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EQUILIBRIUM EXISTENCE THEOREMS IN HADAMARD MANIFOLDS

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Abstract. In this paper, we first prove a generalization of Fan-Browder type fixed point theorem for multimaps on a geodesic convex subset in a Hadamard manifold due to Kim [5], and next we will prove general existence theorems of Nash equilibrium for generalized games $\mathcal{G}$ with geodesic convex values in Hadamard manifolds.

1. Introduction

In 1950, Nash [9] established the pioneering result on the existence of equilibrium for abstract economies. Since then, the classical results of Nash [9] and Debreu [4] have served as basic references for the existence of Nash equilibrium for generalized games. As in [1,2,4,9,13], in most results on the existence of equilibria for abstract economies, the underlying spaces (commodity spaces or choice sets) are always compact and convex sets in topological vector spaces. Till now, there have been a number of generalizations, and also many applications of those theorems have been found in several areas, e.g., see [1] and references therein.

In the last three decades, without assuming the linear structures, several important concepts of nonlinear analysis have been extended from Euclidian spaces to Riemannian manifold settings in order to go further in the studies of convex analysis, fixed point theory, variational problems, and related topics. In recent papers (e.g. [11,12]), there have been some Fan-Browder type fixed

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point theorem for multimaps in a geodesic convex subset of a Hadamard manifold. However, there are some typical problems in the concept of geodesic convex hull as remarked in [8]. In a recent paper [5], the author provided an exact Fan-Browder type fixed point theorems for multimaps on a geodesic convex subset of a Hadamard manifold by using a geodesic KKM theorem for closed valued multimaps. Actually, Fan-Browder type fixed point theorems for geodesic convex sets in Hadamard manifolds can be basic tools for solving nonlinear problems in Hadamard manifolds, and there might have been numerous generalizations and applications in numerous areas of nonlinear analysis where various generalized geodesic convexity concepts are equipped.

In this paper, we first prove a generalization of Fan-Browder type fixed point theorem for multimaps on a geodesic convex subset in a Hadamard manifold due to Kim [5], and next we will prove general existence theorems of Nash equilibrium for generalized games $\mathcal{G} = (X_i; A_i, P_i)_{i\in I}$ with geodesic convex values in Hadamard manifolds.

2. Preliminaries

We begin with some basic definitions and terminologies on Riemannian manifolds in [3,7,8,10]. Let M be a complete finite dimensional Riemannian manifold with the Levi-Civita connection ∇ on M. Let $x \in M$ and let T_xM denote the tangent space at x to M. For $x,\ y \in M$, let $\gamma_{x,y} : [0,1] \to M$ be a piecewise smooth curve joining x to y. Then, a curve $\gamma_{x,y}$ (γ for short) is called a *geodesic* if $\gamma(0) = x$, $\gamma(1) = y$, and $\nabla_{\dot\gamma}\dot\gamma = 0$ for all $t \in [0,1]$. A geodesic $\gamma_{x,y} : [0,1] \to M$ joining x to y is *minimal* if its arc-length equals its Riemannian distance between x and y. And, M is called a *Hadamard manifold* if M is a simply connected complete Riemannian manifold of nonpositive sectional curvature. In a Hadamard manifold, the geodesic between any two points is unique, and the exponential map at each point of M is a global diffeomorphism. Therefore all convexities in a Hadamard manifold as in [8] coincide.

Let X be a nonempty subset of a Riemannian manifold M, we shall denote by 2^X the family of all subsets of X. If $T : X \to 2^M$ and $S : X \to 2^M$ are multimaps (or correspondences), then $S \cap T : X \to 2^M$ is a correspondence defined by $(S \cap T)(x) = S(x) \cap T(x)$ for each $x \in X$. When a multimap $T : X \to 2^X$ is given, we shall denote $T^{-1}(y) := \{x \in X \mid y \in T(x)\}$ for each $y \in X$. A multimap T has *open graph* in X if the graph $Gr\,T := \{(x,y) \in X \times X \mid x \in X \text{ and } y \in T(x)\}$ is open in $X \times X$. When a multimap $T_i : X \to 2^{X_i}$ has open graph in X for each $i \in I$, where $X = \Pi_{i\in I} X_i$, and let $T : X \to 2^X$ be a multimap defined by $T(x) := \Pi_{i \in I} T_i(x)$ for each $x \in X$, then it is easy to see that the graph of T is open in $X \times X$.

