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ON A NONLINEAR INTEGRODIFFERENTIAL EQUATION IN TWO VARIABLES WITH VALUES IN A GENERAL BANACH SPACE

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Abstract. The paper is devoted to the study of a nonlinear integrodifferential equation in two variables with values in a general Banach space. Applying the fixed point theorems together with the definition of a suitable Banach space and the establishment of appropriate conditions for subsets to be relatively compact in this space, the existence and the compactness of the set of solutions are proved. Two illustrative examples are given.

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1. Introduction

In this paper, we consider the following nonlinear integrodifferential equation in two variables

$$u(x,y) = g(x,y) + \iint_{\Omega} K(x,y,s,t;u(s,t),D_1u(s,t))dsdt, \tag{1.1}$$

where $(x,y) \in \Omega = [0,1] \times [0,1]$ and $g : \Omega \to E$, $K : \Omega \times \Omega \times E^2 \to E$ are given functions, E is a Banach space with norm $\|\cdot\|_E$. Denote by $D_1u = \frac{\partial u}{\partial x}$, the partial derivative of a function $u(x,y)$ defined on Ω, with respect to the first variable.

It is well known that integral and integrodifferential equations have attracted the interest of scientists because of a great deal of application in different branches of sciences and engineering. These equations arise naturally in a variety of models from mechanics, physics, population dynamics, economics and other fields of science, for example, see the books writen by Corduneanu [4], Deimling [5].

Many papers have been devoted to the study of some types of (1.1), in one variable, two variables, or N variables, and its special versions by using different methods, in which the fixed point theorems are often applied, see [1] - [16] and the references given therein.

In case the Banach space E is arbitrary, Bica et al. [3] presented a new approach for the following neutral Fredholm integro-differential equations in Banach spaces

$$x(t) = g(t) + \int_a^b f(t,s,x(s),x'(s))ds, \ t \in [a,b], \tag{1.2}$$

where $f : [a,b] \times [a,b] \times E \times E \to E$ is continuous, E is a Banach space and $g \in C^1([a,b];E)$. Here, the authors used Perov's fixed point theorem to obtain the existence, uniqueness and global approximation of the solution of (1.2).

In the case $E = \mathbb{R}^d$, motivated by the results in [3], based on the applications of the well-known Banach fixed point theorem coupled with Bielecki type norm and a certain integral inequality with explicit estimate, Pachpatte [13] proved uniqueness and other properties of solutions of the following Fredholm type integrodifferential equation

$$x(t) = g(t) + \int_a^b f(t,s,x(s),x'(s),\cdots,x^{(n-1)}(s))ds, \ t \in [a,b],$$

where x, g, f are real valued functions and $n \geq 2$ is an integer. With the same methods, Pachpatte studied the existence, uniqueness and some basic properties of solutions of the Fredholm type integral equation in two variables as follows, see [14],

$$u(x,y) = f(x,y) + \int_0^a \int_0^b g\left(x,y,s,t;u(s,t),D_1u(s,t),D_2u(s,t)\right)dtds.$$

M. A. Abdou et al. also considered the existence of integrable solution of nonlinear integral equation, of type Hammerstein–Volterra of the second kind, by using the technique of measure of weak noncompactness and Schauder fixed point theorem, see [1]. In [2], A. Aghajani et al. proved some results on the existence, uniqueness and estimation of the solutions of Fredholm type integro-differential equations in two variables, by using Perov's fixed point theorem. Recently, in [8] - [12], using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, we have investigated solvability and asymptotically stable of nonlinear functional integral equations in one variable or two variables, or N variables.

Based on the above works, we consider (1.1). This paper is organized as follows. In section 2, we present some preliminaries. It consists of the definition of a suitable Banach space and a sufficient condition for relatively compact subsets. In section 3, by applying the Banach theorem and the Schauder theorem, we prove two existence theorems. Furthermore, the compactness of solutions set is proved. In order to illustrate the results obtained here, two examples are given.

2. Preliminaries

First, we construct an appropriate Banach space for (1.1) as follows. Let $X = C(\Omega; E)$ be the space of all continuous functions from Ω into E equipped with the following norm

$$\|u\|_X = \sup_{(x,y)\in\Omega} \|u(x,y)\|_E\,,\ u \in X. \tag{2.1}$$

Put

$$X_1 = \{u \in X : D_1u \in X\}. \tag{2.2}$$

We remark that $C^1(\Omega; E) \underset{\neq}{\subsetneq} X_1 \underset{\neq}{\subsetneq} X$. Indeed, let $e_1 \in E$, $e_1 \neq 0$,

(i) Consider $u = u(x,y) = \left(\left|x - \frac{1}{2}\right| + \left|y - \frac{1}{3}\right|\right)e_1$, we have $u \in X$, but $u \notin X_1$;

(ii) Consider $v = v(x,y) = x^2\left|y - \frac{1}{3}\right|e_1$, we have $v \in X_1$, but $v \notin C^1(\Omega; E)$.

Lemma 2.1. *X_1 is a Banach space with the norm defined by*

$$\|u\|_{X_1} = \|u\|_X + \|D_1u\|_X\,,\ u \in X_1. \tag{2.3}$$

Proof. Let $\{u_p\} \subset X_1$ be a Cauchy sequence in X_1, it means that

$$\|u_p - u_m\|_{X_1} = \|u_p - u_m\|_X + \|D_1u_p - D_1u_m\|_X \to 0,\ \text{as } p,m \to \infty.$$

Then $\{u_p\}$ and $\{D_1 u_p\}$ are also the Cauchy sequences in X. Since X is complete, $\{u_p\}$ converges to u and $\{D_1 u_p\}$ converges to v in X, i.e.,

$$\|u_p - u\|_X \to 0, \|D_1 u_p - v\|_X \to 0, \text{ as } p \to \infty. \tag{2.4}$$

We shall show that $D_1 u = v$. We have

$$u_p(x,y) - u_p(0,y) = \int_0^x D_1 u_p(s,y)ds, \ \forall (x,y) \in \Omega. \tag{2.5}$$

By $\|u_p - u\|_X \to 0$, we get

$$u_p(x,y) - u_p(0,y) \to u(x,y) - u(0,y) \text{ in } E \ \forall (x,y) \in \Omega. \tag{2.6}$$

On the other hand, it follows from $\|D_1 u_p - v\|_X \to 0$ that

$$\int_0^x D_1 u_p(s,y)ds \to \int_0^x v(s,y)ds, \ \forall (x,y) \in \Omega, \tag{2.7}$$

because of

$$\begin{aligned} \left\| \int_0^x D_1 u_p(s,y)ds - \int_0^x v(s,y)ds \right\|_E &\le \int_0^x \|D_1 u_p(s,y) - v(s,y)\|_E \, ds \\ &\le \|D_1 u_p - v\|_X \\ &\to 0. \end{aligned}$$

Combining (2.5)-(2.7) leads to

$$u(x,y) - u(0,y) = \int_0^x v(s,y)ds \text{ in } E \ \forall (x,y) \in \Omega. \tag{2.8}$$

It implies that $D_1 u = v \in X$. Therefore $u \in X_1$ and $u_p \to u$ in X_1. This completes the proof. □

Next, we give a sufficient condition for relatively compact subsets of X_1.

Lemma 2.2. *Let $\mathcal{F} \subset X_1$. Then $\mathcal{F}$ is relatively compact in X_1 if and only if the following conditions are satisfied:*

(i) *For all $(x,y) \in \Omega, \mathcal{F}(x,y) = \{u(x,y) : u \in \mathcal{F}\}$ and $D_1\mathcal{F}(x,y) = \{D_1 u(x,y) : u \in \mathcal{F}\}$ are relatively compact subsets of E;*

(ii) *If for all $\varepsilon > 0$, there exists $\delta > 0$, for all $(x,y), \ (\bar{x},\bar{y}) \in \Omega$,*

$$|x - \bar{x}| + |y - \bar{y}| < \delta,$$

then

$$\sup_{u \in \mathcal{F}} [u(x,y) - u(\bar{x},\bar{y})]_E < \varepsilon,$$

where

$$\begin{aligned} [u(x,y) - u(\bar{x},\bar{y})]_E = {} & \|u(x,y) - u(\bar{x},\bar{y})\|_E \\ & + \|D_1 u(x,y) - D_1 u(\bar{x},\bar{y})\|_E \, . \end{aligned}$$

Proof. **(a)** Let $\mathcal{F}$ be relatively compact in X_1.

First, we show that (i) is true.

