

STABILITY OF A SEXVIGINTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we establish the general solution of sexvigintic functional equation

$$\begin{aligned} & f(x+13y) - 26f(x+12y) + 325f(x+11y) - 2600f(x+10y) + 14950f(x+9y) \\ & - 65780f(x+8y) + 230230f(x+7y) - 657800f(x+6y) + 1562275f(x+5y) \\ & - 3124550f(x+4y) + 5311735f(x+3y) - 7726160f(x+2y) + 9657700f(x+y) \\ & - 10400600f(x) + 9657700f(x-y) - 7726160f(x-2y) + 5311735f(x-3y) \\ & - 3124550f(x-4y) + 1562275f(x-5y) - 657800f(x-6y) + 230230f(x-7y) \\ & - 65780f(x-8y) + 14950f(x-9y) - 2600f(x-10y) + 325f(x-11y) \\ & - 26f(x-12y) + f(x-13y) = 26!f(y) \end{aligned}$$

and investigate the Hyers-Ulam stability of this functional equation in Banach spaces using two different methods.

⁰Received October 2, 2018. Revised February 16, 2019.

⁰2010 Mathematics Subject Classification: 39B52, 32B72, 32B82, 47H10.

⁰Keywords: Sexvigintic functional equation, fixed point method, Hyers-Ulam stability.

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1. INTRODUCTION

In 1940, an interesting topic was presented by Ulam [21] triggered the study of stability problems for various functional equations. He addressed a question concerning the stability of homomorphism. In the following year, 1941, Hyers [7] was able to give a partial solution to Ulam's question. The result of Hyers was the generalized by Aoki [1] for additive mappings and by Rassias [18] for \mathbb{R} -linear mappings by weakening the condition for the Cauchy difference.

The stability phenomenon that was presented by Rassias is called the generalized Hyers-Ulam stability. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [4] by replacing the unbounded Cauchy difference by a general control function. A further generalization of the Hyers-Ulam stability for a large class of mapping was obtained by Isac and Rassias [8]. They also presented some applications in non-linear analysis, especially in fixed point theory. This terminology may also be applied to the cases of other functional equations [2, 13, 17, 20]. Also, the Hyers-Ulam stability of functional equations and inequalities in matrix normed spaces has been studied by a number of authors [3, 6, 9, 10, 11, 14, 22].

In [17], Rassias and Eslamian introduced the nonic functional equation as follows:

$$\begin{aligned} & f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) \\ & + 126f(x+y) - 126f(x) + 84f(x-y) - 36f(x-2y) \\ & + 9f(x-3y) - f(x-4y) = 9!f(y) \end{aligned} \quad (1.1)$$

where $9!=362880$. Recently, the general solution in vector spaces and the Hyers-Ulam stability of the equation (1.1) in Banach spaces and in various normed spaces by using direct and fixed point methods were introduced and investigated in [12, 16].

Ramdoss *et al.* [15] introduced the tredecic functional equation

$$\begin{aligned} & f(x+7y) - 13f(x+6y) + 78f(x+5y) - 286f(x+4y) \\ & + 715f(x+3y) - 1287f(x+2y) + 1716f(x+y) - 1716f(x) \\ & + 1287f(x-y) - 715f(x-2y) + 286f(x-3y) - 78f(x-4y) \\ & + f(x-5y) - f(x-6y) = 13!f(y) \end{aligned}$$

and established the Hyers-Ulam stability of this functional equation in matrix normed spaces by using the fixed point method.

Govindan [5] introduced the icosic functional equation as

$$\begin{aligned} & f(x+10y) - 20f(x+9y) + 190f(x+8y) \\ & - 1140f(x+7y) + 4845f(x+6y) \\ & - 15504f(x+5y) + 38760f(x+4y) - 77520f(x+3y) + 125970f(x+2y) \\ & - 167960f(x+y) + 184756f(x) - 167960f(x-y) + 125970f(x-2y) \\ & - 77520f(x-3y) + 38760f(x-4y) - 15504f(x-5y) + 4845f(x-6y) \\ & - 1140f(x-7y) + 190f(x-8y) - 20f(x-9y) + f(x-10y) = 20!f(y) \end{aligned}$$

and established the Hyers-Ulam stability of this functional equation in quasi β -normed spaces.

In the present work, we derive the general solution for the functional equation

$$\begin{aligned} & f(x+13y) - 26f(x+12y) + 325f(x+11y) - 2600f(x+10y) \\ & + 14950f(x+9y) - 65780f(x+8y) + 230230f(x+7y) \\ & - 657800f(x+6y) + 1562275f(x+5y) - 3124550f(x+4y) \\ & + 5311735f(x+3y) - 7726160f(x+2y) + 9657700f(x+y) \quad (1.2) \\ & - 10400600f(x) + 9657700f(x-y) - 7726160f(x-2y) \\ & + 5311735f(x-3y) - 3124550f(x-4y) + 1562275f(x-5y) \\ & - 657800f(x-6y) + 230230f(x-7y) - 65780f(x-8y) \\ & + 14950f(x-9y) - 2600f(x-10y) + 325f(x-11y) \\ & - 26f(x-12y) + f(x-13y) = 26!f(y). \end{aligned}$$

We also investigate the Hyers-Ulam stability of this functional equation in Banach spaces using the direct method and the fixed point method.

Theorem 1.1. (Banach Contraction Principle) *Let (X, d) be a complete metric space and consider a mapping $T : X \rightarrow X$ which is a strictly contractive mapping, that is,*

$$(A1) \quad d(Tx, Ty) \leq Ld(x, y) \text{ for some (Lipschitz constant) } L < 1.$$

Then

- (1) *the mapping T has one and only fixed point $x^* = T(x^*)$;*
 - (2) *the fixed point for each given element x^* is globally attractive, that is ,*
- $$(A2) \quad \lim_{n \rightarrow \infty} T^n x = x^* \text{ for any starting point } x \in X;$$

(3) *one has the following estimation inequalities:*

$$(A3) \quad d(T^n x, x^*) \leq \frac{1}{1-L} d(T^n x, T^{n+1} x) \text{ for all } n \geq 0, \quad x \in X.$$

$$(A4) \quad d(x, x^*) \leq \frac{1}{1-L} d(x, T x) \text{ for all } x \in X.$$

Theorem 1.2. (The alternative of fixed point) *Let (X, d) be a complete generalized metric space and $T : X \rightarrow Y$ be a strictly contractive mapping with Lipschitz constant L . Then, for each given element $x \in X$, either*

$$(B1) \ d(T^n x, T^{n+1} x) = \infty, \ \forall \ n \geq 0$$

or

(B2) *there exists natural number n_0 such that*

- (1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) the sequence $(T^n x)$ is convergent to a fixed point y^* of T ;
- (3) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^n x, y) < \infty\}$;
- (4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

2. GENERAL SOLUTION OF THE FUNCTIONAL EQUATION (1.2)

Theorem 2.1. *If $f : X \rightarrow Y$ is a mapping satisfying (1.2) for all $x, y \in X$, then f is servigintic.*

Proof. Substituting (x, y) by $(0, 0)$ in (1.2), we see that

$$f(0) = 0. \tag{2.1}$$

Letting $y = x$ in (1.2), we get

$$\begin{aligned} & f(14x) - 26f(13x) + 325f(12x) - 2600f(11x) + 14950f(10x) \\ & - 65780f(9x) + 230230f(8x) - 657800f(7x) + 1562275f(6x) \tag{2.2} \\ & - 3124550f(5x) + 5311735f(4x) - 7726160f(3x) + 9657700f(2x) \\ & - 10400600f(x) + 9657700f(0) - 7726160f(-x) + 5311735f(-2x) \\ & - 3124550f(-3x) + 1562275f(-4x) - 657800f(-5x) + 230230f(-6x) \\ & - 65780f(-7x) + 14950f(-8x) - 2600f(-9x) \\ & + 325f(-10x) - 26f(-11x) + f(-12x) = 26!f(x) \end{aligned}$$

for all $x \in X$. Letting $y = -x$ in (1.2), we obtain

$$\begin{aligned} & f(-12x) - 26f(-11x) + 325f(-10x) - 2600f(-9x) \\ & + 14950f(-8x) - 65780f(-7x) + 230230f(-6x) - 657800f(-5x) \\ & + 1562275f(-4x) - 3124550f(-3x) + 5311735f(-2x) \tag{2.3} \\ & - 7726160f(-x) + 9657700f(0) - 10400600f(x) + 9657700f(2x) \\ & - 7726160f(3x) + 5311735f(4x) - 3124550f(5x) + 1562275f(6x) \\ & - 657800f(7x) + 230230f(8x) - 65780f(9x) + 14950f(10x) \\ & - 2600f(11x) + 325f(12x) - 26f(13x) + f(14x) = 26!f(-x) \end{aligned}$$

for all $x \in X$. Subtracting (2.3) from (2.2), we get

$$26!f(x) - 26!f(-x) = 0 \quad (2.4)$$

for all $x \in X$. It follows from (2.4) that

$$f(-x) = f(x) \quad (2.5)$$

for all $x \in X$. Replacing (x, y) by $(0, 2x)$ in (1.2) and using (2.5), we obtain

$$\begin{aligned} & 2f(26x) - 52f(24x) + 650f(22x) - 5200f(20x) + 29900f(18x) \\ & - 131560f(16x) + 460460f(14x) - 1315600f(12x) \\ & + 3124550f(10x) - 6249100f(8x) + 10623470f(6x) \\ & - 15452320f(4x) - 4.032914611 \times 10^{26}f(2x) = 0 \end{aligned} \quad (2.6)$$

for all $x \in X$. Dividing the equation (2.6) by 2, we get

$$\begin{aligned} & f(26x) - 26f(24x) + 325f(22x) - 2600f(20x) + 14950f(18x) \\ & - 65780f(16x) + 230230f(14x) - 657800f(12x) \\ & + 1562275f(10x) - 3124550f(8x) + 5311735f(6x) \\ & - 7726160f(4x) - 2.016457306 \times 10^{26}f(2x) = 0 \end{aligned} \quad (2.7)$$

for all $x \in X$. Setting (x, y) by $(13x, x)$ in (1.2), we obtain

$$\begin{aligned} & f(26x) - 26f(25x) + 325f(24x) - 2600f(23x) + 14950f(22x) \\ & - 65780f(21x) + 230230f(20x) - 657800f(19x) + 1562275f(18x) \\ & - 3124550f(17x) + 5311735f(16x) - 7726160f(15x) \\ & + 9657700f(14x) - 10400600f(13x) + 9657700f(12x) \\ & - 7726160f(11x) + 5311735f(10x) - 3124550f(9x) \\ & + 1562275f(8x) - 657800f(7x) + 230230f(6x) - 65780f(5x) \\ & + 14950f(4x) - 2600f(3x) + 325f(2x) - 26f(x) + f(0) = 26!f(2x) \end{aligned} \quad (2.8)$$

for all $x \in X$. Subtracting (2.7) from (2.8), we obtain

$$\begin{aligned} & 26f(25x) - 351f(24x) + 2600f(23x) - 14625f(22x) + 65780f(21x) \\ & - 232830f(20x) + 657800f(19x) - 1547325f(18x) + 3124550f(17x) \\ & - 5377515f(16x) + 7726160f(15x) - 9427470f(14x) \\ & + 10400600f(13x) - 10315500f(12x) + 7726160f(11x) \\ & - 3749460f(10x) + 3124550f(9x) - 4686825f(8x) + 657800f(7x) \\ & + 5081505f(6x) + 65780f(5x) - 7741110f(4x) - 2600f(3x) \\ & - 2.016457306 \times 10^{26}f(2x) + 4.032914611 \times 10^{26}f(x) = 0 \end{aligned} \quad (2.9)$$