Recall the following concept which generalize the convex condition in linear spaces to Riemannian manifolds:

Definition 2.1. A nonempty subset X of a Riemannian manifold M is said to be *geodesic convex* if for any $x, y \in X$, the geodesic joining x to y is contained in X. For an arbitrary subset C of M, the minimal geodesic convex subset which contains C is called the *geodesic convex hull* of C, and denoted by $Gco(C)$.

Then the above definition of geodesic convex hull in a Riemannian manifold M overcomes the delicate problems of geodesic convexity remarked in [8]. As shown in [3], note that $Gco\,(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$, and $C_n = \{z \in \gamma_{x,y} \mid x, y \in C_{n-1}\}$ for each $n \in \mathbb{N}$.

If S is geodesic convex, then $Gco\,(S) = S$, and the intersection of two geodesic convex subsets of M is clearly geodesic convex; but the union two geodesic convex subsets need not be geodesic convex.

The following operations are essential in proving the geodesic convexity:

Lemma 2.2. *Let X and Y be nonempty subsets of a Hadamard manifold M, $X \cap Y$ be nonempty, and $Gco\,(X)$ and $Gco\,(Y)$ be two geodesic convex hulls of X and Y in M, respectively. Then we have*

(1) *$Gco\,(X) \cap Gco\,(Y)$ is geodesic convex;*
(2) *$Gco\,(X \cap Y)$ is a geodesic convex subset of $Gco\,(X) \cap Gco\,(Y)$;*
(3) *$Gco\,(X \times Y)$ is a geodesic convex subset of $Gco\,(X) \times Gco\,(Y)$.*

Proof. (1) Let any $x, y \in Gco\,(X) \cap Gco\,(Y)$. Then $x, y \in Gco\,(X)$ and $Gco\,(Y)$, respectively. Then the geodesic $\gamma_{x,y}$ joining x to y is contained in $Gco\,(X)$ and $Gco\,(Y)$ so that $\gamma_{x,y} \in Gco\,(X) \cap Gco\,(Y)$ and hence $Gco\,(X) \cap Gco\,(Y)$ is geodesic convex.
(2) Since $X \cap Y \subseteq Gco\,(X \cap Y) \subseteq Gco\,(X)$ and $X \cap Y \subseteq Gco\,(X \cap Y) \subseteq Gco\,(Y)$ are clear, it follows from (1).
(3) Similarly, we can show that $Gco\,(X) \times Gco\,(Y)$ is geodesic convex which contains $X \times Y$ so that we obtain the conclusion. □

Next, we recall some notions and terminologies on the generalized Nash equilibrium for pure strategic games as in [1,4,13]. Let $I = \{1, 2, \ldots, n\}$ be a finite (or possibly countably infinite) set of players. For each $i \in I$, let X_i be a nonempty set of actions. An *abstract economy* (or *generalized game*) $\Gamma = (X_i; A_i, P_i)_{i \in I}$ is defined as a family of ordered triples $(X_i; A_i, P_i)$ where X_i is a nonempty topological space (a choice set), $A_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a constraint correspondences and $P_i : \Pi_{j\in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\hat{x} \in X = \Pi_{i\in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. When I is singleton, then $\mathcal{G}$ is called an 1-*person game*.

In a recent paper [6], the author proves a Fan-Browder type fixed point theorem for geodesic convex sets in Hadamard manifolds which is a slight generalized form of Theorem 3.1 in [5] as follows:

Lemma 2.3. ([6]) *Let X be a nonempty geodesic convex subset of a Hadamard manifold M, and S, $T : X \to 2^X$ be two multimaps such that*

(1) *for each $x \in X$, $T(x)$ is nonempty;*
(2) *for each $x \in X$, $Gco\,T(x) \subseteq S(x)$;*
(3) *for each $x \in X$, there exists an $y \in X$ such that $x \in int\,T^{-1}(y)$;*
(4) *there exists an $x_o \in X$ such that $X \setminus int\,T^{-1}(x_o)$ is compact.*

Then S has a fixed point $\bar{x} \in X$, i.e., $\bar{x} \in S(\bar{x})$.

From now on, let M be a finite dimensional Hadamard manifold, and X be a nonempty geodesic convex subset of M. For the other standard notations and terminologies, we shall refer to Colao et al. [3], Kim [5,6], Kristály [7], Németh [10], and the references therein.