To prove that $\mathcal{F}(x,y)$ is relatively compact in E, let $\{u_p(x,y)\}$ be a sequence in $\mathcal{F}(x,y)$, we show that $\{u_p(x,y)\}$ contains a convergent subsequence in E. Because $\overline{\mathcal{F}}$ compact in X_1, we have $\{u_p\} \subset \mathcal{F}$ contains a convergent subsequence $\{u_{p_k}\}$ in X_1. So there exists $u \in X_1$ such that

$$\|u_{p_k} - u\|_{X_1} \to 0, \text{ as } k \to \infty.$$

By $\|u_{p_k}(x,y) - u(x,y)\|_E \leq \|u_{p_k} - u\|_X \leq \|u_{p_k} - u\|_{X_1} \to 0$, $u_{p_k}(x,y) \to u(x,y)$ in E. Thus $\mathcal{F}(x,y)$ is relatively compact in E.

Similarly, by

$$\|D_1 u_{p_k}(x,y) - D_1 u(x,y)\|_E \leq \|D_1 u_{p_k} - D_1 u\|_X \leq \|u_{p_k} - u\|_{X_1} \to 0,$$

we have $D_1\mathcal{F}(x,y)$ is also relatively compact in E.

Next, we show that (ii) is also true.

For every $\varepsilon > 0$, considering a collection of open balls in X_1, with center at $u \in \mathcal{F}$ and radius $\frac{\varepsilon}{4}$, as follows:

$$B(u, \frac{\varepsilon}{4}) = \{\bar{u} \in X_1 : \|u - \bar{u}\|_{X_1} < \frac{\varepsilon}{4}\}, \ u \in \mathcal{F}.$$

It is clear that $\overline{\mathcal{F}} \subset \bigcup_{u\in\mathcal{F}} B(u, \frac{\varepsilon}{4})$. Because $\overline{\mathcal{F}}$ compact in X_1, the open cover $\{B(u, \frac{\varepsilon}{4})\}_{u\in\mathcal{F}}$ of $\overline{\mathcal{F}}$ contains a finite subcover, so there are $u_1, \cdots, u_q \in \mathcal{F}$ such that

$$\overline{\mathcal{F}} \subset \bigcup_{j=1}^{q} B(u_j, \frac{\varepsilon}{4}).$$

By the functions u_j, $D_1 u_j$, $j = 1, \cdots, q$ are uniformly continuous on Ω, there exists $\delta > 0$ such that for all (x,y), $(\bar{x},\bar{y}) \in \Omega$,

$$|x - \bar{x}| + |y - \bar{y}| < \delta \Longrightarrow [u_j(x,y) - u_j(\bar{x},\bar{y})]_E < \frac{\varepsilon}{2}, \ \forall j = 1, \cdots, q.$$

For all $u \in \mathcal{F}$, $u \in B(u_{j_0}, \frac{\varepsilon}{4})$ for some $j_0 = 1, \cdots, q$. Thus, for all (x,y), $(\bar{x},\bar{y}) \in \Omega$, if $|x - \bar{x}| + |y - \bar{y}| < \delta$ then we obtain

$$\begin{aligned}
[u(x,y) - u(\bar{x},\bar{y})]_E &\leq [u(x,y) - u_{j_0}(x,y)]_E + [u_{j_0}(x,y) - u_{j_0}(\bar{x},\bar{y})]_E \\
&\quad + [u_{j_0}(\bar{x},\bar{y}) - u(\bar{x},\bar{y})]_E \\
&\leq 2\|u - u_{j_0}\|_{X_1} + [u_{j_0}(x,y) - u_{j_0}(\bar{x},\bar{y})]_E \\
&< \frac{2\varepsilon}{4} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

It implies that (ii) is true.

(b) Conversely, let the conditions (i) and (ii) be correct.

To prove that $\mathcal{F}$ is relatively compact in X_1, let $\{u_p\}$ be a sequence in $\mathcal{F}$, we show that $\{u_p\}$ contains a convergent subsequence.

Put $\mathcal{F}_1 = \{u_p : p \in \mathbb{N}\}$. By (i), $\mathcal{F}_1(x,y) = \{u_p(x,y) : p \in \mathbb{N}\}$ is relatively compact subset of E, for all $(x,y) \in \Omega$ and $\mathcal{F}_1$ is equicontinuous in X. Applying the Ascoli-Arzela theorem to $\mathcal{F}_1$, it is relatively compact in X, so there exists a subsequence $\{u_{p_k}\}$ of $\{u_p\}$ and $u \in X$ such that

$$\|u_{p_k} - u\|_X \to 0, \text{ as } k \to \infty.$$

Similarly, $\mathcal{F}_2 = \{D_1 u_{p_k} : k \in \mathbb{N}\}$ is also relatively compact in X. We obtain the existence of a subsequence of $\{D_1 u_{p_k}\}$, denoted by the same symbol, and $w \in X$, such that

$$\|D_1 u_{p_k} - w\|_X \to 0, \text{ as } k \to \infty.$$

Because of

$$u_{p_k}(x,y) - u_{p_k}(0,y) = \int_0^x D_1 u_{p_k}(s,y)ds, \ \forall (x,y) \in \Omega,$$

furthermore by $\|u_{p_k} - u\|_X \to 0$ and $\|D_1 u_{p_k} - w\|_X \to 0$, we obtain

$$u(x,y) - u(0,y) = \int_0^x w(s,y)ds \text{ in } E, \forall (x,y) \in \Omega.$$

As $w \in X$ we see that the right hand side is continuously differentiable with respect to x and this leads to $D_1 u = w \in X$. Therefore $u \in X_1$ and $u_{p_k} \to u$ in X_1. This completes the proof. □

3. The existence theorems

We make the following assumptions.

(H_1) $g \in X_1$;

(H_2) $K \in C(\Omega \times \Omega \times E^2; E)$ such that $D_1 K \in C(\Omega \times \Omega \times E^2; E)$,

and there exist nonnegative functions k_0, $k_1 : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\beta = \sup\limits_{(x,y)\in\Omega} \iint_\Omega k_0(x,y,s,t)dsdt + \sup\limits_{(x,y)\in\Omega} \iint_\Omega k_1(x,y,s,t)dsdt < 1$,

(ii) $\|K(x,y,s,t;u,v) - K(x,y,s,t;\bar{u},\bar{v})\|_E$
$\leq k_0(x,y,s,t)\left(\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right)$,

(iii) $\|D_1 K(x,y,s,t;u,v) - D_1 K(x,y,s,t;\bar{u},\bar{v})\|_E$
$\leq k_1(x,y,s,t)\left(\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right)$, for all $l(x,y,s,t) \in \Omega\times\Omega$, $\forall (u,v)$, $(\bar{u},\bar{v}) \in E^2$.

Theorem 3.1. *Let the functions g, K in* (1.1) *satisfy the assumptions* (H_1), (H_2). *Then the equation* (1.1) *has a unique solution in* X_1.

Proof. For every $u \in X_1$, we put

$$(Au)(x,y) = g(x,y) + \iint_\Omega K(x,y,s,t;u(s,t),D_1u(s,t))dsdt,\ (x,y) \in \Omega. \tag{3.1}$$

It is obviously that $Au \in X_1$, $\forall u \in X_1$. We shall show that $A : X_1 \to X_1$ is a contraction map, by proving

$$\|Au - A\bar{u}\|_{X_1} \leq \beta \|u - \bar{u}\|_{X_1},\ \forall u, \bar{u} \in X_1. \tag{3.2}$$

For every u, $\bar{u} \in X_1$, for all $(x,y) \in \Omega$, by (H_2, ii), (3.1) leads to

$$\begin{aligned}
&\|(Au)(x,y) - (A\bar{u})(x,y)\|_E \\
&\leq \iint_\Omega \|K(x,y,s,t;u(s,t),D_1u(s,t)) - K(x,y,s,t;\bar{u}(s,t),D_1\bar{u}(s,t))\|_E\, dsdt \\
&\leq \iint_\Omega k_0(x,y,s,t)\left[\|u(s,t) - \bar{u}(s,t)\|_E + \|D_1u(s,t) - D_1\bar{u}(s,t)\|_E\right] dsdt \\
&\leq \left(\sup_{(x,y)\in\Omega} \iint_\Omega k_0(x,y,s,t)dsdt\right) \|u - \bar{u}\|_{X_1}.
\end{aligned}$$

Hence

$$\|Au - A\bar{u}\|_X \leq \left(\sup_{(x,y)\in\Omega} \iint_\Omega k_0(x,y,s,t)dsdt\right) \|u - \bar{u}\|_{X_1}. \tag{3.3}$$