for all $x \in X$. Replacing (x, y) by $(12x, x)$ in (1.2) and multiplying by 26 in the resulting equation, we obtain

$$\begin{aligned}
& 26f(25x) - 676f(24x) + 8450f(23x) \\
& -67600f(22x) + 388700f(21x) \\
& -1710280f(20x) + 5985980f(19x) \\
& -17102800f(18x) + 40619150f(17x) \\
& -81238300f(16x) + 138105110f(15x) - 200880160f(14x) \\
& +251100200f(13x) - 270415600f(12x) + 251100200f(11x) \quad (2.10) \\
& -200880160f(10x) + 138105110f(9x) - 81238300f(8x) \\
& +40619150f(7x) - 17102800f(6x) \\
& +5985980f(5x) - 1710280f(4x) \\
& +388700f(3x) - 67600f(2x) - 1.048557799 \times 10^{28}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.9) from (2.10), we obtain

$$\begin{aligned}
& 325f(24x) - 5850f(23x) + 52975f(22x) - 322920f(21x) \\
& +1477450f(20x) - 5328180f(19x) \\
& +15555475f(18x) - 37494600f(17x) \\
& +75860785f(16x) - 130378950f(15x) + 191452690f(14x) \\
& -240699600f(13x) + 260100100f(12x) - 243374040f(11x) \quad (2.11) \\
& +197130700f(10x) - 134980560f(9x) + 76551475f(8x) \\
& -39961350f(7x) + 22184305f(6x) - 5920200f(5x) - 6030830f(4x) \\
& -386100f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& +1.088886945 \times 10^{28}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(11x, x)$ in (1.2) and multiplying by 325 in the resulting equation, we obtain

$$\begin{aligned}
& 325f(24x) - 8450f(23x) + 105625f(22x) - 845000f(21x) \\
& +4858750f(20x) - 21378500f(19x) \\
& +74824750f(18x) - 213785000f(17x) \\
& +507739375f(16x) - 1015478750f(15x) + 1726313875f(14x) \\
& -2511002000f(13x) + 3138752500f(12x) - 3380195000f(11x) \quad (2.12) \\
& +3138752500f(10x) - 2511002000f(9x) + 1726313875f(8x) \\
& -1015478750f(7x) + 507739375f(6x) \\
& -213785000f(5x) + 74824750f(4x) \\
& -21378500f(3x) + 4859075f(2x) - 1.310697249 \times 10^{29}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.11) from (2.12), we get

$$\begin{aligned}
& 2600f(23x) - 52650f(22x) + 522080f(21x) - 3381300f(20x) \\
& + 16050320f(19x) - 59269275f(18x) + 176290400f(17x) \\
& - 431878590f(16x) + 885099800f(15x) - 1534861185f(14x) \\
& + 2270302400f(13x) - 2878652400f(12x) \tag{2.13} \\
& + 3136820960f(11x) - 2941621800f(10x) + 2376021440f(9x) \\
& - 1649762400f(8x) + 975517400f(7x) - 485555070f(6x) \\
& + 207864800f(5x) - 80855580f(4x) + 20992400f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 1.419585943 \times 10^{29}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(10x, x)$ in (1.2) and multiplying by 2600 in the resulting equation, we have

$$\begin{aligned}
& 2600f(23x) - 67600f(22x) + 845000f(21x) - 6760000f(20x) \\
& + 38870000f(19x) - 171028000f(18x) + 598598000f(17x) \\
& - 1710280000f(16x) + 4061915000f(15x) - 8123830000f(14x) \\
& + 1.3810511 \times 10^{10}f(13x) - 2.0088016 \times 10^{10}f(12x) \tag{2.14} \\
& + 2.511002 \times 10^{10}f(11x) - 2.704156 \times 10^{10}f(10) \\
& + 2.511002 \times 10^{10}f(9x) \\
& - 2.0088016 \times 10^{10}f(8x) \\
& + 1.3810511 \times 10^{10}f(7x) - 8123830000f(6x) \\
& + 4061915000f(5x) - 1710280000f(4x) + 598600600f(3x) \\
& - 171095600f(2x) - 1.048557799 \times 10^{30}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.13) from (2.14), we obtain

$$\begin{aligned}
& 14950f(22x) - 322920f(21x) + 3378700f(20x) - 2281968f(19x) \\
& + 111758725f(18x) - 422307600f(17x) + 1278401410f(16x) \\
& - 3176815200f(15x) + 6588968815f(14x) \\
& - 1.15402086 \times 10^{10}f(13x) \\
& + 1.72093636 \times 10^{10}f(12x) - 2.197319904 \times 10^{10}f(11x) \tag{2.15} \\
& + 2.40999382 \times 10^{10}f(10x) - 2.273399856 \times 10^{10}f(9x) \\
& + 1.84382536 \times 10^{10}f(8x) - 1.28349936 \times 10^{10}f(7x) + 7638274930f(6x) \\
& - 3854050200f(5x) + 1629424420f(4x) - 577608200f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 1.190516393 \times 10^{30}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(9x, x)$ in (1.2) and multiplying by 14950 in the resulting equation, we get

$$\begin{aligned}
& 14950f(22x) - 388700f(21x) + 4858750f(20x) - 38870000f(19x) \\
& + 223502500f(18x) - 983411000f(17x) + 3441938500f(16x) \\
& - 9834110000f(15x) + 2.335601125 \times 10^{10}f(14x) \\
& - 4.67120225 \times 10^{10}f(13x) + 7.941043825 \times 10^{10}f(12x) \\
& - 1.15506092 \times 10^{11}f(11x) + 1.44382615 \times 10^{11}f(10x) \\
& - 1.5548897 \times 10^{11}f(9x) + 1.44382615 \times 10^{11}f(8x) - 1.15506092 \times 10^{11}f(7x) \\
& + 7.941043825 \times 10^{10}f(6x) - 4.67120225 \times 10^{10}f(5x) \\
& + 2.33560262 \times 10^{10}f(4x) - 9834498700f(3x) + 3446797250f(2x) \\
& - 6.029207344 \times 10^{30}f(x) = 0
\end{aligned} \tag{2.16}$$

for all $x \in X$. Subtracting (2.15) from (2.16), we get

$$\begin{aligned}
& 65780f(21x) - 1480050f(20x) + 16050320f(19x) - 111743775f(18x) \\
& + 561103400f(17x) - 2163537090f(16x) + 6657294800f(15x) \\
& - 1.676704244 \times 10^{10}f(14x) + 3.51718139 \times 10^{10}f(13x) \\
& - 6.220107465 \times 10^{10}f(12x) + 9.353289296 \times 10^{10}f(11x) \\
& - 1.202826768 \times 10^{11}f(10x) + 1.327549714 \times 10^{11}f(9x) \\
& - 1.259443614 \times 10^{11}f(8x) + 1.026710984 \times 10^{11}f(7x) \\
& - 7.177216332 \times 10^{10}f(6x) + 4.28579723 \times 10^{10}f(5x) \\
& - 2.172660178 \times 10^{10}f(4x) + 9256890500f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 7.219723737 \times 10^{30}f(x) = 0
\end{aligned} \tag{2.17}$$

for all $x \in X$. Replacing (x, y) by $(8x, x)$ in (1.2) and multiplying by 65780 in the resulting equation, we get

$$\begin{aligned}
& 65780f(21x) - 1710280f(20x) + 21378500f(19x) - 171028000f(18x) \\
& + 983411000f(17x) - 4327008400f(16x) + 1.51445294 \times 10^{10}f(15x) \\
& - 4.3270084 \times 10^{10}f(14x) + 1.027664495 \times 10^{11}f(13x) \\
& - 2.05532899 \times 10^{11}f(12x) + 3.494059283 \times 10^{11}f(11x) \\
& - 5.082268048 \times 10^{11}f(10x) + 6.35283506 \times 10^{11}f(9x) \\
& - 6.84151468 \times 10^{11}f(8x) + 6.35283506 \times 10^{11}f(7x) \\
& - 5.082268048 \times 10^{11}f(6x) + 3.494059941 \times 10^{11}f(5x) \\
& - 2.055346093 \times 10^{11}f(4x) + 1.02787828 \times 10^{11}f(3x) \\
& - 4.3441112 \times 10^{10}f(2x) - 2.652851231 \times 10^{31}f(x) = 0
\end{aligned} \tag{2.18}$$

for all $x \in X$. Subtracting (2.17) from (2.18), we obtain

$$\begin{aligned}
& 230230f(20x) - 5328180f(19x) + 59284225f(18x) - 422307600f(17x) \\
& + 2163471310f(16x) - 8487234600f(15x) + 2.650304157 \times 10^{10}f(14x) \\
& - 6.75946356 \times 10^{10}f(13x) + 1.433318244 \times 10^{11}f(12x) \\
& - 2.558730353 \times 10^{11}f(11x) + 3.87944128 \times 10^{11}f(10x) \tag{2.19} \\
& - 5.025285346 \times 10^{11}f(9x) + 5.582071066 \times 10^{11}f(8x) \\
& - 5.326124076 \times 10^{11}f(7x) + 4.364546415 \times 10^{11}f(6x) \\
& - 3.065480218 \times 10^{11}f(5x) + 1.838080075 \times 10^{11}f(4x) \\
& - 9.35309375 \times 10^{10}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 3.374823606 \times 10^{31}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(7x, x)$ in (1.2) and multiplying by 230230 in the resulting equation, we obtain

$$\begin{aligned}
& 230230f(20x) - 5985980f(19x) + 74824750f(18x) - 598598000f(17x) \\
& + 3441938500f(16x) - 1.51445294 \times 10^{10}f(15x) \\
& + 5.30058529 \times 10^{10}f(14x) - 1.51445294 \times 10^{11}f(13x) \tag{2.20} \\
& + 3.596825733 \times 10^{11}f(12x) - 7.193651465 \times 10^{11}f(11x) \\
& + 1.222920749 \times 10^{12}f(10x) - 1.778793817 \times 10^{12}f(9x) \\
& + 2.223492271 \times 10^{12}f(8x) - 2.394530138 \times 10^{12}f(7x) \\
& + 2.223492501 \times 10^{12}f(6x) - 1.778799803 \times 10^{12}f(5x) \\
& + 1.222995574 \times 10^{12}f(4x) - 7.199637445 \times 10^{11}f(3x) \\
& + 3.631245118 \times 10^{11}f(2x) - 9.28497931 \times 10^{31}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.19) from (2.20), we get

$$\begin{aligned}
& 657800f(19x) - 15540525f(18x) + 176290400f(17x) - 1278467190f(16x) \\
& + 6657294800f(15x) - 2.650281134 \times 10^{10}f(14x) \\
& + 8.38506584 \times 10^{10}f(13x) - 2.163507489 \times 10^{11}f(12x) \\
& + 4.634921112 \times 10^{11}f(11x) - 8.349766211 \times 10^{11}f(10x) \tag{2.21} \\
& + 1.276265282 \times 10^{12}f(9x) - 1.665285164 \times 10^{12}f(8x) \\
& + 1.86191773 \times 10^{12}f(7x) - 1.78703786 \times 10^{12}f(6x) \\
& + 1.472251781 \times 10^{12}f(5x) - 1.039187566 \times 10^{12}f(4x) \\
& + 6.26432807 \times 10^{11}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 1.265980291 \times 10^{32}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(6x, x)$ in (1.2) and multiplying by 657800 in the resulting equation, we get