3. Equilibrium existence in Hadamard manifolds

First we begin with a generalization of Lemma 2.3 to geodesic convex n-sets in a Hadamard manifold as follows:

Theorem 3.1. *For each $i \in I = \{1, \ldots, n\}$, let X_i be a nonempty geodesic convex subset of a Hadamard manifold M, and let $X := \Pi_{i\in I} X_i = X_i \times X_{-i}$ where $X_{-i} = \Pi_{j\in I, j\neq i} X_j$. For each $i \in I$, let S_i, $T_i : X \to 2^{X_i}$ be two multimaps such that*

(1) *for each $x \in X$, $T_i(x)$ is nonempty, and $Gco\,T_i(x) \subseteq S_i(x)$;*
(2) *for each $y \in X$, there exists an $\bar{x}_i \in X_i$ such that $y \in int\,T_i^{-1}(\bar{x}_i)$;*
(3) *there exists an $\hat{x}_i \in X_i$ such that $X \setminus int\,T_i^{-1}(\hat{x}_i)$ is compact.*

Then there exists a fixed point $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ for S_i, i.e., for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$.

Proof. We first define two multimaps S, $T : X \to 2^X$ by for each $x \in X$,

$$S(x) := \bigcap_{i=1}^{n} S_i'(x) \quad \text{where} \quad S_i'(x) := S_i(x) \times X_{-i};$$

$$T(x) := \bigcap_{i=1}^{n} T_i'(x) \quad \text{where} \quad T_i'(x) := T_i(x) \times X_{-i}.$$

Then, by the assumption (1), for each $x \in X$, $T(x)$ is nonempty, and by Lemma 2.2, we have

$$Gco\, T(x) = Gco\big(\bigcap_{i\in I} T_i'(x)\big) \subseteq \bigcap_{i\in I} Gco\, T_i'(x) \subseteq \bigcap_{i\in I} S_i'(x) = S(x).$$

Note that for each $y = (y_i)_{i\in I} \in X$, we have

$$T^{-1}(y) = \big(\bigcap_{i\in I} T_i'\big)^{-1}(y) = \bigcap_{i\in I} T_i^{-1}(y_i).$$

Indeed,

$$\begin{aligned} T^{-1}(y) &= \{x \in X \mid y_i \in T_i(x) \quad \text{for each} \;\; i \in I\} \\ &= \{x \in X \mid x \in T_i^{-1}(y_i) \quad \text{for each} \;\; i \in I\} \\ &= \bigcap_{i\in I} T_i^{-1}(y_i). \end{aligned}$$

For each $y \in X$, by the assumption (2), there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that

$$y \in \bigcap_{i\in I} int\, T_i^{-1}(\bar{x}_i) \subseteq int \bigcap_{i\in I} T_i^{-1}(\bar{x}_i) = int\, T^{-1}(\bar{x});$$

so that the assumption (2) of Lemma 2.3 is satisfied.

Next, if we let $\hat{x} := (\hat{x}_i)_{i\in I} \in X$, by the assumption (3),

$$\begin{aligned} X \setminus int\, T^{-1}(\hat{x}) &= X \setminus int\,\big(\bigcap_{i\in I} T_i^{-1}(\hat{x}_i)\big) \\ &= X \setminus \bigcap_{i\in I} int\, T_i^{-1}(\hat{x}_i) = \bigcup_{i\in I} \big(X \setminus int\, T_i^{-1}(\hat{x}_i)\big) \end{aligned}$$

is compact so that the assumption (3) of Lemma 2.3 is satisfied. Therefore, the whole assumptions of Lemma 2.3 are satisfied so that S has a fixed point $\bar{x} \in X$, i.e., $\bar{x} \in S(\bar{x})$. Indeed, $\bar{x} = (\bar{x}_i)_{i\in I} \in S(\bar{x}) = \bigcap_{i\in I} S_i'(\bar{x})$, i.e., $\bar{x}_i \in S_i(\bar{x})$ for each $i \in I$ which completes the proof. □

Remark 3.2. When I is singleton, Theorem 3.1 further generalizes Theorem 3.1 due to Kim [5] in the following aspects:

(a) for each $y \in X$, $T^{-1}(y)$ need not be open in X;
(b) for each $x \in X$, $S(x)$ need not be geodesic convex.