Similarly, by for all $(x,y) \in \Omega$,

$$D_1(Au)(x,y) = D_1g(x,y) + \iint_\Omega D_1K(x,y,s,t;u(s,t),D_1u(s,t))dsdt,$$

and (H_2)-(ii) we obtain

$$\begin{aligned}
&\|D_1(Au)(x,y) - D_1(A\bar{u})(x,y)\|_E \\
&\leq \iint_\Omega ||D_1K(x,y,s,t;u(s,t),D_1u(s,t)) \\
&\quad - D_1K(x,y,s,t;\bar{u}(s,t),D_1\bar{u}(s,t))||_E dsdt \\
&\leq \iint_\Omega k_1(x,y,s,t)\left[\|u(s,t) - \bar{u}(s,t)\|_E + \|D_1u(s,t) - D_1\bar{u}(s,t)\|_E\right] dsdt \\
&\leq \left(\sup_{(x,y)\in\Omega} \iint_\Omega k_1(x,y,s,t)dsdt\right) \|u - \bar{u}\|_{X_1},
\end{aligned}$$

it implies that

$$\|D_1(Au) - D_1(A\bar{u})\|_X \leq \left(\sup_{(x,y)\in\Omega} \iint_\Omega k_1(x,y,s,t)dsdt\right) \|u - \bar{u}\|_{X_1}. \tag{3.4}$$

From (3.3) and (3.4), we have (3.2). Applying the Banach fixed point theorem, Theorem 3.1 is proved. □

We also obtain the existence of solutions of (1.1) in X_1 via the Schauder fixed point theorem, by making the following assumptions.

$(\bar{H}_2)$ $K \in C(\Omega \times \Omega \times E^2; E)$ such that $D_1 K \in C(\Omega \times \Omega \times E^2; E)$,

and there exist nonnegative functions $\bar{k}_0, \bar{k}_1 : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\bar{\beta} = \sup_{(x,y)\in\Omega} \iint_\Omega \bar{k}_0(x,y,s,t)dsdt + \sup_{(x,y)\in\Omega} \iint_\Omega \bar{k}_1(x,y,s,t)dsdt < 1$,

(ii) $\|K(x,y,s,t;u,v)\|_E \leq \bar{k}_0(x,y,s,t)\left(1 + \|u\|_E + \|v\|_E\right)$,

and

(iii) $\|D_1 K(x,y,s,t;u,v)\|_E \leq \bar{k}_1(x,y,s,t)\left(1 + \|u\|_E + \|v\|_E\right)$,

for all $(x,y,s,t) \in \Omega \times \Omega$, $\forall (u,v) \in E^2$;

$(\bar{H}_3)$ $K, D_1K : \Omega \times \Omega \times E^2 \to E$, are completely continuous suchthat for any bounded subset J of E^2, for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\begin{aligned}
&\forall (x,y,s,t),\ (\bar{x},\bar{y},s,t) \in \Omega \times \Omega,\ |x-\bar{x}| + |y-\bar{y}| < \delta \\
&\Longrightarrow \|K(x,y,s,t;u,v) - K(\bar{x},\bar{y},s,t;u,v)\|_E \\
&\qquad + \|D_1K(x,y,s,t;u,v) - D_1K(\bar{x},\bar{y},s,t;u,v)\|_E < \varepsilon,\ \forall (u,v) \in J.
\end{aligned}$$

Theorem 3.2. *Let the functions g, K in* (1.1) *satisfy the assumptions* (H_1), $(\bar{H}_2)$, $(\bar{H}_3)$. *Then the equation* (1.1) *has a solution in X_1. Furthermore, the set of solutions is compact.*

Proof. With the operator A as in (3.1), it is clear that $A : X_1 \to X_1$. For $\rho > 0$, we define a closed ball in X_1 as follows

$$B_\rho = \{u \in X_1 : \|u\|_{X_1} \leq \rho\}.$$

We shall show that there exists $\rho > 0$ such that $A : B_\rho \to B_\rho$. For every $u \in B_\rho$, for all $(x,y) \in \Omega$, we have

$$\begin{aligned}
\|(Au)(x,y)\|_E &\leq \|g(x,y)\|_E + \iint_\Omega \|K(x,y,s,t;u(s,t),D_1u(s,t))\|_E\, dsdt \\
&\leq \|g\|_X + \iint_\Omega \bar{k}_0(x,y,s,t)\left(1 + \|u(s,t)\|_E + \|D_1u(s,t)\|_E\right) dsdt \\
&\leq \|g\|_X + \iint_\Omega \bar{k}_0(x,y,s,t)\left(1 + \|u\|_{X_1}\right) dy \\
&\leq \|g\|_X + (1+\rho)\left(\sup_{(x,y)\in\Omega} \iint_\Omega \bar{k}_0(x,y,s,t)dsdt\right),
\end{aligned}$$

it gives

$$\|Au\|_X \le \|g\|_X + (1+\rho)\left(\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_0(x,y,s,t)dsdt\right). \tag{3.5}$$

Similarly, we have

$$\begin{aligned}\|D_1(Au)(x,y)\|_E &\le \|D_1 g(x,y)\|_E \\ &\quad + \iint_\Omega \|D_1 K(x,y,s,t;u(s,t),D_1u(s,t))\|_E\, dsdt \\ &\le \|D_1 g\|_X + (1+\rho)\left(\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_1(x,y,s,t)dsdt\right),\end{aligned}$$

so

$$\|D_1(Au)\|_X \le \|D_1 g\|_X + (1+\rho)\left(\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_1(x,y,s,t)dsdt\right). \tag{3.6}$$

This gives

$$\|Au\|_{X_1} \le \|g\|_{X_1} + (1+\rho)\,\bar{\beta}. \tag{3.7}$$

Choosing $\rho \ge \|g\|_{X_1} + (1+\rho)\,\bar{\beta}$, i.e. $\rho \ge \frac{\|g\|_{X_1}+\bar{\beta}}{1-\bar{\beta}}$. Thefore, $A : B_\rho \to B_\rho$.

Now we show that two conditions as below are satisfied.

(a) $A : B_\rho \to B_\rho$ is continuous.
(b) $\mathcal{F} = A(B_\rho)$ is relatively compact in X_1.

To prove (a), let $\{u_p\} \subset B_\rho$, $\|u_p - u_0\|_{X_1} \to 0$, as $p \to \infty$, we need to show that

$$||Au_p - Au_0||_X \to 0 \text{ and } \|D_1(Au_p) - D_1(Au_0)\|_X \to 0, \text{ as } p \to \infty. \tag{3.8}$$

Note that

$$\begin{aligned}&||(Au_p)(x,y) - (Au_0)(x,y)||_E \\ &\le \iint_\Omega ||K(x,y,s,t;u_p(s,t),D_1u_p(s,t)) \\ &\quad - K(x,y,s,t;u_0(s,t),D_1u_0(s,t))||_E dsdt.\end{aligned} \tag{3.9}$$

Put

$$\begin{aligned} S_1 &= \{u_p(s,t) : (s,t) \in \Omega,\ p \in \mathbb{Z}_+\}, \\ S_2 &= \{D_1u_p(s,t) : (s,t) \in \Omega,\ p \in \mathbb{Z}_+\}, \end{aligned} \tag{3.10}$$

then we have S_1, S_2 are compact in E, since $\|u_p - u_0\|_{X_1} \to 0$.

(i) S_1 is compact in E.

Indeed, let $\{u_{p_j}(s_j,t_j)\}_j$, be a sequence in S_1. We can assume that $\lim_{j\to\infty}(s_j,t_j)=(s_0,t_0)$ and $\lim_{j\to\infty}\left\|u_{p_j}-u_0\right\|_{X_1}=0$. We have

$$\begin{aligned}
&\left\|u_{p_j}(s_j,t_j)-u_0(s_0,t_0)\right\|_E\\
&\leq \left\|u_{p_j}(s_j,t_j)-u_0(s_j,t_j)\right\|_E+\left\|u_0(s_j,t_j)-u_0(s_0,t_0)\right\|_E\\
&\leq \left\|u_{p_j}-u_0\right\|_{X_1}+\left\|u_0(s_j,t_j)-u_0(s_0,t_0)\right\|_E\\
&\to 0,\ \text{as } j\to\infty,
\end{aligned}\tag{3.11}$$

which shows that $\lim_{j\to\infty}u_{p_j}(s_j,t_j)=u_0(s_0,t_0)$ in E. This means that S_1 is compact in E.

(ii) Similarly S_2 is also compact in E.