$$\begin{aligned}
& 657800f(19x) - 17102800f(18x) \\
& + 213785000f(17x) - 1710280000f(16x) \\
& + 9834110000f(15x) - 4.3270084 \times 10^{10}f(14x) \\
& + 1.51445249 \times 10^{11}f(13x) \\
& - 4.3270084 \times 10^{11}f(12x) + 1.027664495 \times 10^{12}f(11x) \quad (2.22) \\
& - 2.05532899 \times 10^{12}f(10x) + 3.494059283 \times 10^{12}f(9x) \\
& - 5.082268048 \times 10^{12}f(8x) + 6.352835718 \times 10^{12}f(7x) \\
& - 6.841531783 \times 10^{12}f(6x) + 6.353048845 \times 10^{12}f(5x) \\
& - 5.083978328 \times 10^{12}f(4x) + 3.503893393 \times 10^{12}f(3x) \\
& - 2.098599074 \times 10^{12}f(2x) - 2.652851231 \times 10^{32}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.21) from (2.22), we obtain

$$\begin{aligned}
& 1562275f(18x) - 37494600f(17x) \\
& + 431812810f(16x) - 3176815200f(15x) \\
& + 1.676727267 \times 10^{10}f(14x) - 6.75946356 \times 10^{10}f(13x) \\
& + 2.163500911 \times 10^{11}f(12x) - 5.641723838 \times 10^{11}f(11x) \quad (2.23) \\
& + 1.220352369 \times 10^{12}f(10x) - 2.217794001 \times 10^{12}f(9x) \\
& + 3.416982884 \times 10^{12}f(8x) - 4.490917987 \times 10^{12}f(7x) \\
& + 5.054493923 \times 10^{12}f(6x) - 4.880797064 \times 10^{12}f(5x) \\
& + 4.044790762 \times 10^{12}f(4x) - 2.877460586 \times 10^{12}f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 3.918831523 \times 10^{32}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(5x, x)$ in (1.2) and multiplying by 1562275 in the resulting equation, we have

$$\begin{aligned}
& 1562275f(18x) - 40619150f(17x) \\
& + 507739375f(16x) - 4061915000f(15x) \\
& + 2.335601125 \times 10^{10}f(14x) - 1.027664495 \times 10^{11}f(13x) \\
& + 3.596825733 \times 10^{11}f(12x) - 1.027664495 \times 10^{12}f(11x) \quad (2.24) \\
& + 2.440703176 \times 10^{12}f(10x) - 4.881406351 \times 10^{12}f(9x) \\
& + 8.298392359 \times 10^{12}f(8x) - 1.207042723 \times 10^{13}f(7x) \\
& + 1.508849101 \times 10^{13}f(6x) - 1.625265928 \times 10^{13}f(5x) \\
& + 1.511133928 \times 10^{13}f(4x) - 1.217315306 \times 10^{13}f(3x) \\
& + 8.65807337 \times 10^{12}f(2x) - 6.300521674 \times 10^{32}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.23) from (2.24), we obtain

$$\begin{aligned}
& 3124550f(17x) - 75926565f(16x) + 885099800f(15x) \\
& - 6588738585f(14x) + 3.51718139 \times 10^{10}f(13x) \\
& - 1.433324822 \times 10^{11}f(12x) + 4.634921112 \times 10^{11}f(11x) \\
& - 1.220350807 \times 10^{12}f(10x) + 2.66361235 \times 10^{12}f(9x) \quad (2.25) \\
& - 4.881409476 \times 10^{12}f(8x) + 7.579509246 \times 10^{12}f(7x) \\
& - 1.003399708 \times 10^{13}f(6x) + 1.137186222 \times 10^{13}f(5x) \\
& - 1.106654852 \times 10^{13}f(4x) + 9.295692478 \times 10^{12}f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 1.02193532 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(4x, x)$ in (1.2) and multiplying by 3124550 in the resulting equation, we get

$$\begin{aligned}
& 3124550f(17x) - 81238300f(16x) + 1015478750f(15x) \\
& - 8123830000f(14x) + 4.67120225 \times 10^{10}f(13x) \\
& - 2.05532899 \times 10^{11}f(12x) \\
& + 7.193651465 \times 10^{11}f(11x) - 2.05532899 \times 10^{12}f(10x) \\
& + 4.881409476 \times 10^{12}f(9x) - 9.762893941 \times 10^{12}f(8x) \quad (2.26) \\
& + 1.659779707 \times 10^{13}f(7x) - 2.414889706 \times 10^{13}f(6x) \\
& + 3.022267856 \times 10^{13}f(5x) - 3.270272763 \times 10^{13}f(4x) \\
& + 3.089533168 \times 10^{13}f(3x) - 2.619610222 \times 10^{13}f(2x) \\
& - 1.260104335 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.25) from (2.26), we obtain

$$\begin{aligned}
& 5311735f(16x) - 130378950f(15x) + 1535091415f(14x) \\
& - 1.15402086 \times 10^{10}f(13x) + 6.220041686 \times 10^{10}f(12x) \\
& - 2.558730353 \times 10^{11}f(11x) + 8.349781833 \times 10^{11}f(10x) \\
& - 2.217797125 \times 10^{12}f(9x) + 4.881484465 \times 10^{12}f(8x) \quad (2.27) \\
& - 9.018287827 \times 10^{12}f(7x) + 1.411489997 \times 10^{13}f(6x) \\
& - 1.885081634 \times 10^{13}f(5x) + 2.163617911 \times 10^{13}f(4x) \\
& - 2.15996392 \times 10^{13}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 2.282039655 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(3x, x)$ in (1.2) and multiplying by 5311735 in the resulting equation, we obtain

$$\begin{aligned}
& 5311735f(16x) - 138105110f(15x) + 1726313875f(14x) \\
& - 1.3810511 \times 10^{10}f(13x) + 7.941043825 \times 10^{10}f(12x) \\
& - 3.494059283 \times 10^{11}f(11x) + 1.222926061 \times 10^{12}f(10x) \\
& - 3.494197388 \times 10^{12}f(9x) + 8.300117111 \times 10^{12}f(8x) \quad (2.28) \\
& - 1.661059211 \times 10^{13}f(7x) + 2.829393915 \times 10^{13}f(6x) \\
& - 4.138872042 \times 10^{13}f(5x) + 5.252206386 \times 10^{13}f(4x) \\
& - 5.873929032 \times 10^{13}f(3x) + 5.959753391 \times 10^{13}f(2x) \\
& - 2.142177369 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.27) from (2.28), we get

$$\begin{aligned}
& 7726160f(15x) - 191222460f(14x) + 2270302400f(13x) \\
& - 1.72100214 \times 10^{10}f(12x) + 9.353289296 \times 10^{10}f(11x) \\
& - 3.879478775 \times 10^{11}f(10x) + 1.276400263 \times 10^{12}f(9x) \\
& - 3.418632646 \times 10^{12}f(8x) + 7.592304278 \times 10^{12}f(7x) \quad (2.29) \\
& - 1.417903917 \times 10^{13}f(6x) + 2.253790407 \times 10^{13}f(5x) \\
& - 3.088588475 \times 10^{13}f(4x) + 3.713965112 \times 10^{13}f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 4.424217024 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(2x, x)$ in (1.2) and multiplying by 7726160 in the resulting equation, we have

$$\begin{aligned}
& 7726160f(15x) - 200880160f(14x) + 2511002000f(13x) \\
& - 2.0088016 \times 10^{10}f(12x) + 1.155138182 \times 10^{11}f(11x) \\
& - 5.08427685 \times 10^{11}f(10x) + 1.781304819 \times 10^{12}f(9x) \quad (2.30) \\
& - 5.102356064 \times 10^{12}f(8x) + 1.218589271 \times 10^{13}f(7x) \\
& - 2.464900003 \times 10^{13}f(6x) + 4.28181083 \times 10^{13}f(5x) \\
& - 6.477581639 \times 10^{13}f(4x) + 8.668732205 \times 10^{13}f(3x) \\
& - 1.044974729 \times 10^{14}f(2x) - 3.115894355 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.29) from (2.30), we get

$$\begin{aligned}
& 9657700f(14x) - 240699600f(13x) + 2877994600f(12x) \\
& - 2.19809252 \times 10^{10}f(11x) + 1.204798075 \times 10^{11}f(10x) \\
& - 5.04904556 \times 10^{11}f(9x) + 1.683723418 \times 10^{12}f(8x) \quad (2.31) \\
& - 4.593588428 \times 10^{12}f(7x) + 1.046996086 \times 10^{13}f(6x) \\
& - 2.028020423 \times 10^{13}f(5x) + 3.388993165 \times 10^{13}f(4x) \\
& - 4.954767093 \times 10^{13}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 7.540111379 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by (x, x) in (1.2) and multiplying by 9657700 in the resulting equation, we get

$$\begin{aligned}
& 9657700f(14x) - 251100200f(13x) + 3148410200f(12x) \\
& - 2.53611202 \times 10^{10}f(11x) + 1.475213675 \times 10^{11}f(10x) \\
& - 6.60393526 \times 10^{11}f(9x) + 2.367874886 \times 10^{12}f(8x) \quad (2.32) \\
& - 6.988118566 \times 10^{12}f(7x) + 1.731147554 \times 10^{13}f(6x) \\
& - 3.65288016 \times 10^{13}f(5x) + 6.638712638 \times 10^{13}f(4x) \\
& - 1.04792902 \times 10^{14}f(3x) + 1.445703124 \times 10^{14}f(2x) \\
& - 3.894867944 \times 10^{33}f(x) = 0
\end{aligned}$$

for all $x \in X$. Subtracting (2.31) from (2.32), we obtain

$$\begin{aligned}
& 10400600f(13x) - 270415600f(12x) + 3380195000f(11x) \\
& - 2.704156 \times 10^{10}f(10x) + 1.5548897 \times 10^{11}f(9x) \\
& - 6.84151468 \times 10^{11}f(8x) \\
& + 2.394530138 \times 10^{12}f(7x) - 6.84151468 \times 10^{12}f(6x) \quad (2.33) \\
& + 1.624859737 \times 10^{13}f(5x) - 3.249719473 \times 10^{13}f(4x) \\
& + 5.524523104 \times 10^{13}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 1.143497932 \times 10^{34}f(x) = 0
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(0, x)$ in (1.2) and dividing 2, we obtain

$$\begin{aligned}
& f(13x) - 26f(12x) + 325f(11x) - 2600f(10x) + 14950f(9x) \\
& - 65780f(8x) + 230230f(7x) - 657800f(6x) + 1562275f(5x) \\
& - 3124550f(4x) + 5311735f(3x) - 7726160f(2x) \quad (2.34) \\
& - 2.016457306 \times 10^{26}f(x) = 0
\end{aligned}$$

for all $x \in X$. Multiplying by 10400600 in (2.34), we obtain

$$\begin{aligned} &10400600f(13x) - 270415600f(12x) + 3380195000f(11x) \\ &- 2.704156 \times 10^{10}f(10x) + 1.5548897 \times 10^{11}f(9x) \\ &- 6.84151468 \times 10^{11}f(8x) + 2.394530138 \times 10^{12}f(7x) \\ &- 6.84151468 \times 10^{12}f(6x) + 1.624859737 \times 10^{13}f(5x) \\ &- 3.249719473 \times 10^{13}f(4x) + 5.524523104 \times 10^{13}f(3x) \\ &- 8.03566997 \times 10^{13}f(2x) - 2.097236585 \times 10^{33}f(x) = 0 \end{aligned} \quad (2.35)$$

for all $x \in X$. Subtracting (2.33) from (2.34) and dividing by 4, we get

$$-5.041143265 \times 10^{25}f(2x) + 3.383053978 \times 10^{33}f(x) = 0 \quad (2.36)$$

for all $x \in X$. From (2.36), we obtain

$$f(2x) = 2^{26}f(x), \quad (2.37)$$

as desired. \square

3. STABILITY RESULT OF THE FUNCTIONAL EQUATION (1.2): DIRECT METHOD

In this section, we investigate the stability result for the new type of functional equation (1.2) in Banach spaces using the direct method.