In Theorem 3.1, when the set X_i is assumed to be compact geodesic convex for each $i \in I$, then the coercive assumption (3) is not needed anymore:

Corollary 3.3. *For each $i \in I = \{1, \dots, n\}$ $(n \geq 2)$, let X_i be a nonempty compact geodesic convex subset of a Hadamard manifold M, and let $X := \Pi_{i \in I} X_i$, $X_{-i} = \Pi_{j \in I, j \neq i} X_j$. For each $i \in I$, let $S_i, T_i : X \to 2^{X_i}$ be two multimaps such that*

(1) *for each $x \in X$, $T_i(x)$ is nonempty, and $Gco\, T_i(x) \subseteq S_i(x)$;*
(2) *for each $y \in X$, there exists an $\bar{x}_i \in X_i$ such that $y \in int\, T_i^{-1}(\bar{x}_i)$.*

Then there exists a fixed point $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ for S_i, i.e., for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$.

When I is singleton and the set X is assumed to be compact geodesic convex, $S = T$, and $T^{-1}(y)$ is open in X for each $y \in X$ in Theorem 3.1, we can obtain the Fan-Browder fixed point theorem in a Hadamard manifold as follow:

Corollary 3.4. *Let X be a nonempty compact geodesic convex subset of a Hadamard manifold M, and $T : X \to 2^X$ be a multimap such that*

(1) *for each $x \in X$, $T(x)$ is a nonempty geodesic convex subset of X;*
(2) *for each $y \in X$, $T^{-1}(y)$ is an open subset of X.*

Then T has a fixed point $\bar{x} \in X$, i.e., $\bar{x} \in T(\bar{x})$.

As an application of Theorem 3.1, we shall prove a basic equilibrium existence theorem for an abstract economy with geodesic convex values in a Hadamard manifold as follow:

Theorem 3.5. *Let $\Gamma = (X_i; A_i, P_i)_{i \in I}$ be an abstract economy where I is a finite set of agents such that for each $i \in I$,*

(1) *X_i is a nonempty compact geodesic convex subset of a Hadamard manifold M;*
(2) *for each $x \in X := \Pi_{i \in I} X_i$, $A_i(x)$ is a nonempty geodesic convex subset of X_i, and $A_i^{-1}(y)$ is open in X for each $y \in X_i$;*
(3) *for each $x \in X$, $(A_i \cap P_i)(x)$ is geodesic convex and $x_i \notin P_i(x)$;*
(4) *for each $y \in X_i$, $(A_i \cap P_i)^{-1}(y)$ is open in X;*
(5) *the set $W_i := \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is (possibly empty) such that $(X \setminus W_i) \cap A_i^{-1}(y)$ is open for each $y \in X_i$.*

Then Γ has an equilibrium choice $\hat{x} \in X$, i.e. for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Proof. Suppose that $W_i = \emptyset$ for all $i \in I$. Then we first define a multimap $A : X \to 2^X$ by

$$A(x) := \Pi_{i \in I} A_i(x) \quad \text{for each } x \in X.$$

Then, by the assumption (2), we have

(i) for each $x \in X$, $A(x)$ is a nonempty geodesic convex subset of X;

(ii) for each $y = (y_i)_{i\in I} \in X$,

$$\begin{aligned} A^{-1}(y) &= \{x \in X \mid y \in A(x) = \Pi_{i\in I} A_i(x)\} \\ &= \{x \in X \mid y_i \in A_i(x) \quad \text{for all} \;\; i \in I\} \\ &= \{x \in X \mid x \in A_i^{-1}(y_i) \quad \text{for all} \;\; i \in I\} \\ &= \bigcap_{i\in I} A_i^{-1}(y_i). \end{aligned}$$

Here, by the assumption (4), for each $y = (y_i)_{i\in I} \in X$ such that $A^{-1}(y)$ is open in X. Therefore, by Corollary 3.4, there exists a fixed point $\hat{x} \in X$ for A, i.e. for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$, and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Next, suppose that I_o be a nonempty maximal subset of I such that W_i is nonempty for each $i \in I_o$. Then, for each $i \in I_o$, we define a correspondence $\phi_i : X \to 2^{X_i}$ by

$$\phi_i(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if} \quad x \in W_i; \\ A_i(x), & \text{if} \quad x \notin W_i. \end{cases}$$