Give $\varepsilon>0$. Since K is uniformly continuous on $\Omega\times\Omega\times S_1\times S_2$, there exists $\delta>0$ such that for all (u,v), $(\bar{u},\bar{v})\in S_1\times S_2$,

$$\left\|u-\bar{u}\right\|_E+\left\|v-\bar{v}\right\|_E<\delta\Longrightarrow\left\|K(x,y,s,t;u,v)-K(x,y,s,t;\bar{u},\bar{v})\right\|_E<\varepsilon,$$

for all $(x,y,s,t)\in\Omega\times\Omega$.

Because of $\left\|u_p-u_0\right\|_X\to 0$ and $\left\|D_1u_p-D_1u_0\right\|_X\to 0$, there is $p_0\in\mathbb{N}$ such that

$$\forall p\in\mathbb{N},\ p\geq p_0\Longrightarrow\left\|u_p-u_0\right\|_X+\left\|D_1u_p-D_1u_0\right\|_X<\delta.$$

It implies that for all $p\in\mathbb{N}$, $p\geq p_0$,

$$\left\|K(x,y,s,t;u_p(s,t),D_1u_p(s,t))-K(x,y,s,t;u_0(s,t),D_1u_0(s,t))\right\|_E<\varepsilon,$$

for all $(x,y,s,t)\in\Omega\times\Omega$, consequently

$$\left\|(Au_p)(x,y)-(Au_0)(x,y)\right\|_E<\varepsilon,\ \forall(x,y)\in\Omega,\ \forall p\geq p_0,$$

it means that

$$\left\|Au_p-Au_0\right\|_X<\varepsilon,\ \forall p\geq p_0,\tag{3.12}$$

that is, $\left\|Au_p-Au_0\right\|_X\to 0$, as $p\to\infty$.

By the same argument, we obtain that $\left\|D_1(Au_p)-D_1(Au_0)\right\|_X\to 0$, as $p\to\infty$. The continuity of A is proved.

To prove (b), we use Lemma 2.2.

First, we prove the condition (i) in Lemma 2.2: $A(B_\rho)(x,y)=\{Au(x,y):u\in B_\rho\}$ and $D_1A(B_\rho)(x,y)=\{D_1(Au)(x,y):u\in B_\rho\}$ are relatively compact in E.

Put

$$\begin{aligned}
R_1&=\{u(s,t):(s,t)\in\Omega,\ u\in B_\rho\},\\
R_2&=\{D_1u(s,t):(s,t)\in\Omega,\ u\in B_\rho\}.
\end{aligned}\tag{3.13}$$

Then R_1, R_2 are bounded in E. Since K is completely continuous, $K\left(\Omega\times\Omega\times R_1\times R_2\right)$ is relatively compact in E, it implies that

$\overline{K(\Omega \times \Omega \times R_1 \times R_2)}$ is compact in E, and so is $\overline{conv}\left(K\left(\Omega \times \Omega \times R_1 \times R_2\right)\right)$, where $\overline{conv}\left(K\left(\Omega \times \Omega \times R_1 \times R_2\right)\right)$ is the closure of convex hull of $K(\Omega \times \Omega \times R_1 \times R_2)$.

For every $(x, y) \in \Omega$, for all $u \in B_\rho$, it follows from

$$K(x,y,s,t;u(s,t),D_1u(s,t)) \in K\left(\Omega \times \Omega \times R_1 \times R_2\right), \forall (s,t) \in \Omega, \tag{3.14}$$

that

$$\begin{aligned} \overline{A(B_\rho)(x,y)} &\subset g(x,y) + |\Omega|\, \overline{conv}\left(K\left(\Omega \times \Omega \times R_1 \times R_2\right)\right) \\ &= g(x,y) + \overline{conv}\left(K\left(\Omega \times \Omega \times R_1 \times R_2\right)\right). \end{aligned} \tag{3.15}$$

Hence, the set $A(B_\rho)(x,y)$ is relatively compact in E.

Similarly, $\overline{D_1A(B_\rho)(x,y)} \subset D_1g(x,y) + \overline{conv}\left(D_1K\left(\Omega \times \Omega \times R_1 \times R_2\right)\right)$, so, the set $D_1A(B_\rho)(x,y)$ is relatively compact in E.

Next, we prove the condition (ii) in Lemma 2.2:

Give $\varepsilon > 0$. By $(\bar{H}_3)$, there exists $\delta_1 > 0$ such that $\forall (x,y)$, $(\bar{x},\bar{y}) \in \Omega$, $|x-\bar{x}| + |y-\bar{y}| < \delta_1 \Longrightarrow$

$$\begin{aligned} &[K(x,y,s,t;u,v) - K(\bar{x},\bar{y},s,t;u,v)]_E \\ &= \|K(x,y,s,t;u,v) - K(\bar{x},\bar{y},s,t;u,v)\|_E \\ &\quad + \|D_1K(x,y,s,t;u,v) - D_1K(\bar{x},\bar{y},s,t;u,v)\|_E \\ &< \frac{\varepsilon}{2},\ \forall (s,t) \in \Omega,\ \forall (u,v) \in R_1 \times R_2. \end{aligned}$$

Since g, D_1g are uniformly continuous on Ω, there is $\delta_2 > 0$ such that

$$\forall (x,y),\ (\bar{x},\bar{y}) \in \Omega,\ \ |x-\bar{x}| + |y-\bar{y}| < \delta_2 \Longrightarrow [g(x,y) - g(\bar{x},\bar{y})]_E < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$, it gives, $\forall (x,y)$, $(\bar{x},\bar{y}) \in \Omega$, $|x-\bar{x}| + |y-\bar{y}| < \delta$,

$$\begin{aligned} [(Au)(x,y) - (Au)(\bar{x},\bar{y})]_E &\le [g(x,y) - g(\bar{x},\bar{y})]_E \\ &\quad + \iint_\Omega [K(x,y,s,t;u(s,t),D_1u(s,t)) \\ &\quad - K(\bar{x},\bar{y},s,t;u(s,t),D_1u(s,t))]_E ds dt \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ \forall u \in B_\rho. \end{aligned} \tag{3.16}$$

Using Lemma 2.2, $\mathcal{F} = A(B_\rho)$ is relatively compact in X_1. And applying the Schauder fixed point theorem, the existence of a solution is proved.

Next, we show that the set of solutions, $S = \{u \in B_\rho : u = Au\}$, is compact in X_1. By the compactness of the operator $A : B_\rho \to B_\rho$ and $S = A(S)$, we only prove that S is closed. Let $\{u_p\} \subset S$, $\|u_p - u\|_{X_1} \to 0$. The continuity of A leads to

$$\begin{aligned}\|u - Au\|_{X_1} &\leq \|u - u_p\|_{X_1} + \|u_p - Au\|_{X_1} \\ &= \|u - u_p\|_{X_1} + \|Au_p - Au\|_{X_1} \\ &\to 0,\end{aligned}$$

so $u = Au \in S$. Theorem 3.2 is proved. □

4. Examples

For the end, we illustrate the results obtained here by two examples.

Let $E = C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $v : [0,1] \to \mathbb{R}$ with the norm

$$\|v\|_E = \sup_{0\leq\eta\leq1} |v(\eta)|\,,\ v \in E. \tag{4.1}$$

Let $X = C(\Omega; E)$ be the space of all continuous functions from Ω into E equipped with the following norm

$$\|u\|_X = \sup_{(x,y)\in\Omega} \|u(x,y)\|_E\,,\ u \in X. \tag{4.2}$$

Put

$$X_1 = \{u \in X : D_1 u \in X\}. \tag{4.3}$$

Then, for all $u \in X_1$ and $(x, y) \in \Omega$, $u(x, y)$ is an element of E and we denote

$$u(x, y)(\eta) = u(x, y; \eta),\ 0 \leq \eta \leq 1. \tag{4.4}$$

We also have the following lemma, it is clear, so we omit its proof.