Throughout the paper, let X be a normed space and Y be a Banach space. Define a mapping $Df_{26} : X \rightarrow Y$ by

$$\begin{aligned} Df_{26}(x, y) = &f(x + 13y) - 26f(x + 12y) + 325f(x + 11y) \\ &- 2600f(x + 10y) + 14950f(x + 9y) - 65780f(x + 8y) \\ &+ 230230f(x + 7y) - 657800f(x + 6y) + 1562275f(x + 5y) \\ &- 3124550f(x + 4y) + 5311735f(x + 3y) - 7726160f(x + 2y) \\ &+ 9657700f(x + y) - 10400600f(x) + 9657700f(x - y) \\ &- 7726160f(x - 2y) + 5311735f(x - 3y) - 3124550f(x - 4y) \\ &+ 1562275f(x - 5y) - 657800f(x - 6y) + 230230f(x - 7y) \\ &- 65780f(x - 8y) + 14950f(x - 9y) - 2600f(x - 10y) \\ &+ 325f(x - 11y) - 26f(x - 12y) + f(x - 13y) - 26!f(y). \end{aligned}$$

Theorem 3.1. *Let $j = \pm 1$ and $\alpha : X^2 \rightarrow [0, \infty)$ be a function such that $\sum_{i=0}^{\infty} \frac{\alpha(2^{ij}x, 2^{ij}y)}{2^{26ij}}$ converges in \mathbb{R} for all $x, y \in X$. Let $Df_{26} : X \rightarrow Y$ be a mapping satisfying the inequality*

$$\|Df_{26}(x, y)\| \leq \alpha(x, y) \quad (3.1)$$

for all $x, y \in X$. Then, there exists a unique mapping $G : X \rightarrow Y$ which satisfies (1.2) and

$$\|f(x) - G(x)\| \leq \frac{1}{2^{26}} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{ij}x, 2^{ij}y)}{2^{26ij}} \quad (3.2)$$

where $G(x)$ is defined by

$$G(x) = \lim_{i \rightarrow \infty} \frac{f(2^{ij}x)}{2^{26ij}} \quad (3.3)$$

for all $x \in X$.

Proof. Let $j = 1$. Replacing (x, y) by $(0, 2x)$ in (3.1), we obtain

$$\begin{aligned} & \|f(26x) - 26f(24x) + 325f(22x) - 2600f(20x) + 14950f(18x) \\ & - 65780f(16x) + 230230f(14x) - 657800f(12x) + 1562275f(10x) \\ & - 3124550f(8x) + 5311735f(6x) - 7726160f(4x) + 9657700f(2x) \\ & - 10400600f(0) + 9657700f(-2x) - 7726160f(-4x) + 5311735f(-6x) \\ & - 3124550f(-8x) + 1562275f(-10x) - 657800f(-12x) \\ & + 230230f(-14x) - 65780f(-16x) + 14950f(-18x) - 2600f(-20x) \\ & + 325f(-22x) - 26f(-24x) + f(-26x) - 26!f(2x)\| \leq \alpha(0, 2x) \end{aligned} \quad (3.4)$$

for all $x \in X$. Replacing (x, y) by $(13x, x)$ in (1.2), we get

$$\begin{aligned} & \|f(26x) - 26f(25x) + 325f(24x) - 2600f(23x) + 14950f(22x) \\ & - 65780f(21x) + 230230f(20x) - 657800f(19x) + 1562275f(18x) \\ & - 3124550f(17x) + 5311735f(16x) - 7726160f(15x) \\ & + 9657700f(14x) - 10400600f(13x) + 9657700f(12x) - 7726160f(11x) \\ & + 5311735f(10x) - 3124550f(9x) + 1562275f(8x) - 657800f(7x) \\ & + 230230f(6x) - 65780f(5x) + 14950f(4x) - 2600f(3x) \\ & + 325f(2x) - 26f(x) + f(0) - 26!f(2x)\| \leq \alpha(13x, x) \end{aligned} \quad (3.5)$$

for all $x \in X$. Subtracting (3.4) from (3.5), we have

$$\begin{aligned} & \|26f(25x) - 351f(24x) + 2600f(23x) - 14625f(22x) + 65780f(21x) \\ & - 232830f(20x) + 657800f(19x) - 1547325f(18x) + 3124550f(17x) \\ & - 5377515f(16x) + 7726160f(15x) - 9427470f(14x) + 10400600f(13x) \\ & - 10315500f(12x) + 7726160f(11x) - 3749460f(10x) + 3124550f(9x) \\ & - 4686825f(8x) + 657800f(7x) + 5081505f(6x) + 65780f(5x) \\ & - 7741110f(4x) - 2600f(3x) - 2.016457306 \times 10^{26}f(2x) \\ & + 4.032914611 \times 10^{26}f(x)\| \leq \alpha(0, 2x) + \alpha(13x, x) \end{aligned} \quad (3.6)$$

for all $x \in X$. Replacing (x, y) by $(12x, x)$ in (3.1) and multiplying by 26 on both sides of the resulting equation, we obtain

$$\begin{aligned}
& \|26f(25x) - 676f(24x) + 8450f(23x) \\
& - 67600f(22x) + 388700f(21x) \\
& - 1710280f(20x) + 5985980f(19x) \\
& - 17102800f(18x) + 40619150f(17x) \\
& - 81238300f(16x) + 138105110f(15x) - 200880160f(14x) \quad (3.7) \\
& + 251100200f(13x) - 270415600f(12x) + 251100200f(11x) \\
& - 200880160f(10x) + 138105110f(9x) - 81238300f(8x) \\
& + 40619150f(7x) - 17102800f(6x) + 5985980f(5x) - 1710280f(4x) \\
& + 388700f(3x) - 67600f(2x) - 1.048557799 \cdot 10^{28}f(x)\| \leq 26\alpha(12x, x)
\end{aligned}$$

for all $x \in X$. Subtracting (3.6) from (3.7), we obtain

$$\begin{aligned}
& \|325f(24x) - 5850f(23x) + 52975f(22x) - 322920f(21x) \\
& + 1477450f(20x) - 5328180f(19x) + 15555475f(18x) - 37494600f(17x) \\
& + 75860785f(16x) - 130378950f(15x) + 191452690f(14x) \quad (3.8) \\
& - 240699600f(13x) + 260100100f(12x) - 243374040f(11x) \\
& + 197130700f(10x) - 134980560f(9x) + 76551475f(8x) \\
& - 39961350f(7x) + 22184305f(6x) - 5920200f(5x) - 6030830f(4x) \\
& - 386100f(3x) - 2.016457306 \times 10^{26}f(2x) + 1.088886945 \times 10^{28}f(x)\| \\
& \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(11x, x)$ in (3.1) and multiplying by 325 on both sides of the consequence equation, we obtain

$$\begin{aligned}
& \|325f(24x) - 8450f(23x) + 105625f(22x) - 845000f(21x) \\
& + 4858750f(20x) - 21378500f(19x) \\
& + 74824750f(18x) - 213785000f(17x) \\
& + 507739375f(16x) - 1015478750f(15x) + 1726313875f(14x) \\
& - 2511002000f(13x) + 3138752500f(12x) - 3380195000f(11x) \quad (3.9) \\
& + 3138752500f(10x) - 2511002000f(9x) + 1726313875f(8x) \\
& - 1015478750f(7x) + 507739375f(6x) - 213785000f(5x) \\
& + 74824750f(4x) - 21378500f(3x) + 4859075f(2x) \\
& - 1.310697249 \times 10^{29}f(x)\| \leq 325\alpha(11x, x)
\end{aligned}$$

for all $x \in X$. Subtracting (3.8) from (3.9), we get

$$\begin{aligned}
& \|2600f(23x) - 52650f(22x) + 522080f(21x) - 3381300f(20x) \\
& + 16050320f(19x) - 59269275f(18x) + 176290400f(17x) \\
& - 431878590f(16x) + 885099800f(15x) - 1534861185f(14x) \\
& + 2270302400f(13x) - 2878652400f(12x) + 3136820960f(11x) \quad (3.10) \\
& - 2941621800f(10x) + 2376021440f(9x) - 1649762400f(8x) \\
& + 975517400f(7x) - 485555070f(6x) + 207864800f(5x) - 80855580f(4x) \\
& + 20992400f(3x) - 2.016457306 \times 10^{26}f(2x) + 1.419585943 \times 10^{29}f(x)\| \\
& \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) + 325\alpha(11x, x)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(10x, x)$ in (3.1) and multiplying by 2600 on both sides of the resulting equation, we have

$$\begin{aligned}
& \|2600f(23x) - 67600f(22x) + 845000f(21x) - 6760000f(20x) \\
& + 38870000f(19x) - 171028000f(18x) + 598598000f(17x) \\
& - 1710280000f(16x) + 4061915000f(15x) \quad (3.11) \\
& - 8123830000f(14x) + 1.3810511 \times 10^{10}f(13x) - 2.0088016 \times 10^{10}f(12x) \\
& + 2.511002 \times 10^{10}f(11x) - 2.704156 \cdot 10^{10}f(10) + 2.511002 \times 10^{10}f(9x) \\
& - 2.0088016 \times 10^{10}f(8x) + 1.3810511 \cdot 10^{10}f(7x) - 8123830000f(6x) \\
& + 4061915000f(5x) - 1710280000f(4x) + 598600600f(3x) \\
& - 171095600f(2x) - 1.048557799 \times 10^{30}f(x)\| \leq 2600\alpha(10x, x)
\end{aligned}$$

for all $x \in X$. Subtracting (3.10) from (3.11), we obtain

$$\begin{aligned}
& \|14950f(22x) - 322920f(21x) + 3378700f(20x) - 2281968f(19x) \\
& + 111758725f(18x) - 422307600f(17x) + 1278401410f(16x) \\
& - 3176815200f(15x) + 6588968815f(14x) - 1.15402086 \times 10^{10}f(13x) \\
& + 1.72093636 \times 10^{10}f(12x) - 2.197319904 \times 10^{10}f(11x) \quad (3.12) \\
& + 2.40999382 \times 10^{10}f(10x) - 2.273399856 \times 10^{10}f(9x) \\
& + 1.84382536 \times 10^{10}f(8x) - 1.28349936 \times 10^{10}f(7x) + 7638274930f(6x) \\
& - 3854050200f(5x) + 1629424420f(4x) - 577608200f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 1.190516393 \times 10^{30}f(x)\| \\
& \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) + 325\alpha(11x, x) + 2600\alpha(10x, x)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(9x, x)$ in (3.1) and multiplying by 14950 on both sides of the resulting equation, we get

$$\begin{aligned}
& \|14950f(22x) - 388700f(21x) \\
& + 4858750f(920x) - 38870000f(19x) \\
& + 223502500f(18x) - 983411000f(17x) \\
& + 3441938500f(16x) \\
& - 9834110000f(15x) + 2.335601125 \times 10^{10}f(14x) \\
& - 4.67120225 \times 10^{10}f(13x) \\
& + 7.941043825 \times 10^{10}f(12x) \\
& - 1.15506092 \times 10^{11}f(11x) \\
& + 1.44382615 \times 10^{11}f(10x) - 1.5548897 \times 10^{11}f(9x) \\
& + 1.44382615 \times 10^{11}f(8x) \\
& - 1.15506092 \times 10^{11}f(7x) + 7.941043825 \times 10^{10}f(6x) \\
& - 4.67120225 \times 10^{10}f(5x) \\
& + 2.33560262 \times 10^{10}f(4x) - 9834498700f(3x) \\
& + 3446797250f(2x) \\
& - 6.029207344 \times 10^{30}f(x)\| \leq 14950\alpha(9x, x)
\end{aligned} \tag{3.13}$$