Then, for each $i \in I_o$, by the assumptions (2) and (3), we have $\phi_i(x)$ is a nonempty geodesic convex subset of X_i for each $x \in X$. Also, by the assumptions (4) and (5), for each $y_i \in X_i$, we have

$$\begin{aligned} \phi_i^{-1}(y_i) &= \{x \in X \mid y_i \in \phi_i(x)\} \\ &= \{x \in W_i \mid y_i \in \phi_i(x)\} \cup \{x \in X \setminus W_i \mid y_i \in \phi_i(x)\} \\ &= \{x \in W_i \mid y_i \in (A_i \cap P_i)(x)\} \cup \{x \in X \setminus W_i \mid y_i \in A_i(x)\} \\ &= \{x \in X \mid y_i \in (A_i \cap P_i)(x)\} \cup \{x \in X \setminus W_i \mid y_i \in A_i(x)\} \\ &= (A_i \cap P_i)^{-1}(y_i) \cup [(X \setminus W_i) \cap A_i^{-1}(y_i)] \end{aligned}$$

is open in X.

Finally we define a multimap $\Psi : X \to 2^X$ by

$$\Psi(x) := \Pi_{i\in I}\, \psi_i(x) \qquad \text{for each } x \in X,$$

where $\psi_i : X \to 2^{X_i}$ is defined by

$$\psi_i(x) = \begin{cases} \phi_i(x), & \text{if} \quad i \in I_o; \\ A_i(x), & \text{if} \quad i \notin I_o. \end{cases}$$

Then $\Psi(x)$ is a nonempty geodesic convex subset of X for each $x \in X$. We shall show that $\Psi^{-1}(y)$ is an open subset of X for each $y = (y_i) \in X$. Indeed, for each $y = (y_i) \in X$, we have

$$\begin{aligned}\Psi^{-1}(y) &= \{x \in X \mid y = (y_i) \in \Psi(x) = \Pi_{i\in I}\, \psi_i(x)\} \\ &= \bigcap_{i\in I} \{x \in X \mid y_i \in \psi_i(x)\} \\ &= \big(\bigcap_{i\in I_o} \{x \in X \mid y_i \in \psi_i(x)\}\big) \bigcap \big(\bigcap_{i\notin I_o} \{x \in X \mid y_i \in \psi_i(x)\}\big) \\ &= \big(\bigcap_{i\in I_o} \{x \in X \mid y_i \in \phi_i(x)\}\big) \cap \big(\bigcap_{i\notin I_o} \{x \in X \mid y_i \in A_i(x)\}\big) \\ &= \big(\bigcap_{i\in I_o} \phi_i^{-1}(y_i)\big) \bigcap \big(\bigcap_{i\notin I_o} A_i^{-1}(y_i)\big)\end{aligned}$$

is open in X. Therefore, the multimap $\Psi : X \to 2^X$ satisfies the whole assumptions of Corollary 3.4 so that there exists $\hat{x} = (\hat{x}_i)_{i\in I} \in X$ such that $\hat{x} \in \Psi(\hat{x})$, i.e., $\hat{x}_i \in \psi_i(\hat{x})$ for each $i \in I$. Then, for each $i \in I_o$, $\hat{x}_i \in \psi_i(\hat{x}) = \phi_i(\hat{x})$. If $\hat{x} \in W_i$, then

$$\hat{x}_i \in \phi_i(\hat{x}) = (A_i \cap P_i)(\hat{x}) \subseteq P_i(\hat{x})$$

which contradicts the assumption (3). Therefore, when $i \in I_o$, we have $\hat{x} \notin W_i$ so that $\hat{x}_i \in \psi_i(\hat{x}) = \phi_i(\hat{x}) = A_i(\hat{x})$ and $(A_i \cap P_i)(\hat{x}) = \emptyset$. In case of $i \notin I_o$, we have $W_i = \emptyset$ so that $(A_i \cap P_i)(\hat{x}) = \emptyset$, and $\hat{x}_i \in \psi_i(\hat{x}) = A_i(\hat{x})$. This completes the proof. $\square$

We now give a simple example of a non-convex 2-person game which is suitable for Theorem 3.5, but the previous equilibrium existence theorems in Border [1], Colao et al.[3], Yang-Pu [12], and Yannelis-Prabhakar [13] for compact games can not be applied:

Example 3.6. Let $\mathcal{G} = (X_i; A_i, P_i)_{i\in I}$ be a non-convex generalized game such that the pure strategic space X_i for each player i is defined by

$$\begin{aligned} X_1 &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1, x_2 \le 1\}; \\ X_2 &:= \{(\sqrt{2}\cos t,\ \sqrt{2}\sin t) \in \mathbb{R}^2 \mid \frac{\pi}{4} \le t \le \frac{3\pi}{4}\}. \end{aligned}$$

Then, X_1 is a compact (geodesic) convex subset of $\mathbb{R}^2$ in the usual sense, and X_2 is compact but not a convex subset of $\mathbb{R}^2$ in the usual sense. However, as remarked in [7], if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, then the set X_2 is geodesic convex with respect to the metric $g_{\mathbb{H}}$ being the image of a geodesic segment from $(\mathbb{H}^2, g_{\mathbb{H}})$.

For each player $i = 1, 2$, let the constraint correspondence $A_i : X = X_1 \times X_2 \to 2^{X_i}$ and the preference correspondence $P_i : X \to 2^{X_i}$ are defined as follows:

For each $\big((x_1,x_2),(y_1,y_2)\big) \in X = X_1 \times X_2$,

$$A_1\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} \big\{(\bar{x}_1,\bar{x}_2) \in X_1 \mid \bar{x}_1 < x_1\big\}, & \text{if} \quad x_1 \neq 0; \\ X_1, & \text{if} \quad x_1 = 0; \end{cases}$$

$$A_2\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} \big\{(\bar{y}_1,\bar{y}_2) \in X_2 \mid \bar{y}_1 < y_1\big\}, & \text{if} \quad y_1 \neq -1; \\ X_2, & \text{if} \quad y_1 = -1; \end{cases}$$

$$P_1\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} \big\{(\bar{x}_1,\bar{x}_2) \in X_1 \mid \bar{x}_1 > x_1\big\}, & \text{if} \quad x_1 \neq 0; \\ \emptyset, & \text{if} \quad x_1 = 0; \end{cases}$$

$$P_2\big((x_1,x_2),(y_1,y_2)\big) := \begin{cases} \big\{(\bar{y}_1,\bar{y}_2) \in X_2 \mid \bar{y}_1 > y_1\big\}, & \text{if} \quad y_1 \neq -1; \\ \emptyset, & \text{if} \quad y_1 = -1. \end{cases}$$

Then, it is clear that for each $i = 1,2$, $A_i\big((x_1,x_2),(y_1,y_2)\big)$ is nonempty geodesic convex subset of X_i, and $(x_1,x_2) \notin P_1\big((x_1,x_2),(y_1,y_2)\big)$ and $(y_1,y_2) \notin P_2\big((x_1,x_2),(y_1,y_2)\big)$. And we have

$$A_1^{-1}(\bar{x}_1,\bar{x}_2) = \begin{cases} \big\{\big((x_1,x_2),(y_1,y_2)\big) \in X \mid \bar{x}_1 < x_1\big\}, & \text{if} \quad x_1 \neq 0; \\ X_1 \times X_2, & \text{if} \quad \bar{x}_1 = 0; \end{cases}$$

$$A_2^{-1}(\bar{y}_1,\bar{y}_2) = \begin{cases} \big\{\big((x_1,x_2),(y_1,y_2)\big) \in X \mid \bar{y}_1 < y_1\big\}, & \text{if} \quad \bar{y}_1 \neq -1; \\ X_1 \times X_2, & \text{if} \quad \bar{y}_1 = -1; \end{cases}$$

so that $A_1^{-1}(x_1,x_2)$ and $A_2^{-1}(y_1,y_2)$ are both open in X, and also

$$P_1^{-1}(\bar{x}_1,\bar{x}_2) = \begin{cases} \big\{\big((x_1,x_2),(y_1,y_2)\big) \in X \mid \bar{x}_1 > x_1\big\}, & \text{if} \quad \bar{x}_1 \neq 0; \\ \emptyset, & \text{if} \quad \bar{x}_1 = 0; \end{cases}$$

$$P_2^{-1}(\bar{y}_1,\bar{y}_2) = \begin{cases} \big\{\big((x_1,x_2),(y_1,y_2)\big) \in X \mid \bar{y}_1 > y_1\big\}, & \text{if} \quad \bar{y}_1 \neq -1; \\ \emptyset, & \text{if} \quad \bar{y}_1 = -1; \end{cases}$$