Lemma 4.1. *Let positive constants* α, γ_2, γ_1 *satisfy* $0 < \alpha < 1$, $0 < \gamma_2 \leq 1 < \gamma_1$. *Then*

$$\begin{aligned} 0 &\leq x^{\gamma_1} |y - \alpha|^{\gamma_2} \leq \max\{\alpha^{\gamma_2}, (1-\alpha)^{\gamma_2}\}, \\ 0 &\leq x^{\gamma_1 - 1} |y - \alpha|^{\gamma_2} \leq \max\{\alpha^{\gamma_2}, (1-\alpha)^{\gamma_2}\},\ \forall x,\ y \in [0,1]. \end{aligned}$$

Example 4.2. We consider (1.1), with the functions $g : \Omega \to E$, $K : \Omega \times \Omega \times E^2 \to E$ as follows:

(i) Function $K : \Omega \times \Omega \times E^2 \to E$,

$$K(x, y, s, t; u, v)(\eta) = k(x, y; \eta)[(st)^{\alpha_0} \sin(\tfrac{\pi u(\eta)}{2w_0(s,t;\eta)}) + (st)^{\alpha_1} \cos(\tfrac{2\pi v(\eta)}{D_1 w_0(s,t;\eta)})], \tag{4.5}$$

for $0 \le \eta \le 1$, $(x, y, s, t; u, v) \in \Omega \times \Omega \times E^2$, with

$$\begin{cases} k,\ w_0 : \Omega \to E, \\ k(x, y; \eta) = \frac{1}{1+\eta} x^{\tilde{\gamma}_1} \left| y - \tilde{\alpha} \right|^{\tilde{\gamma}_2}, \\ w_0(x, y; \eta) = \frac{1}{1+\eta} \left[e^x + x^{\gamma_1} \left| y - \alpha \right|^{\gamma_2} \right],\ 0 \le \eta \le 1,\ (x, y) \in \Omega. \end{cases} \tag{4.6}$$

(ii) Function $g : \Omega \to E$,

$$g(x, y; \eta) = w_0(x, y; \eta) - \left[\frac{1}{(1 + \alpha_0)^2} + \frac{1}{(1 + \alpha_1)^2} \right] k(x, y; \eta), \tag{4.7}$$

for $0 \le \eta \le 1$, $(x, y) \in \Omega$, where α, γ_1, γ_2, $\tilde{\alpha}$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, α_0, α_1 are positive constants satisfying

$$\begin{cases} 0 < \alpha < 1,\ 0 < \tilde{\alpha} < 1,\ 0 < \gamma_2 \le 1 < \gamma_1,\ 0 < \tilde{\gamma}_2 \le 1 < \tilde{\gamma}_1, \\ \pi(1 + \tilde{\gamma}_1) \left[\frac{1}{(1+\alpha_0)^2} + \frac{4}{(1+\alpha_1)^2} \right] \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1 - \tilde{\alpha})^{\tilde{\gamma}_2}\} < 1. \end{cases} \tag{4.8}$$

Then we have

$$\begin{aligned} w_0(x, y; \eta) &= \frac{1}{1 + \eta} \left[e^x + x^{\gamma_1} \left| y - \alpha \right|^{\gamma_2} \right], \\ D_1 w_0(x, y; \eta) &= \frac{1}{1 + \eta} \left[e^x + \gamma_1 x^{\gamma_1 - 1} \left| y - \alpha \right|^{\gamma_2} \right],\ 0 \le \eta \le 1,\ (x, y) \in \Omega, \end{aligned}$$

and so w_0, $D_1 w_0 \in X$ and $w_0(x, y; \eta) \ge \frac{1}{2}$, $D_1^i w_0(x, y; \eta) \ge \frac{1}{2}$.

We now prove that (H_1), (H_2) hold. It is obvious that (H_1) holds, by w_0, $k \in X_1$.

Assumption (H_2) holds, by the fact that:

First, we show that $K \in C(\Omega \times \Omega \times E^2; E)$.
For all $(x, y, s, t; u, v)$, $(\bar{x}, \bar{y}, \bar{s}, \bar{t}; \bar{u}, \bar{v}) \in \Omega \times \Omega \times E^2$, for all $\eta \in [0, 1]$,

$$\begin{aligned} & K(x, y, s, t; u, v)(\eta) - K(\bar{x}, \bar{y}, \bar{s}, \bar{t}; \bar{u}, \bar{v})(\eta) \\ =\ & [k(x, y; \eta) - k(\bar{x}, \bar{y}; \eta)] \\ & \times \left[(st)^{\alpha_0} \sin\left(\frac{\pi u(\eta)}{2 w_0(s, t; \eta)} \right) + (st)^{\alpha_1} \cos\left(\frac{2\pi v(\eta)}{D_1 w_0(s, t; \eta)} \right) \right] \\ & + k(\bar{x}, \bar{y}; \eta) \left[(st)^{\alpha_0} - (\bar{s}\bar{t})^{\alpha_0} \right] \sin\left(\frac{\pi u(\eta)}{2 w_0(s, t; \eta)} \right) \\ & + k(\bar{x}, \bar{y}; \eta) \left[(st)^{\alpha_1} - (\bar{s}\bar{t})^{\alpha_1} \right] \cos\left(\frac{2\pi v(\eta)}{D_1 w_0(s, t; \eta)} \right) \\ & + k(\bar{x}, \bar{y}; \eta) (\bar{s}\bar{t})^{\alpha_0} \left[\sin\left(\frac{\pi u(\eta)}{2 w_0(s, t; \eta)} \right) - \sin\left(\frac{\pi \bar{u}(\eta)}{2 w_0(\bar{s}, \bar{t}; \eta)} \right) \right] \\ & + k(\bar{x}, \bar{y}; \eta) (\bar{s}\bar{t})^{\alpha_1} \left[\cos\left(\frac{2\pi v(\eta)}{D_1 w_0(s, t; \eta)} \right) - \cos\left(\frac{2\pi \bar{v}(\eta)}{D_1 w_0(\bar{s}, \bar{t}; \eta)} \right) \right]. \end{aligned}$$

Then

$$\begin{aligned}
&|K(x,y,s,t;u,v)(\eta)-K(\bar{x},\bar{y},\bar{s},\bar{t};\bar{u},\bar{v})(\eta)| \\
\leq\;& 2\left\|k(x,y)-k(\bar{x},\bar{y})\right\|_E+\left\|k(\bar{x},\bar{y})\right\|_E\left|(st)^{\alpha_0}-(\bar{s}\bar{t})^{\alpha_0}\right| \\
&+\left\|k(\bar{x},\bar{y})\right\|_E\left|(st)^{\alpha_1}-(\bar{s}\bar{t})^{\alpha_1}\right|+\left\|k(\bar{x},\bar{y})\right\|_E\left|\frac{\pi u(\eta)}{2w_0(s,t;\eta)}-\frac{\pi\bar{u}(\eta)}{2w_0(\bar{s},\bar{t};\eta)}\right| \\
&+\left\|k(\bar{x},\bar{y})\right\|_E\left|\frac{2\pi v(\eta)}{D_1w_0(s,t;\eta)}-\frac{2\pi\bar{v}(\eta)}{D_1w_0(\bar{s},\bar{t};\eta)}\right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left|\frac{\pi u(\eta)}{2w_0(s,t;\eta)}-\frac{\pi\bar{u}(\eta)}{2w_0(\bar{s},\bar{t};\eta)}\right| \\
=\;&\frac{\pi}{2}\left|\frac{w_0(\bar{s},\bar{t};\eta)\left[u(\eta)-\bar{u}(\eta)\right]+\left[w_0(\bar{s},\bar{t};\eta)-w_0(s,t;\eta)\right]\bar{u}(\eta)}{w_0(s,t;\eta)w_0(\bar{s},\bar{t};\eta)}\right| \\
\leq\;& 2\pi\left[\left\|w_0(\bar{s},\bar{t})\right\|_E\left\|u-\bar{u}\right\|_E+\left\|w_0(\bar{s},\bar{t})-w_0(s,t)\right\|_E\left\|\bar{u}\right\|_E\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{2\pi v(\eta)}{D_1w_0(s,t;\eta)}-\frac{2\pi\bar{v}(\eta)}{D_1w_0(\bar{s},\bar{t};\eta)}\right| \\
\leq\;& 8\pi\left[\left\|D_1w_0(\bar{s},\bar{t})\right\|_E\left\|v-\bar{v}\right\|_E+\left\|D_1w_0(\bar{s},\bar{t})-D_1w_0(s,t)\right\|_E\left\|\bar{v}\right\|_E\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|K(x,y,s,t;u,v)-K(\bar{x},\bar{y},\bar{s},\bar{t};\bar{u},\bar{v})\right\|_E \\
\leq\;& 2\left\|k(x,y)-k(\bar{x},\bar{y})\right\|_E \\
&+\left\|k(\bar{x},\bar{y})\right\|_E\left|(st)^{\alpha_0}-(\bar{s}\bar{t})^{\alpha_0}\right|+\left\|k(\bar{x},\bar{y})\right\|_E\left|(st)^{\alpha_1}-(\bar{s}\bar{t})^{\alpha_1}\right| \\
&+2\pi\left\|k(\bar{x},\bar{y})\right\|_E\left[\left\|w_0(\bar{s},\bar{t})\right\|_E\left\|u-\bar{u}\right\|_E+\left\|w_0(\bar{s},\bar{t})-w_0(s,t)\right\|_E\left\|\bar{u}\right\|_E\right] \\
&+8\pi\left\|k(\bar{x},\bar{y})\right\|_E\left[\left\|D_1w_0(\bar{s},\bar{t})\right\|_E\left\|v-\bar{v}\right\|_E\right. \\
&\left.+||D_1w_0(\bar{s},\bar{t})-D_1w_0(s,t)||_E||\bar{v}||_E\right] \\
\to\;& 0,
\end{aligned}$$

as $|x-\bar{x}|+|y-\bar{y}|+|s-\bar{s}|+|t-\bar{t}|+\left\|u-\bar{u}\right\|_E+\left\|v-\bar{v}\right\|_E\to 0$. Thus $K\in C(\Omega\times\Omega\times E^2;E)$.