for all $x \in X$. Subtracting (3.13) from (3.12), we have

$$\begin{aligned}
& \|65780f(21x) - 1480050f(20x) \\
& + 16050320f(19x) - 111743775f(18x) \\
& + 561103400f(17x) - 2163537090f(16x) + 6657294800f(15x) \\
& - 1.676704244 \times 10^{10}f(14x) + 3.51718139 \times 10^{10}f(13x) \\
& - 6.220107465 \times 10^{10}f(12x) + 9.353289296 \times 10^{10}f(11x) \\
& - 1.202826768 \times 10^{11}f(10x) + 1.327549714 \times 10^{11}f(9x) \\
& - 1.259443614 \times 10^{11}f(8x) + 1.026710984 \times 10^{11}f(7x) \\
& - 7.177216332 \times 10^{10}f(6x) + 4.28579723 \times 10^{10}f(5x) \\
& - 2.172660178 \times 10^{10}f(4x) \\
& + 9256890500f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 7.219723737 \times 10^{30}f(x)\| \\
& \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) \\
& + 325\alpha(11x, x) + 2600\alpha(10x, x) + 14950\alpha(9x, x)
\end{aligned} \tag{3.14}$$

for all $x \in X$. Replacing (x, y) by $(8x, x)$ in (3.1) and multiplying by 65780 on both sides of the resulting equation, we obtain

$$\begin{aligned}
& \|65780f(21x) - 1710280f(20x) + 21378500f(19x) - 171028000f(18x) \\
& + 983411000f(17x) - 4327008400f(16x) + 1.51445294 \times 10^{10}f(15x) \\
& - 4.3270084 \times f(14x) + 1.027664495 \times 10^{11}f(13x) \\
& - 2.05532899 \times 10^{11}f(12x) + 3.494059283 \times 10^{11}f(11x) \quad (3.15) \\
& - 5.082268048 \times 10^{11}f(10x) + 6.35283506 \times 10^{11}f(9x) \\
& - 6.84151468 \times 10^{11}f(8x) + 6.35283506 \times 10^{11}f(7x) \\
& - 5.082268048 \times 10^{11}f(6x) + 3.494059941 \times 10^{11}f(5x) \\
& - 2.055346093 \times 10^{11}f(4x) + 1.02787828 \times 10^{11}f(3x) \\
& - 4.3441112 \times 10^{10}f(2x) - 2.652851231 \times 10^{31}f(x)\| \leq 65780\alpha(8x, x)
\end{aligned}$$

for all $x \in X$. Subtracting (3.15) from (3.14), we get

$$\begin{aligned}
& \|230230f(20x) - 5328180f(19x) + 59284225f(18x) - 422307600f(17x) \\
& + 2163471310f(16x) - 8487234600f(15x) + 2.650304157 \times 10^{10}f(14x) \\
& - 6.75946356 \times 10^{10}f(13x) + 1.433318244 \times 10^{11}f(12x) \\
& - 2.558730353 \times 10^{11}f(11x) + 3.87944128 \times 10^{11}f(10x) \\
& - 5.025285346 \times 10^{11}f(9x) + 5.582071066 \times 10^{11}f(8x) \quad (3.16) \\
& - 5.326124076 \times 10^{11}f(7x) + 4.364546415 \times 10^{11}f(6x) \\
& - 3.065480218 \times 10^{11}f(5x) + 1.838080075 \times 10^{11}f(4x) \\
& - 9.35309375 \times 10^{10}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 3.374823606 \times 10^{31}f(x)\| \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) \\
& + 325\alpha(11x, x) + 2600\alpha(10x, x) + 14950\alpha(9x, x) + 65780\alpha(8x, x)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(7x, x)$ in (1.2) and multiplying by 230230 on both sides of the resulting equation, we obtain

$$\begin{aligned}
& \|230230f(20x) - 5985980f(19x) + 74824750f(18x) - 598598000f(17x) \\
& + 3441938500f(16x) - 1.51445294 \times 10^{10}f(15x) \\
& + 5.30058529 \times 10^{10}f(14x) \\
& - 1.51445294 \times 10^{11}f(13x) + 3.596825733 \times 10^{11}f(12x) \\
& - 7.193651465 \times 10^{11}f(11x) + 1.222920749 \times 10^{12}f(10x) \\
& - 1.778793817 \times 10^{12}f(9x) + 2.223492271 \times 10^{12}f(8x) \quad (3.17)
\end{aligned}$$

$$\begin{aligned}
& -2.394530138 \times 10^{12} f(7x) + 2.223492501 \times 10^{12} f(6x) \\
& -1.778799803 \times 10^{12} f(5x) + 1.222995574 \times 10^{12} f(4x) \\
& -7.199637445 \times 10^{11} f(3x) + 3.631245118 \times 10^{11} f(2x) \\
& -9.28497931 \times 10^{31} f(x) \|\leq 230230\alpha(7x, x)
\end{aligned}$$

for all $x \in X$. Subtracting (2.14) from (3.16), we obtain

$$\begin{aligned}
& \|657800f(19x) - 15540525f(18x) \\
& +176290400f(17x) - 1278467190f(16x) \\
& +6657294800f(15x) - 2.650281134 \times 10^{10}f(14x) \\
& +8.38506584 \times 10^{10}f(13x) \\
& -2.163507489 \times 10^{11}f(12x) + 4.634921112 \times 10^{11}f(11x) \\
& -8.349766211 \times 10^{11}f(10x) + 1.276265282 \times 10^{12}f(9x) \quad (3.18) \\
& -1.665285164 \times 10^{12}f(8x) + 1.86191773 \times 10^{12}f(7x) \\
& -1.78703786 \times 10^{12}f(6x) + 1.472251781 \times 10^{12}f(5x) \\
& -1.039187566 \times 10^{12}f(4x) + 6.26432807 \times 10^{11}f(3x) \\
& -2.016457306 \times 10^{26}f(2x) + 1.265980291 \times 10^{32}f(x) \|\leq \\
& \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) + 325\alpha(11x, x) \\
& +2600\alpha(10x, x) + 14950\alpha(9x, x) \\
& +65780\alpha(8x, x) + 230230\alpha(7x, x)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(6x, x)$ in (1.2) and multiplying by 657800 on both sides of the resulting equation, we have

$$\begin{aligned}
& \|657800f(19x) - 17102800f(18x) + 213785000f(17x) \\
& -1710280000f(16x) + 9834110000f(15x) \\
& -4.3270084 \times 10^{10}f(14x) \\
& +1.51445249 \times 10^{11}f(13x) - 4.3270084 \times 10^{11}f(12x) \quad (3.19) \\
& +1.027664495 \times 10^{12}f(11x) - 2.05532899 \times 10^{12}f(10x) \\
& +3.494059283 \times 10^{12}f(9x) - 5.082268048 \times 10^{12}f(8x) \\
& +6.352835718 \times 10^{12}f(7x) - 6.841531783 \times 10^{12}f(6x) \\
& +6.353048845 \times 10^{12}f(5x) - 5.083978328 \times 10^{12}f(4x) \\
& +3.503893393 \times 10^{12}f(3x) - 2.098599074 \times 10^{12}f(2x) \\
& -2.652851231 \times 10^{32}f(x) \|\leq 657800\alpha(6x, x)
\end{aligned}$$

for all $x \in X$. Subtracting (3.19) from (3.18), we get

$$\begin{aligned}
& \|1562275f(18x) - 37494600f(17x) + 431812810f(16x) \\
& - 3176815200f(15x) + 1.676727267 \times 10^{10}f(14x) \\
& - 6.75946356 \times 10^{10}f(13x) + 2.163500911 \times 10^{11}f(12x) \\
& - 5.641723838 \times 10^{11}f(11x) + 1.220352369 \times 10^{12}f(10x) \\
& - 2.217794001 \times 10^{12}f(9x) + 3.416982884 \times 10^{12}f(8x) \\
& - 4.490917987 \times 10^{12}f(7x) + 5.054493923 \times 10^{12}f(6x) \\
& - 4.880797064 \times 10^{12}f(5x) + 4.044790762 \times 10^{12}f(4x) \\
& - 2.877460586 \times 10^{12}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 3.918831523 \times 10^{32}f(x)\| \\
& \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) + 325\alpha(11x, x) \\
& + 2600\alpha(10x, x) + 14950\alpha(9x, x) \\
& + 65780\alpha(8x, x) + 230230\alpha(7x, x) + 657800\alpha(6x, x)
\end{aligned} \tag{3.20}$$

for all $x \in X$. Replacing (x, y) by $(5x, x)$ in (3.1) and multiplying by 1562275 on both sides of the resulting equation, we have

$$\begin{aligned}
& \|1562275f(18x) - 40619150f(17x) + 507739375f(16x) \\
& - 4061915000f(15x) + 2.335601125 \times 10^{10}f(14x) \\
& - 1.027664495 \times 10^{11}f(13x) \\
& + 3.596825733 \times 10^{11}f(12x) - 1.027664495 \times 10^{12}f(11x) \\
& + 2.440703176 \times 10^{12}f(10x) - 4.881406351 \times 10^{12}f(9x) \\
& + 8.298392359 \times 10^{12}f(8x) - 1.207042723 \times 10^{13}f(7x) \\
& + 1.508849101 \times 10^{13}f(6x) - 1.625265928 \times 10^{13}f(5x) \\
& + 1.511133928 \times 10^{13}f(4x) - 1.217315306 \times 10^{13}f(3x) \\
& + 8.65807337 \times 10^{12}f(2x) - 6.300521674 \times 10^{32}f(x)\| \\
& \leq 1562275\alpha(5x, x)
\end{aligned} \tag{3.21}$$

for all $x \in X$. Subtracting (3.21) from (3.20), we get

$$\begin{aligned}
& \|3124550f(17x) - 75926565f(16x) + 885099800f(15x) \\
& - 6588738585f(14x) + 3.51718139 \times 10^{10}f(13x) \\
& - 1.433324822 \times 10^{11}f(12x) \\
& + 4.634921112 \times 10^{11}f(11x) - 1.220350807 \times 10^{12}f(10x)
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&+2.66361235 \times 10^{12} f(9x) - 4.881409476 \times 10^{12} f(8x) \\
&+7.579509246 \times 10^{12} f(7x) - 1.003399708 \times 10^{13} f(6x) \\
&+1.137186222 \times 10^{13} f(5x) - 1.106654852 \times 10^{13} f(4x) \\
&+9.295692478 \times 10^{12} f(3x) - 2.016457306 \times 10^{26} f(2x) \\
&+1.02193532 \times 10^{33} f(x) \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) \\
&+325\alpha(11x, x) + 2600\alpha(10x, x) + 14950\alpha(9x, x) + 65780\alpha(8x, x) \\
&+230230\alpha(7x, x) + 657800\alpha(6x, x) + 1562275\alpha(5x, x)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(4x, x)$ in (3.1) and multiplying by 3124550 on both sides of the resulting equation, we obtain