so that $P_1^{-1}(x_1,x_2)$ and $P_2^{-1}(y_1,y_2)$ are both open in X. Therefore, the assumptions (2)-(4) are satisfied. Applying Theorem 3.5, it remains to show the assumption (5) of Theorem 3.5. Indeed, the set $W_1 = \{x \in X \mid (A_1 \cap P_1)(x) \neq \emptyset\}$ is empty so that $X \setminus W_1$ is open, and the set $W_2 = \{x \in X \mid (A_2 \cap P_2)(x) \neq \emptyset\}$ is also empty so that $X \setminus W_2$ is open in X; thus the assumption (5) of Theorem 3.5 is satisfied. Therefore, all the assumptions of Theorem 3.5 for the generalized game $\mathcal{G} = (X_i; A_i, P_i)_{i\in I}$ are satisfied. Hence, we can obtain an equilibrium point $\big((0,1),(-1,1)\big) \in X = X_1 \times X_2$ for $\mathcal{G}$ such that for each $i = 1,2$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

That is,

$$(0,1) \in A_1\big((0,1),(-1,1)\big) \quad \text{and} \quad (A_1 \cap P_1)\big((0,1),(-1,1)\big) = \emptyset;$$

$$(-1,1) \in A_2\big((0,1),(-1,1)\big) \quad \text{and} \quad (A_2 \cap P_2)\big((0,1),(-1,1)\big) = \emptyset.$$

Finally, we provide a noncompact intersection theorem which is essential in proving the equilibrium existence theorem in noncompact Hadamard manifolds settings:

Theorem 3.7. *For each $i \in I = \{1, \ldots, n\}$, let X_i be a nonempty geodesic convex subset of a Hadamard manifold M, and let $X := \Pi_{i\in I} X_i = X_i \times X_{-i}$. Let $A_1, \cdots, A_n$, and $B_1, \cdots, B_n$ be nonempty $2n$ subsets of X, and let*

$$A_i(x_i) := \{x_{-i} \in X_{-i} \mid (x_i, x_{-i}) \in A_i\}; A_i(x_{-i}) := \{x_i \in X_i \mid (x_i, x_{-i}) \in A_i\};$$

$$B_i(x_i) := \{x_{-i} \in X_{-i} \mid (x_i, x_{-i}) \in B_i\}; B_i(x_{-i}) := \{x_i \in X_i \mid (x_i, x_{-i}) \in B_i\}.$$

Suppose that

(1) *for each $i \in I$, and any $x_{-i} \in X_{-i}$, there exists an $y_i \in X_i$ such that $x_{-i} \in int\, A_i(y_i)$;*
(2) *for each $i \in I$, and any $x_{-i} \in X_{-i}$, $\quad Gco\, A_i(x_{-i}) \subseteq B_i(x_{-i})$;*
(3) *there exists an $\bar{x} \in X$ such that $X \setminus int\,\big(\Pi_{i\in I} A_i^{-1}(\bar{x}_{-i})\big)$ is compact.*

Then we have $\quad \bigcap_{i=1}^n B_i \neq \emptyset$.

Proof. We first introduce two multimaps $T, S : X \to 2^{X_i}$ by for each $x = (x_i, x_{-i}) \in X$,

$$T(x) := \Pi_{i=1}^n A_i(x_{-i}); \quad S(x) := \Pi_{i=1}^n B_i(x_{-i}).$$

Then, by the assumption (1), for each $x \in X$, there exists an $y \in X$ such that $x \in int\, T(y)$, and by the assumption (2) and Lemma 2.2, we have $Gco\, T(x) \subseteq S(x)$ for all $x \in X$. Here we note that since $A_i(x_i) = \{x_{-i} \in X_{-i} \mid (x_i, x_{-i}) \in A_i\}$ for each $x_i \in X_i$, we obtain that the inverse set $A_i^{-1}(x_{-i}) = \{x_i \in X_i \mid (x_i, x_{-i}) \in A_i\}$ for each $x_{-i} \in X_{-i}$. Since $T^{-1}(x) = \Pi_{i=1}^n A_i^{-1}(x_{-i})$ for each $x \in X$, by the assumption (3), we have that there exists an $x_o \in X$ such that $X \setminus int\, T^{-1}(x_o)$ is compact. Then all the assumptions of Lemma 2.3 are satisfied so that there exists a fixed point $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$, i.e., $\quad \bar{x} \in \bigcap_{i=1}^n B_i \neq \emptyset$. This completes the proof. □

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