Similarly $D_1K\in C(\Omega\times\Omega\times E^2;E)$.

Next, the assumptions (H_2)-(i) and (ii) hold, by the fact that

$$\begin{aligned}
&| \, K(x,y,s,t;u,v)(\eta) - K(x,y,s,t;\bar{u},\bar{v})(\eta)| \\
\leq \;& k(x,y;\eta)\left((st)^{\alpha_0}\frac{\pi\,|u(\eta)-\bar{u}(\eta)|}{2w_0(s,t;\eta)} + (st)^{\alpha_1}\frac{2\pi\,|v(\eta)-\bar{v}(\eta)|}{D_1w_0(s,t;\eta)}\right) \\
\leq \;& \pi k(x,y;\eta)\left((st)^{\alpha_0}\,|u(\eta)-\bar{u}(\eta)| + 4(st)^{\alpha_1}\,|v(\eta)-\bar{v}(\eta)|\right) \\
\leq \;& \pi k(x,y;\eta)\left[(st)^{\alpha_0}+4(st)^{\alpha_1}\right]\left[\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\|K(x,y,s,t;u,v) - K(x,y,s,t;\bar{u},\bar{v})\|_E \\
\leq \;& \pi\,\|k(x,y)\|_E\left[(st)^{\alpha_0}+4(st)^{\alpha_1}\right]\left[\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right] \\
= \;& k_0(x,y,s,t)\left[\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right],
\end{aligned} \tag{4.9}$$

in which

$$k_0(x,y,s,t) = \pi x^{\tilde{\gamma}_1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[(st)^{\alpha_0}+4(st)^{\alpha_1}\right]. \tag{4.10}$$

Similarly, with

$$\begin{aligned}
&D_1K(x,y,s,t;u,v)(\eta) \\
= \;& D_1k(x,y;\eta)\left[(st)^{\alpha_0}\sin\left(\frac{\pi u(\eta)}{2w_0(s,t;\eta)}\right) + (st)^{\alpha_1}\cos\left(\frac{2\pi v(\eta)}{D_1w_0(s,t;\eta)}\right)\right],
\end{aligned}$$

we have

$$\begin{aligned}
&\|D_1K(x,y,s,t;u,v) - D_1K(x,y,s,t;\bar{u},\bar{v})\|_E \\
\leq \;& k_1(x,y,s,t)\left[\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right],
\end{aligned} \tag{4.11}$$

with

$$k_1(x,y,s,t) = \pi\tilde{\gamma}_1 x^{\tilde{\gamma}_1-1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[(st)^{\alpha_0}+4(st)^{\alpha_1}\right]. \tag{4.12}$$

Using Lemma 4.2, we get

$$\begin{aligned}
\iint_\Omega k_0(x,y,s,t)dsdt &= \pi x^{\tilde{\gamma}_1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\iint_\Omega\left[(st)^{\alpha_0}+4(st)^{\alpha_1}\right]dsdt \\
&= \pi x^{\tilde{\gamma}_1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[\frac{1}{(1+\alpha_0)^2}+\frac{4}{(1+\alpha_1)^2}\right] \\
&\leq \pi\left[\frac{1}{(1+\alpha_0)^2}+\frac{4}{(1+\alpha_1)^2}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}; \\
\iint_\Omega k_1(x,y,s,t)dsdt &= \pi\tilde{\gamma}_1 x^{\tilde{\gamma}_1-1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\iint_\Omega\left[(st)^{\alpha_0}+4(st)^{\alpha_1}\right]dsdt \\
&\leq \pi\tilde{\gamma}_1\left[\frac{1}{(1+\alpha_0)^2}+\frac{4}{(1+\alpha_1)^2}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}.
\end{aligned}$$

Therefore

$$\begin{aligned}\beta &= \sup_{(x,y)\in\Omega}\iint_\Omega k_0(x,y,s,t)dsdt + \sup_{(x,y)\in\Omega}\iint_\Omega k_1(x,y,s,t)dsdt \\ &\le \pi(1+\tilde{\gamma}_1)\left[\frac{1}{(1+\alpha_0)^2}+\frac{4}{(1+\alpha_1)^2}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\} \qquad (4.13)\\ &< 1.\end{aligned}$$

Hence, the assumption (H_2)-(i) holds. Then, Theorem 3.1 is fulfilled. Morever, $w_0 \in X_1$ is also a unique solution of (1.1).

Example 4.3. We consider (1.1), with the functions $g : \Omega \to E$, $K : \Omega \times \Omega \times E^2 \to E$ defined by

(i) Function $K : \Omega \times \Omega \times E^2 \to E$,

$$K(x,y,s,t;u,v)(\eta) = k(x,y;\eta)\left[(st)^{\alpha_0}\int_0^1\left|\frac{u(\zeta)}{w_0(s,t;\zeta)}\right|^{1/4}d\zeta + (st)^{\alpha_1}\int_0^1\left(\frac{v(\zeta)}{D_1w_0(s,t;\zeta)}\right)^{1/3}d\zeta\right], \qquad (4.14)$$

for $0 \le \eta \le 1$, $(x,y,s,t;u,v) \in \Omega\times\Omega\times E^2$, with

$$\begin{cases} k,\, w_0 : \Omega \to E \\ k(x,y;\eta) = \frac{1}{1+\eta}x^{\tilde{\gamma}_1}\left|y-\tilde{\alpha}\right|^{\tilde{\gamma}_2}, \\ w_0(x,y;\eta) = \frac{1}{1+\eta}\left[e^x + x^{\gamma_1}\left|y-\alpha\right|^{\gamma_2}\right],\ 0\le\eta\le1,\ (x,y)\in\Omega. \end{cases} \qquad (4.15)$$

(ii) Function $g : \Omega \to E$,

$$g(x,y;\eta) = w_0(x,y;\eta) - \left[\frac{1}{(1+\alpha_0)^2}+\frac{1}{(1+\alpha_1)^2}\right]k(x,y;\eta), \qquad (4.16)$$

for $0 \le \eta \le 1$, $(x,y) \in \Omega$, where α, γ_1, γ_2, $\tilde{\alpha}$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, α_0, α_1 are positive constants satisfying

$$\begin{cases} 0<\alpha<1,\ 0<\tilde{\alpha}<1,\ 0<\gamma_2\le 1<\gamma_1,\ 0<\tilde{\gamma}_2\le1<\tilde{\gamma}_1, \\ 4(1+\tilde{\gamma}_1)\left[\frac{1}{(1+\alpha_0)^2}+\frac{1}{(1+\alpha_1)^2}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}<1. \end{cases} \qquad (4.17)$$

We can prove that (H_1), $(\bar{H}_2)$, $(\bar{H}_3)$ hold, by the following:

By k, $w_0 \in X_1$, we have (H_1) holds.

Assumption $(\bar{H}_2)$ holds, by the fact that:

First, we show that $K \in C(\Omega\times\Omega\times E^2; E)$.