$$\begin{aligned}
&\|3124550f(17x) - 81238300f(16x) + 1015478750f(15x) \\
&-8123830000f(14x) + 4.67120225 \times 10^{10} f(13x) \\
&-2.05532899 \times 10^{11} f(12x) \\
&+7.193651465 \times 10^{11} f(11x) - 2.05532899 \times 10^{12} f(10x) \\
&+4.881409476 \times 10^{12} f(9x) - 9.762893941 \times 10^{12} f(8x) \\
&+1.659779707 \times 10^{13} f(7x) - 2.414889706 \times 10^{13} f(6x) \quad (3.23) \\
&+3.022267856 \times 10^{13} f(5x) - 3.270272763 \times 10^{13} f(4x) \\
&+3.089533168 \times 10^{13} f(3x) - 2.619610222 \times 10^{13} f(2x) \\
&-1.260104335 \times 10^{33} f(x)\| \leq 3124550\alpha(4x, x)
\end{aligned}$$

for all $x \in X$. Subtracting (3.23) from (3.22), we obtain

$$\begin{aligned}
&\|5311735f(16x) - 130378950f(15x) + 1535091415f(14x) \\
&-1.15402086 \times 10^{10} f(13x) + 6.220041686 \times 10^{10} f(12x) \\
&-2.558730353 \times 10^{11} f(11x) + 8.349781833 \times 10^{11} f(10x) \\
&-2.217797125 \times 10^{12} f(9x) + 4.881484465 \times 10^{12} f(8x) \quad (3.24) \\
&-9.018287827 \times 10^{12} f(7x) + 1.411489997 \times 10^{13} f(6x) \\
&-1.885081634 \times 10^{13} f(5x) + 2.163617911 \times 10^{13} f(4x) \\
&-2.15996392 \times 10^{13} f(3x) - 2.016457306 \times 10^{26} f(2x) \\
&+2.282039655 \times 10^{33} f(x)\| \\
&\leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) + 325\alpha(11x, x) \\
&+2600\alpha(10x, x) + 14950\alpha(9x, x) + 65780\alpha(8x, x) \\
&+230230\alpha(7x, x) + 657800\alpha(6x, x) \\
&+1562275\alpha(5x, x) + 3124550\alpha(4x, x)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(3x, x)$ in (3.1) and multiplying by 5311735 on both sides of the resulting equation, we obtain

$$\begin{aligned}
& \|5311735f(16x) - 138105110f(15x) + 1726313875f(14x) \\
& - 1.3810511 \times 10^{10}f(13x) + 7.941043825 \times 10^{10}f(12x) \\
& - 3.494059283 \times 10^{11}f(11x) + 1.222926061 \times 10^{12}f(10x) \\
& - 3.494197388 \times 10^{12}f(9x) + 8.300117111 \times 10^{12}f(8x) \\
& - 1.661059211 \times 10^{13}f(7x) + 2.829393915 \times 10^{13}f(6x) \\
& - 4.138872042 \times 10^{13}f(5x) + 5.252206386 \times 10^{13}f(4x) \\
& - 5.873929032 \times 10^{13}f(3x) + 5.959753391 \times 10^{13}f(2x) \\
& - 2.142177369 \times 10^{33}f(x)\| \leq 5311735\alpha(3x, x)
\end{aligned} \tag{3.25}$$

for all $x \in X$. Subtracting (3.25) from (3.24), we get

$$\begin{aligned}
& \|7726160f(15x) - 191222460f(14x) + 2270302400f(13x) \\
& - 1.72100214 \times 10^{10}f(12x) + 9.353289296 \times 10^{10}f(11x) \\
& - 3.879478775 \times 10^{11}f(10x) + 1.276400263 \times 10^{12}f(9x) \\
& - 3.418632646 \times 10^{12}f(8x) + 7.592304278 \times 10^{12}f(7x) \\
& - 1.417903917 \times 10^{13}f(6x) + 2.253790407 \times 10^{13}f(5x) \\
& - 3.088588475 \times 10^{13}f(4x) + 3.713965112 \times 10^{13}f(3x) \\
& - 2.016457306 \times 10^{26}f(2x) + 4.424217024 \times 10^{33}f(x)\| \\
& \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) \\
& + 325\alpha(11x, x) + 2600\alpha(10x, x) \\
& + 14950\alpha(9x, x) + 65780\alpha(8x, x) \\
& + 230230\alpha(7x, x) + 657800\alpha(6x, x) \\
& + 1562275\alpha(5x, x) + 3124550\alpha(4x, x) + 5311735\alpha(3x, x)
\end{aligned} \tag{3.26}$$

for all $x \in X$. Replacing (x, y) by $(2x, x)$ in (3.1) and multiplying by 7726160 on both sides of the resulting equation, we have

$$\begin{aligned}
& \|7726160f(15x) - 200880160f(14x) + 2511002000f(13x) \\
& - 2.0088016 \times 10^{10}f(12x) + 1.155138182 \times 10^{11}f(11x) \\
& - 5.08427685 \times 10^{11}f(10x) + 1.781304819 \times 10^{12}f(9x) \\
& - 5.102356064 \times 10^{12}f(8x) + 1.218589271 \times 10^{13}f(7x) \\
& - 2.464900003 \times 10^{13}f(6x) + 4.28181083 \times 10^{13}f(5x) \\
& - 6.477581639 \times 10^{13}f(4x) + 8.668732205 \times 10^{13}f(3x) \\
& - 1.044974729 \times 10^{14}f(2x) - 3.115894355 \times 10^{33}f(x)\| \\
& \leq 7726160\alpha(2x, x)
\end{aligned} \tag{3.27}$$

for all $x \in X$. Subtracting (3.27) from (3.26), we obtain

$$\begin{aligned}
& \|9657700f(14x) - 240699600f(13x) + 2877994600f(12x) \\
& - 2.19809252 \times 10^{10}f(11x) + 1.204798075 \times 10^{11}f(10x) \\
& - 5.04904556 \times 10^{11}f(9x) + 1.683723418 \times 10^{12}f(8x) \\
& - 4.593588428 \times 10^{12}f(7x) + 1.046996086 \times 10^{13}f(6x) \\
& - 2.028020423 \times 10^{13}f(5x) + 3.388993165 \times 10^{13}f(4x) \\
& - 4.954767093 \times 10^{13}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 7.540111379 \times 10^{33}f(x)\| \\
& \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) \\
& + 325\alpha(11x, x) + 2600\alpha(10x, x) + 14950\alpha(9x, x) + 65780\alpha(8x, x) \\
& + 230230\alpha(7x, x) + 657800\alpha(6x, x) + 1562275\alpha(5x, x) \\
& + 3124550\alpha(4x, x) + 5311735\alpha(3x, x) + 7726160\alpha(2x, x)
\end{aligned} \tag{3.28}$$

for all $x \in X$. Replacing (x, y) by (x, x) in (3.1) and multiplying by 9657700 on both sides of the resulting equation, we obtain

$$\begin{aligned}
& \|9657700f(14x) - 251100200f(13x) + 3148410200f(12x) \\
& - 2.53611202 \times 10^{10}f(11x) + 1.475213675 \times 10^{11}f(10x) \\
& - 6.60393526 \times 10^{11}f(9x) + 2.367874886 \times 10^{12}f(8x) \\
& - 6.988118566 \times 10^{12}f(7x) + 1.731147554 \times 10^{13}f(6x) \\
& - 3.65288016 \times 10^{13}f(5x) + 6.638712638 \times 10^{13}f(4x) \\
& - 1.04792902 \times 10^{14}f(3x) + 1.445703124 \times 10^{14}f(2x) \\
& - 3.894867944 \times 10^{33}f(x)\| \leq 9657700\alpha(x, x)
\end{aligned} \tag{3.29}$$

for all $x \in X$. Subtracting (3.29) from (3.28), we have

$$\begin{aligned}
& \|10400600f(13x) - 270415600f(12x) + 3380195000f(11x) \\
& - 2.704156 \times 10^{10}f(10x) + 1.5548897 \times 10^{11}f(9x) \\
& - 6.84151468 \times 10^{11}f(8x) \\
& + 2.394530138 \times 10^{12}f(7x) - 6.84151468 \times 10^{12}f(6x) \\
& + 1.624859737 \times 10^{13}f(5x) - 3.249719473 \times 10^{13}f(4x) \\
& + 5.524523104 \times 10^{13}f(3x) - 2.016457306 \times 10^{26}f(2x) \\
& + 1.143497932 \times 10^{34}f(x)\| \leq \alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) \\
& + 325\alpha(11x, x) + 2600\alpha(10x, x) + 14950\alpha(9x, x) + 65780\alpha(8x, x) \\
& + 230230\alpha(7x, x) + 657800\alpha(6x, x) + 1562275\alpha(5x, x) + 3124550\alpha(4x, x) \\
& + 5311735\alpha(3x, x) + 7726160\alpha(2x, x) + 9657700\alpha(x, x)
\end{aligned} \tag{3.30}$$

for all $x \in X$. Replacing (x, y) by $(0, x)$ in (3.1) and multiplying by 10400600 on both sides of the resulting equation, we obtain

$$\begin{aligned} & \|10400600f(13x) - 270415600f(12x) + 3380195000f(11x) \\ & - 2.704156 \times 10^{10}f(10x) + 1.5548897 \times 10^{11}f(9x) \\ & - 6.84151468 \times 10^{11}f(8x) \\ & + 2.394530138 \times 10^{12}f(7x) - 6.84151468 \times 10^{12}f(6x) \\ & + 1.624859737 \times 10^{13}f(5x) - 3.249719473 \times 10^{13}f(4x) \\ & + 5.524523104 \times 10^{13}f(3x) - 8.03566997 \times 10^{13}f(2x) \\ & - 2.097236585 \times 10^{33}f(x)\| \leq 10400600\alpha(0, x) \end{aligned} \quad (3.31)$$

for all $x \in X$. Subtracting (3.31) from (3.30), we obtain

$$\begin{aligned} \|f(2x) - 2^{26}f(x)\| & \leq \frac{1}{2.016457306 \times 10^{26}} [\alpha(0, 2x) + \alpha(13x, x) \\ & + 26\alpha(12x, x) + 325\alpha(11x, x) + 2600\alpha(10x, x) + 14950\alpha(9x, x) \\ & + 65780\alpha(8x, x) + 230230\alpha(7x, x) + 657800\alpha(6x, x) \\ & + 1562275\alpha(5x, x) + 3124550\alpha(4x, x) + 5311735\alpha(3x, x) \\ & + 7726160\alpha(2x, x) + 9657700\alpha(x, x) + 10400600\alpha(0, x)] \end{aligned} \quad (3.32)$$

for all $x \in X$. Define

$$\begin{aligned} \alpha(x, x) & = \frac{1}{2.016457306 \times 10^{26}} [\alpha(0, 2x) + \alpha(13x, x) + 26\alpha(12x, x) \\ & + 325\alpha(11x, x) + 2600\alpha(10x, x) + 14950\alpha(9x, x) + 65780\alpha(8x, x) \\ & + 230230\alpha(7x, x) + 657800\alpha(6x, x) + 1562275\alpha(5x, x) + 3124550\alpha(4x, x) \\ & + 5311735\alpha(3x, x) + 7726160\alpha(2x, x) + 9657700\alpha(x, x) + 10400600\alpha(0, x)] \end{aligned}$$

for all $x \in X$. From (3.32), we have

$$\|f(2x) - 2^{26}f(x)\| \leq \alpha(x, x) \quad (3.33)$$

for all $x \in X$. It follows from (3.33) that

$$\left\| \frac{f(2x)}{2^{26}} - f(x) \right\| \leq \frac{\alpha(x, x)}{2^{26}} \quad (3.34)$$

for all $x \in X$. Now, replacing x by $2x$ and dividing 2^{26} in (3.34), we obtain

