclusion

In this paper, we have introduced and discussed a new extended generalized linear operator of Noor-type $\mathcal{GN}_p[\mu_l]\,\omega(z)$ on the class $\mathbb{S}_{\mathbb{H}(p)}$. Moreover, a certain subclass $\mathcal{H}_{p,b}(\tau,\varsigma)$ including the above operator is considered. In addition, some results are gained by involving coefficient conditions and by showing the significance of these conditions for negative coefficient, growth bounds, extreme points, convolution property, convex linear combination and a class-preserving integral operator.

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