For all $(x,y,s,t;u,v)$, $(\bar{x},\bar{y},\bar{s},\bar{t};\bar{u},\bar{v}) \in \Omega\times\Omega\times E^2$, for all $\eta\in[0,1]$,

$$\begin{aligned}
&K(x,y,s,t;u,v)(\eta) - K(\bar{x},\bar{y},\bar{s},\bar{t};\bar{u},\bar{v})(\eta) \\
&= [k(x,y;\eta) - k(\bar{x},\bar{y};\eta)] \left[(st)^{\alpha_0} \int_0^1 \left| \frac{u(\zeta)}{w_0(s,t;\zeta)} \right|^{1/4} d\zeta \right. \\
&\qquad \left. + (st)^{\alpha_1} \int_0^1 \left(\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)} \right)^{1/3} d\zeta \right] \\
&\quad + k(\bar{x},\bar{y};\eta)\left[(st)^{\alpha_0} - (\bar{s}\bar{t})^{\alpha_0}\right] \int_0^1 \left| \frac{u(\zeta)}{w_0(s,t;\zeta)} \right|^{1/4} d\zeta \\
&\quad + k(\bar{x},\bar{y};\eta)\left[(st)^{\alpha_1} - (\bar{s}\bar{t})^{\alpha_1}\right] \int_0^1 \left(\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)} \right)^{1/3} d\zeta \\
&\quad + k(\bar{x},\bar{y};\eta)(\bar{s}\bar{t})^{\alpha_0} \left[\int_0^1 \left(\left| \frac{u(\zeta)}{w_0(s,t;\zeta)} \right|^{1/4} - \left| \frac{\bar{u}(\zeta)}{w_0(\bar{s},\bar{t};\zeta)} \right|^{1/4} \right) d\zeta \right] \\
&\quad + k(\bar{x},\bar{y};\eta)(\bar{s}\bar{t})^{\alpha_1} \int_0^1 \left[\left(\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)} \right)^{1/3} - \left(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{s},\bar{t};\zeta)} \right)^{1/3} \right] d\zeta.
\end{aligned}$$

Note that, w_0, $D_1 w_0 \in X$ and $w_0(x,y;\eta) \geq \frac{1}{2}$, $D_1 w_0(x,y;\eta) \geq \frac{1}{2}$, we obtain

$$\begin{aligned}
&\left| K(x,y,s,t;u,v)(\eta) - K(\bar{x},\bar{y},\bar{s},\bar{t};\bar{u},\bar{v})(\eta) \right| \\
&\leq 2\left\| k(x,y) - k(\bar{x},\bar{y}) \right\|_E \left[\|u\|_E^{1/4} + \|v\|_E^{1/3} \right] \\
&\quad + 2\left\| k(\bar{x},\bar{y}) \right\|_E \left| (st)^{\alpha_0} - (\bar{s}\bar{t})^{\alpha_0} \right| \|u\|_E^{1/4} \\
&\quad + 2\left\| k(\bar{x},\bar{y}) \right\|_E \left| (st)^{\alpha_1} - (\bar{s}\bar{t})^{\alpha_1} \right| \|v\|_E^{1/3} \\
&\quad + \left\| k(\bar{x},\bar{y}) \right\|_E \int_0^1 \left| \left| \frac{u(\zeta)}{w_0(s,t;\zeta)} \right|^{1/4} - \left| \frac{\bar{u}(\zeta)}{w_0(\bar{s},\bar{t};\zeta)} \right|^{1/4} \right| d\zeta \\
&\quad + \left\| k(\bar{x},\bar{y}) \right\|_E \int_0^1 \left| \left(\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)} \right)^{1/3} - \left(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{s},\bar{t};\zeta)} \right)^{1/3} \right| d\zeta.
\end{aligned}$$

Applying the following inequalities:

$$\begin{aligned}
\left| |a|^q - |b|^q \right| &\leq |a-b|^q, \ \forall a,b\in\mathbb{R}, \ \forall q\in(0,1], \qquad (4.18)\\
\left| |a|^{q-1} a - |b|^{q-1} b \right| &\leq 2^{1-q} |a-b|^q, \ \forall a,b\in\mathbb{R}, \ \forall q\in(0,1], \\
(a+b)^q &\leq a^q + b^q, \ \forall a,b\geq 0, \ \forall q\in(0,1],
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left|\frac{u(\eta)}{w_0(s,t;\eta)} - \frac{\bar{u}(\eta)}{w_0(\bar{s},\bar{t};\eta)}\right| \\
&= \left|\frac{w_0(\bar{s},\bar{t};\eta)\left[u(\eta)-\bar{u}(\eta)\right] + \left[w_0(\bar{s},\bar{t};\eta) - w_0(s,t;\eta)\right]\bar{u}(\eta)}{w_0(s,t;\eta)w_0(\bar{s},\bar{t};\eta)}\right| \\
&\leq 4\left\|w_0(\bar{s},\bar{t})\right\|_E \left\|u-\bar{u}\right\|_E + 4\left\|w_0(\bar{s},\bar{t}) - w_0(s,t)\right\|_E \left\|\bar{u}\right\|_E,
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{v(\eta)}{D_1 w_0(s,t;\eta)} - \frac{\bar{v}(\eta)}{D_1 w_0(\bar{s},\bar{t};\eta)}\right| \\
&\leq 4\left\|D_1 w_0(\bar{s},\bar{t})\right\|_E \left\|v-\bar{v}\right\|_E + 4\left\|D_1 w_0(\bar{s},\bar{t}) - D_1 w_0(s,t)\right\|_E \left\|\bar{v}\right\|_E
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left|\frac{u(\zeta)}{w_0(s,t;\zeta)}\right|^{1/4} - \left|\frac{\bar{u}(\zeta)}{w_0(\bar{s},\bar{t};\zeta)}\right|^{1/4}\right| \\
&\leq \left|\frac{u(\zeta)}{w_0(s,t;\zeta)} - \frac{\bar{u}(\zeta)}{w_0(\bar{s},\bar{t};\zeta)}\right|^{1/4} \\
&\leq 4^{1/4}\Big[\left\|w_0(\bar{s},\bar{t})\right\|_E \left\|u-\bar{u}\right\|_E + \left\|w_0(\bar{s},\bar{t}) - w_0(s,t)\right\|_E \left\|\bar{u}\right\|_E\Big]^{1/4} \\
&\leq \sqrt{2}\left[\left\|w_0(\bar{s},\bar{t})\right\|_E^{1/4} \left\|u-\bar{u}\right\|_E^{1/4} + \left\|w_0(\bar{s},\bar{t}) - w_0(s,t)\right\|_E^{1/4} \left\|\bar{u}\right\|_E^{1/4}\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left|\Big(\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)}\Big)^{1/3} - \Big(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{s},\bar{t};\zeta)}\Big)^{1/3}\right| \\
&= \Big|\Big(\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)}\Big)^{-2/3}\Big(\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)}\Big) \\
&\qquad -\Big(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{s},\bar{t};\zeta)}\Big)^{-2/3}\Big(\frac{\bar{v}(\zeta)}{D_1 w_0(\bar{s},\bar{t};\zeta)}\Big)\Big| \\
&\leq 2^{2/3}\left|\frac{v(\zeta)}{D_1 w_0(s,t;\zeta)} - \frac{\bar{v}(\zeta)}{D_1 w_0(\bar{s},\bar{t};\zeta)}\right|^{1/3} \\
&\leq 2^{2/3}4^{1/3}\Big[\left\|D_1 w_0(\bar{s},\bar{t})\right\|_E \left\|v-\bar{v}\right\|_E + \left\|D_1 w_0(\bar{s},\bar{t}) - D_1 w_0(s,t)\right\|_E \left\|\bar{v}\right\|_E\Big]^{1/3} \\
&\leq 2^{4/3}\left[\left\|D_1 w_0(\bar{s},\bar{t})\right\|_E^{1/3} \left\|v-\bar{v}\right\|_E^{1/3} + \left\|D_1 w_0(\bar{s},\bar{t}) - D_1 w_0(s,t)\right\|_E^{1/3} \left\|\bar{v}\right\|_E^{1/3}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\|K(x,y,s,t;u,v)-K(\bar{x},\bar{y},\bar{s},\bar{t};\bar{u},\bar{v})\|_E\\
&\le 2\,\|k(x,y)-k(\bar{x},\bar{y})\|_E\left[\|u\|_E^{1/4}+\|v\|_E^{1/3}\right]\\
&\quad+2\,\|k(\bar{x},\bar{y})\|_E\,|(st)^{\alpha_0}-(\bar{s}\bar{t})^{\alpha_0}|\,\|u\|_E^{1/4}+2\,\|k(\bar{x},\bar{y})\|_E\,|(st)^{\alpha_1}-(\bar{s}\bar{t})^{\alpha_1}|\,\|v\|_E^{1/3}\\
&\quad+\sqrt{2}\,\|k(\bar{x},\bar{y})\|_E\left[\|w_0(\bar{s},\bar{t})\|_E^{1/4}\,\|u-\bar{u}\|_E^{1/4}+\|w_0(\bar{s},\bar{t})-w_0(s,t)\|_E^{1/4}\,\|\bar{u}\|_E^{1/4}\right]\\
&\quad+2^{4/3}\,\|k(\bar{x},\bar{y})\|_E\\
&\quad\times\left[\|D_1w_0(\bar{s},\bar{t})\|_E^{1/3}\,\|v-\bar{v}\|_E^{1/3}+\|D_1w_0(\bar{s},\bar{t})-D_1w_0(s,t)\|_E^{1/3}\,\|\bar{v}\|_E^{1/3}\right]\\
&\to 0,
\end{aligned}$$

as $|x-\bar{x}|+|y-\bar{y}|+|s-\bar{s}|+|t-\bar{t}|+\|u-\bar{u}\|_E+\|v-\bar{v}\|_E\to 0$. Thus $K\in C(\Omega\times\Omega\times E^2;E)$.