$$\left\| \frac{f(2^2x)}{2^{52}} - \frac{f(2x)}{2^{26}} \right\| \leq \frac{\alpha(2x, 2x)}{2^{52}} \quad (3.35)$$

for all $x \in X$. From (3.34) and (3.35), we obtain

$$\left\| \frac{f(2^2x)}{2^{52}} - f(x) \right\| \leq \frac{1}{2^{26}} \left[\alpha(x, x) + \frac{\alpha(2x, 2x)}{2^{26}} \right] \quad (3.36)$$

for all $x \in X$. For any positive integer a , we get

$$\left\| \frac{f(2^a x)}{2^{a(26)}} - f(x) \right\| \leq \frac{1}{2^{26}} \sum_{i=0}^{a-1} \frac{\alpha(2^i x, 2^i x)}{2^{i(26)}} \quad (3.37)$$

for all $x \in X$. To prove the convergence of the sequence $\left\{ \frac{f(2^a x)}{2^{a(26)}} \right\}$, we replace x by $2^i x$ in (3.37) and divide the resulting inequality by $2^{i(26)}$ for any $a, i > 0$. Then we get

$$\begin{aligned} \left\| \frac{f(2^{a+i} x)}{2^{(a+i)(26)}} - \frac{f(2^i x)}{2^{i(26)}} \right\| &= \frac{1}{2^{26i}} \left\| \frac{f(2^a \cdot 2^i x)}{2^{a(26)}} - f(2^i x) \right\| \\ &\leq \frac{1}{2^{i(26)}} \frac{1}{2^{26}} \sum_{c=0}^{a-1} \frac{\alpha(2^c \cdot 2^i x, 2^c \cdot 2^i x)}{2^{c(26)}} \\ &\leq \frac{1}{2^{26}} \sum_{c=0}^{\infty} \frac{\alpha(2^{c+i} x, 2^{c+i} x)}{2^{26(c+i)}} \\ &\rightarrow 0 \text{ as } i \rightarrow \infty \end{aligned}$$

for all $x \in X$. Thus it follows that the sequence $\left\{ \frac{f(2^a x)}{2^{a(26)}} \right\}$ is Cauchy in Y and so it converges. Therefore, we see that a mapping $G : X \rightarrow Y$ defined by

$$G(x) = \lim_{a \rightarrow \infty} \frac{f(2^a x)}{2^{a(26)}}$$

is well defined for all $x \in X$. In order to show that G satisfies (1.2) by replacing (x, y) by $(2^a x, 2^a y)$ in (3.1) and then dividing by $2^{a(26)}$, we get

$$\|G(x, y)\| = \lim_{a \rightarrow \infty} \frac{1}{2^{a(26)}} \|Df_{26}(2^a x, 2^a y)\| \leq \lim_{a \rightarrow \infty} \frac{1}{2^{a(26)}} \alpha(2^a x, 2^a y)$$

for all $x, y \in X$ and so the mapping G is sexvigintic.

Taking the limit as a approaches to infinity in (3.37), we find that the mapping G is a sexvigintic mapping satisfying the inequality (3.1) near the approximate mapping $f : X \rightarrow Y$ of equation (1.2). Hence G satisfies (1.2), for all $x, y \in X$. To prove that G is unique, we assume now that there is U as another sexvigintic mapping satisfying (1.2) and the inequality (3.2). Then

$$\begin{aligned} \|G(x) - U(x)\| &= \frac{1}{2^{a(26)}} \|G(2^a x) - U(2^a x)\| \\ &\leq \frac{1}{2^{a(26)}} \{ \|G(2^a x) - f(2^a x)\| + \|f(2^a x) - U(2^a x)\| \} \\ &\leq \frac{1}{2^{26}} \sum_{i=0}^{\infty} \frac{\alpha(2^{c+a} x, 2^{c+a} x)}{2^{26(c+a)}} \end{aligned}$$

for all $x \in X$. Therefore, as $a \rightarrow \infty$ in the above inequality, we conclude that $G(x) = U(x)$ for all $x \in X$. Now, replacing x by $\frac{x}{2}$ in (3.33), we have

$$\left\| f(x) - 2^{27} f\left(\frac{x}{2}\right) \right\| \leq \alpha\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. The rest of the proof is similar to that of the case $j = 1$. Thus, for $j = -1$ the assertion holds as well. This completes the proof. \square

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.2).

Corollary 3.2. *Let $Df_{26} : X \rightarrow Y$ be a mapping. If there exist real numbers ψ and λ such that*

$$\|Df_{26}(x, y)\| \leq \begin{cases} \psi, & \lambda \neq 26 \\ \psi \{ \|x\|^\lambda + \|y\|^\lambda \}, & \lambda \neq \frac{26}{2} \\ \psi \{ \|x\|^\lambda \|y\|^\lambda, & \lambda \neq \frac{26}{2} \\ \psi \{ \|x\|^\lambda \|y\|^\lambda + \{ \|x\|^{2\lambda} + \|y\|^{2\lambda} \} \}, & \lambda \neq \frac{26}{2} \end{cases}$$

for all $x, y \in X$, then there exists a unique sexvigintic mapping $G : X \rightarrow Y$ such that

$$\|f(x) - G(x)\| \leq \begin{cases} \frac{\psi_C}{|2^{26}-1|} \\ \frac{\psi_S \|x\|^\lambda}{|2^{26}-2^\lambda|} \\ \frac{\psi_P \|x\|^{2\lambda}}{|2^{26}-2^{2\lambda}|} \\ \frac{\psi_{SP} \|x\|^{2\lambda}}{|2^{26}-2^{2\lambda}|} \end{cases}$$

for all $x \in X$. where

$$\begin{aligned} \psi_C &= \frac{38754733 \psi}{2.016457306 \times 10^{26}}, \\ \psi_S &= \frac{\psi}{2.016457306 \times 10^{26}} \left[13^\lambda + 26 \times 12^\lambda + 325 \times 11^\lambda + 2600 \times 10^\lambda \right. \\ &\quad + 14950 \times 9^\lambda + 65780 \times 8^\lambda + 230230 \times 7^\lambda \\ &\quad + 657800 \times 6^\lambda + 1562275 \times 5^\lambda \\ &\quad \left. + 3124550 \times 4^\lambda + 5311735 \times 3^\lambda + 7726161 \times 2^\lambda + 48412432 \right], \\ \psi_P &= \frac{\psi}{2.016457306 \times 10^{26}} \left[13^\lambda + 26 \times 12^\lambda + 325 \cdot 11^\lambda + 2600 \times 10^\lambda \right. \\ &\quad + 14950 \times 9^\lambda + 65780 \times 8^\lambda + 230230 \times 7^\lambda \\ &\quad + 657800 \times 6^\lambda + 1562275 \times 5^\lambda \\ &\quad \left. + 3124550 \times 4^\lambda + 5311735 \times 3^\lambda + 7726161 \times 2^\lambda + 9657700 \right], \end{aligned}$$

$$\begin{aligned} \psi_{SP} = & \frac{\psi}{2.016457306 \times 10^{26}} \{ [13^\lambda + 26 \times 12^\lambda + 325 \times 11^\lambda + 2600 \times 10^\lambda \\ & + 14950 \times 9^\lambda + 65780 \times 8^\lambda + 230230 \times 7^\lambda + 657800 \times 6^\lambda + 1562275 \times 5^\lambda \\ & + 3124550 \times 4^\lambda + 5311735 \times 3^\lambda + 7726161 \times 2^\lambda + 9657700] + [13^{2\lambda} \\ & + 26 \times 12^{2\lambda} + 325 \times 11^{2\lambda} + 2600 \times 10^{2\lambda} + 14950 \times 9^{2\lambda} + 65780 \times 8^{2\lambda} \\ & + 230230 \times 7^{2\lambda} + 657800 \times 6^{2\lambda} + 1562275 \times 5^{2\lambda} + 3124550 \times 4^{2\lambda} \\ & + 5311735 \times 3^{2\lambda} + 7726161 \times 2^{2\lambda} + 48412432] \}. \end{aligned}$$

4. STABILITY RESULT OF THE FUNCTIONAL EQUATION (1.2): FIXED POINT METHOD

In this section, we establish the stability result for the new type of functional equation (1.2) in Banach spaces using the fixed point method.

Theorem 4.1. *Let $Df_{26} : X \rightarrow Y$ be a mapping for which there exists a function $\gamma : X^2 \rightarrow [0, \infty)$ with the condition*

$$\lim_{n \rightarrow \infty} \frac{1}{\eta_i^{26n}} \gamma(\eta_i^n x, \eta_i^n y) = 0 \quad (4.1)$$

where $\eta_i = \begin{cases} 2 & \text{if } i = 0 \\ \frac{1}{2} & \text{if } i = 1 \end{cases}$, such that the functional inequality

$$\|Df_{26}(x, y)\| \leq \gamma(x, y) \quad (4.2)$$

holds for all $x, y \in X$. Assume that there exists $L = L(i)$ such that the mapping

$$D(x, x) = \alpha\left(\frac{x}{2}, \frac{x}{2}\right)$$

where $\alpha(x, x)$ is defined in (3.3) with the property

$$\frac{1}{\eta_i^{26}} D(\eta_i x, \eta_i x) = LD(x, x) \quad (4.3)$$

for all $x \in X$. Then there exists a unique sexvigintic mapping $G : X \rightarrow Y$ satisfying the functional equation (1.2) and

$$\|f(x) - G(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) D(x, x) \quad (4.4)$$

for all $x \in X$.

Proof. Consider the set $Z = \{h|h : X \rightarrow Y, h(0) = 0\}$ and introduce the generalized metric $d : Z \times Z \rightarrow [0, \infty]$ as follows:

$$d(g, h) = \inf \{\omega \in (0, \infty) : \|g(x) - h(x)\| \leq \omega D(x, x), x \in X\} \quad (4.5)$$

It is easy to see that (Z, d) is complete with respect to the defined metric. Let us define the linear mapping $J : Z \rightarrow Z$ by

$$Jh(x) = \frac{1}{\eta_i^{26}} h(\eta_i x)$$

for all $x \in X$. For given $h, g \in Z$, let $\omega \in [0, \infty)$ be an arbitrary constant with $d(g, h) \leq \omega$, that is,

$$\|h(x) - g(x)\| \leq \omega D(x, x)$$

for all $x \in X$. So we have

$$\begin{aligned} \|Jh(x) - Jg(x)\|_X &= \left\| \frac{1}{\eta_i^{26}} h(\eta_i x) - \frac{1}{\eta_i^{26}} g(\eta_i x) \right\| \\ &\leq \frac{\omega}{\eta_i^{26}} D(\eta_i x, \eta_i x) \\ &\leq L\omega D(x, x) \end{aligned}$$

for all $x \in X$, that is, $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in Z$. This implies that J is a strictly contractive mapping on Z with Lipschitz constant L . From (3.33), (4.3) and (4.5) for the case $i = 0$, we have

$$\|f(2x) - 2^{26} f(x)\| \leq \alpha(x, x)$$

and

$$\left\| \frac{f(2x)}{2^{26}} - f(x) \right\| \leq \frac{1}{2^{26}} \alpha(x, x)$$

for all $x \in X$. So we obtain

$$\|Jf(x) - f(x)\| \leq LD(x, x)$$

for all $x \in X$. Hence

$$d(Jf, f) \leq L^{1-0} \quad (4.6)$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (4.3) and (4.6) for the case $i = 1$, we find

$$\left\| f(x) - 2^{26} f\left(\frac{x}{2}\right) \right\| \leq \alpha\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Then

$$\|f(x) - Jf(x)\| \leq D(x, x)$$

and

$$\|f(x) - Jf(x)\| \leq L^{1-1} D(x, x)$$

for all $x \in X$. Thus we obtain

$$d(Jf, f) \leq L^{1-1} \quad (4.7)$$

for all $f \in Z$. Therefore, from (4.6) and (4.7), we obtain

$$d(Jf, f) \leq L^{1-i}$$

for all $f \in Z$, where $i = 0, 1$. Hence the property (B1) holds. It follows from property (B2) that there exists a fixed point G of J in Z such that

$$G(x) = \lim_{n \rightarrow \infty} \frac{1}{\eta_i^{26n}} f(\eta_i^n x) \tag{4.8}$$

for all $x \in X$. In order to show that G satisfies (1.2), replace (x, y) by $(\eta_i^n x, \eta_i^n y)$ in (4.2) and divide by η_i^{26n} , we get

$$\begin{aligned} \|G_{26}(x, y)\| &= \lim_{n \rightarrow \infty} \frac{1}{\eta_i^{26n}} \|Df_{26}(\eta_i^n x, \eta_i^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{\eta_i^{26n}} \gamma(\eta_i^n x, \eta_i^n y) = 0 \end{aligned}$$

for all $x, y \in X$ and so the mapping G is sexvigintic. By property (B3), G is the unique fixed point of J in the set

$$\Delta = \{G \in Z : d(f, G) < \infty\}.$$

Finally, by property (B4), we arrive

$$\|f(x) - G(x)\| \leq \frac{L^{1-i}}{1-L}.$$

Hence the proof is completed. □

Using Theorem 4.1, we prove the following corollary concerning the stability of (1.2).

Corollary 4.2. *Let $Df_{26} : X \rightarrow Y$ be a mapping. If there exist real numbers ψ and λ such that*

$$\|Df_{26}(x, y)\| \leq \begin{cases} \psi, & \lambda \neq 26 \\ \psi \{ \|x\|^\lambda + \|y\|^\lambda \}, & \lambda \neq \frac{26}{2} \\ \psi \|x\|^\lambda \|y\|^\lambda, & \lambda \neq \frac{26}{2} \\ \psi \{ \|x\|^\lambda \|y\|^\lambda + \{ \|x\|^{2\lambda} + \|y\|^{2\lambda} \} \}, & \lambda \neq \frac{26}{2} \end{cases}$$

for all $x, y \in X$, then there exists a unique sexvigintic mapping $G : X \rightarrow Y$ such that

$$\|f(x) - G(x)\| \leq \begin{cases} \frac{\psi_C}{|2^{26}-1|} \\ \frac{\psi_S \|x\|^\lambda}{|2^{26}-2^\lambda|} \\ \frac{\psi_P \|x\|^{2\lambda}}{|2^{26}-2^{2\lambda}|} \\ \frac{\psi_{SP} \|x\|^{2\lambda}}{|2^{26}-2^{2\lambda}|} \end{cases}$$

for all $x \in X$, where $\psi_C, \psi_S, \psi_P, \psi_{SP}$ are given in Corollary 3.2.

Proof. Let

$$\gamma(x, y) = \begin{cases} \psi \\ \psi \{ \|x\|^\lambda + \|y\|^\lambda \} \\ \psi \|x\|^\lambda \|y\|^\lambda \\ \psi \{ \|x\|^\lambda \|y\|^\lambda + \{ \|x\|^{2\lambda} + \|y\|^{2\lambda} \} \} \end{cases}$$

for all $x, y \in X$. Now

$$\frac{1}{\eta_i^{26n}} \gamma(\eta_i^n x, \eta_i^n y) = \begin{cases} \frac{\psi}{\eta_i^{26n}} \\ \frac{\psi}{\eta_i^{27n}} \{ \|\eta_i^n x\|^\lambda + \|\eta_i^n y\|^\lambda \} \\ \frac{\psi}{\eta_i^{26n}} \|\eta_i^n x\|^\lambda \|\eta_i^n y\|^\lambda \\ \frac{\psi}{\eta_i^{26n}} \{ \|\eta_i^n x\|^\lambda \|\eta_i^n y\|^\lambda + \{ \|\eta_i^n x\|^{2\lambda} + \|\eta_i^n y\|^{2\lambda} \} \} \end{cases}$$

which tends to zero as $n \rightarrow \infty$. Thus (4.1) holds. On the other hand

$$D(x, x) = \alpha \left(\frac{x}{2}, \frac{x}{2} \right) \tag{4.9}$$

for all $x \in X$. It follows from (4.9) that

$$D(x, x) = \alpha \left(\frac{x}{2}, \frac{x}{2} \right) = \begin{cases} \psi_C \\ \psi_S \\ \psi_P \\ \psi_{SP} \end{cases}$$

for all $x \in X$. Similarly by (4.9), we prove

$$\frac{1}{\eta_i^{26}} D(\eta_i x, \eta_i x) = \begin{cases} \eta_i^{-26} \psi_C \\ \eta_i^{\lambda-26} \psi_S \\ \eta_i^{2\lambda-26} \psi_P \\ \eta_i^{2\lambda-26} \psi_{SP} \end{cases} .$$

Hence the inequality (4.4) holds for

- (i) $L = \eta_i^{-26}$ if $i = 0$ and $L = \eta_i^{26}$ if $i = 1$;
- (ii) $L = \eta_i^{\lambda-26}$ for $\lambda < 26$ if $i = 0$ and $L = \eta_i^{26-\lambda}$ for $\lambda > 26$ if $i = 1$;
- (iii) $L = \eta_i^{2\lambda-26}$ for $2\lambda < 26$ if $i = 0$ and $L = \eta_i^{26-2\lambda}$ for $2\lambda > 26$ if $i = 1$;
- (iv) $L = \eta_i^{2\lambda-26}$ for $2\lambda < 26$ if $i = 0$ and $L = \eta_i^{26-2\lambda}$ for $2\lambda > 26$ if $i = 1$.

Now, from (4.4), we show the following cases for condition (i).

- (i) $L = \eta_i^{-26}$ if $i = 0$

$$\|f(x) - G(x)\| \leq \left(\frac{L^{1-i}}{1-L} \right) D(x, x) = \left(\frac{2^{-26}}{1-2^{-26}} \right) \psi_C = \frac{\psi_C}{2^{26}-1}$$

and for $L = \eta_i^{26}$ if $i = 1$

$$\|f(x) - G(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) D(x, x) = \left(\frac{1}{1-2^{26}}\right) \psi_C = \frac{\psi_C}{1-2^{26}}.$$

(ii) $L = \eta_i^{\lambda-26}$ for $p < 26$ if $i = 0$

$$\|f(x) - G(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) D(x, x) = \left(\frac{2^{\lambda-26}}{1-2^{\lambda-26}}\right) \psi_S = \frac{2^\lambda \cdot \psi_S}{2^{26} - 2^\lambda}$$

and for $L = \eta_i^{26-\lambda}$ for $p > 26$ if $i = 1$

$$\|f(x) - G(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) D(x, x) = \left(\frac{1}{1-2^{26-\lambda}}\right) \psi_S = \frac{2^\lambda \cdot \psi_S}{2^\lambda - 2^{26}}.$$

(iii) $L = \eta_i^{2\lambda-26}$ for $2\lambda < 26$ if $i = 0$

$$\|f(x) - G(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) D(x, x) = \left(\frac{2^{2\lambda-26}}{1-2^{2\lambda-26}}\right) \psi_P = \frac{2^{2\lambda} \cdot \psi_P}{2^{26} - 2^{2\lambda}}$$

and for $L = \eta_i^{26-2\lambda}$ for $2\lambda > 26$ if $i = 1$

$$\|f(x) - G(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) D(x, x) = \left(\frac{1}{1-2^{26-2\lambda}}\right) \psi_P = \frac{2^{2\lambda} \cdot \psi_P}{2^{2\lambda} - 2^{26}}.$$

Finally, the proof of (4.4) for condition (iv) is similar to the condition (ii) and (iii). \square

REFERENCES

- [1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.
- [2] L. Cadariu and V. Radu, *Fixed points and the stability of Jensens functional equation*, J. Inequal. Pure Appl. Math., **4** (2003), 1–7.
- [3] J. Gao, *On the stability of functional equations in 2-normed spaces*, Nonlinear Funct. Anal. Appl., **15** (2010), 635–645.
- [4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.
- [5] V. Govindan, S. Murthy and G. Gokila, *Fixed point and stability of Icosic functional equation in quasi β -normed spaces*, Malaya J. Matematik, **6** (2018), 261–275.
- [6] V. Govindan, S. Murthy and M. Saravanan, *Solution and stability of a cubic type functional equation: using direct and fixed point methods*, Kragujevac J. Math., **44** (2018), 7–26.
- [7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224.
- [8] G. Isac and Th. M. Rassias, *Stability of ϕ -additive mappings: Applications to nonlinear analysis*, J. Funct. Anal., **19** (1996), 219–228.

- [9] Y. Manar, E. Elqorachi and Th. M. Rassias, *Hyers-Ulam stability of the Jensen functional equation in quasi-Banach spaces*, *Nonlinear Funct. Anal. Appl.* **15** (2010), 581–603.
- [10] R. Murali and V. Vithya, *Hyers-Ulam-Rassias stability of functional equations in matrix normed spaces: A fixed point approach*, *Asian J. Math. Comput. Research*, **4** (2015), 155–163.
- [11] S. Murthy, M. Arunkumar and V. Govindan, *General solution and generalizes Hyers-Ulam stability of a generalized n -type additive quadratic functional equation in Banach space and Banach algebra: using direct and fixed point methods*, *Int. J. Adv. Math. Sci.*, **3** (2015), 24–64.
- [12] A. Najati and C. Park, *On the stability of an n -dimensional functional equation originating from quadratic forms*, *Taiwanese J. Math.*, **12**, (2008), 1609–1624.
- [13] C. Park, *Stability of an AQCQ-functional equation in paranormed spaces*, *Adv. Difference Equ.*, **2012**, 2012:148, 20 pages.
- [14] M. Ramdoss, C. Park, V. Veeramani and Y. Cho, *A septendecic functional equation in matrix normed spaces*, *J. Comput. Anal. Appl.*, **26** (2019), 318–333.
- [15] M. Ramdoss, S. Pinelas and V. Veeramani, *Stability of tredecic functional equation in matrix normed spaces*, *J. Adv. Math.*, **13** (2017), 7135–7145.
- [16] J. M. Rassias, M. Arunkumar, E. Sathya and T. Namachivayam, *Various generalized Ulam-Hyers stabilities of a nonic functional equation* (preprint).
- [17] J. M. Rassias and M. Eslamian, *Fixed points and stability of nonic functional equation in quasi β -normed spaces*, *Cont. Anal. Appl. Math.*, **3** (2015), 293–309.
- [18] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300.
- [19] K. Ravi, J. M. Rassias and B. V. Senthil Kumar, *Ulam-Hyers stability of undecic functional equation in quasi β -normed spaces: Fixed point method*, *Tbilisi Math. Sci.*, **9** (2016), 83–103.
- [20] Y. Shen and W. Chen, *On the stability of septic and octic functional equations*, *J. Comput. Anal. Appl.*, **18** (2015), 277–290.
- [21] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, 1964.
- [22] Z. Wang and P. K. Sahoo, *Stability of an ACQ-functional equation in various matrix normed spaces*, *J. Nonlinear Sci. Appl.*, **8** (2015), 64–85.
- [23] T. Z. Xu , J. M. Rassias, M. J. Rassias and W. X. Xu, *A fixed point approach to the stability of quintic and sextic functional equations in quasi β -normed spaces*, *J. Inequal. Appl.*, **2010**, Art. ID 423231, 23 pages.