Similarly $D_1K\in C(\Omega\times\Omega\times E^2;E)$.

Assumptions $(\bar{H}_2)$-(ii) and (iii) also hold, the proof is as follows:
Applying the inequality

$$a\le 1+a^q,\ \ \forall a\ge 0,\ \ \forall q\ge 1, \tag{4.19}$$

we obtain

$$\begin{aligned}
&|K(x,y,s,t;u,v)(\eta)|\\
&\le |k(x,y;\eta)|\left[(st)^{\alpha_0}\left(1+\frac{2\,|u(\eta)|}{w_0(s,t;\eta)}\right)+(st)^{\alpha_1}\left(1+\frac{2\,|v(\eta)|}{D_1w_0(s,t;\eta)}\right)\right]\\
&\le 4x^{\tilde{\gamma}_1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[(st)^{\alpha_0}+(st)^{\alpha_1}\right]\left[1+\|u\|_E+\|v\|_E\right],
\end{aligned}$$

it leads to

$$\|K(x,y,s,t;u,v)\|_E\le \bar{k}_0(x,y,s,t)\left[1+\|u\|_E+\|v\|_E\right], \tag{4.20}$$

in which

$$\bar{k}_0(x,y,s,t)=4x^{\tilde{\gamma}_1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[(st)^{\alpha_0}+(st)^{\alpha_1}\right]. \tag{4.21}$$

Similarly, we have

$$\|D_1K(x,y,s,t;u,v)\|_E\le \bar{k}_1(x,y,s,t)\left[1+\|u\|_E+\|v\|_E\right], \tag{4.22}$$

with

$$\bar{k}_1(x,y,s,t)=4\tilde{\gamma}_1x^{\tilde{\gamma}_1-1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\left[(st)^{\alpha_0}+(st)^{\alpha_1}\right]. \tag{4.23}$$

Next,

$$\begin{aligned}
\iint_\Omega \bar{k}_0(x,y,s,t)dsdt &= 4x^{\tilde{\gamma}_1}\,|y-\tilde{\alpha}|^{\tilde{\gamma}_2}\iint_\Omega\left[(st)^{\alpha_0}+(st)^{\alpha_1}\right]dsdt\\
&\le 4\left[\frac{1}{(1+\alpha_0)^2}+\frac{1}{(1+\alpha_1)^2}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\};
\end{aligned}$$

$$\begin{aligned}\iint_\Omega \bar{k}_1(x,y,s,t)dsdt &= 4\tilde{\gamma}_1 x^{\tilde{\gamma}_1-1}\left|y-\tilde{\alpha}\right|^{\tilde{\gamma}_2}\iint_\Omega \left[(st)^{\alpha_0}+(st)^{\alpha_1}\right]dsdt \\ &\le 4\tilde{\gamma}_1\left[\frac{1}{(1+\alpha_0)^2}+\frac{1}{(1+\alpha_1)^2}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}.\end{aligned}$$

Thus

$$\begin{aligned}\bar{\beta} &= \sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_0(x,y,s,t)dsdt+\sup_{(x,y)\in\Omega}\iint_\Omega \bar{k}_1(x,y,s,t)dsdt \qquad (4.24)\\ &= 4(1+\tilde{\gamma}_1)\left[\frac{1}{(1+\alpha_0)^2}+\frac{1}{(1+\alpha_1)^2}\right]\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}<1.\end{aligned}$$

Hence, assumption $(\bar{H}_2)$-(i) holds.

Assumption $(\bar{H}_3)$ holds; the proof is as below:

(a) We prove that $K:\Omega\times\Omega\times E^2\to E$, is completely continuous.

By K, $D_1K\in C(\Omega\times\Omega\times E^2;E)$, we have to prove that K, $D_1K:\Omega\times\Omega\times E^2\to E$ are compact.

Let B be bounded in $\Omega\times\Omega\times E^2$. We have

$$\begin{aligned}\|K(x,y,s,t;u,v)\|_E &\le \bar{k}_0(x,y,s,t)\left(1+\|u\|_E+\|v\|_E\right) \qquad (4.25)\\ &\le \sup_{(x,y,s,t;u,v)\in B}\bar{k}_0(x,y,s,t)\left(1+\|u\|_E+\|v\|_E\right)\\ &\equiv M_1,\end{aligned}$$

for all $(x,y,s,t;u,v)\in B$, which implies that $K(B)$ is uniformly bounded in E.

For all η, $\bar{\eta}\in[0,1]$, for all $(x,y,s,t;u,v)\in B$,

$$\begin{aligned}&\left|K(x,y,s,t;u,v)(\eta)-K(x,y,s,t;u,v)(\bar{\eta})\right|\\ &=\left|k(x,y;\eta)-k(x,y;\bar{\eta})\right|\\ &\quad\times\left|(st)^{\alpha_0}\int_0^1\left|\frac{u(\zeta)}{w_0(s,t;\zeta)}\right|^{1/4}d\zeta+(st)^{\alpha_1}\int_0^1\left(\frac{v(\zeta)}{D_1w_0(s,t;\zeta)}\right)^{1/3}d\zeta\right|\\ &\le 2\left|k(x,y;\eta)-k(x,y;\bar{\eta})\right|\left[\|u\|_E^{1/4}+\|v\|_E^{1/3}\right].\end{aligned}$$

On the other hand

$$\left|k(x,y;\eta)-k(x,y;\bar{\eta})\right|=\frac{|\bar{\eta}-\eta|}{(1+\eta)(1+\bar{\eta})}x^{\tilde{\gamma}_1}\left|y-\tilde{\alpha}\right|^{\tilde{\gamma}_2}\le\left|\bar{\eta}-\eta\right|.$$

Hence

$$\begin{aligned}\||K(x,y,s,t;u,v)-K(x,y,s,t;u,v)||_E &\le 2\left|\bar{\eta}-\eta\right|\left[\|u\|_E^{1/4}+\|v\|_E^{1/3}\right]\\ &\le C\left|\bar{\eta}-\eta\right|, \qquad (4.26)\end{aligned}$$

for all $(x, y, s, t; u, v) \in B$, for all η, $\bar{\eta} \in [0, 1]$. Consequently, $K(B)$ is equicontinuous.

(b) Similarly, we have also $D_1K : \Omega \times \Omega \times E^2 \to E$, is completely continuous.

(c) Finally, for all bounded subset J of E^2, for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all (x, y, s, t), $(\bar{x}, \bar{y}, s, t) \in \Omega \times \Omega$, $|x - \bar{x}| + |y - \bar{y}| < \delta \Longrightarrow$

$$\begin{aligned}
&\|K(x, y, s, t; u, v) - K(\bar{x}, \bar{y}, s, t; u, v)\|_E \\
&+ \|D_1K(x, y, s, t; u, v) - D_1K(\bar{x}, \bar{y}, s, t; u, v)\|_E \\
&< \varepsilon, \ \forall (u, v) \in J.
\end{aligned}$$

Indeed, we get the above property, since

$$\begin{aligned}
&\|K(x, y, s, t; u, v) - K(\bar{x}, \bar{y}, s, t; u, v)\|_E \\
&+ \|D_1K(x, y, s, t; u, v) - D_1K(\bar{x}, \bar{y}, s, t; u, v)\|_E \\
&\leq 2 \left(\|k(x, y) - k(\bar{x}, \bar{y})\|_E + \|D_1k(x, y) - D_1k(\bar{x}, \bar{y})\|_E\right) \left[\|u\|_E^{1/4} + \|v\|_E^{1/3}\right] \\
&\leq C \left(\|k(x, y) - k(\bar{x}, \bar{y})\|_E + \|D_1k(x, y) - D_1k(\bar{x}, \bar{y})\|_E\right),
\end{aligned}$$

for all $(s, t; u, v) \in \Omega \times J$, and $(x, y), (\bar{x}, \bar{y}) \in \Omega$, where k, $D_1k : \Omega \to E$ are uniformly continuous on Ω. Theorem 3.2 is true. Furthermore, $w_0 \in X_1$ is also a solution of (1.1) in this case.

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