Nonlinear Functional Analysis and Applications
Vol. 24, No. 2 (2019), pp. 361-387
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

COMMON FIXED POINTS FOR TWO PAIRS OF MAPPINGS SATISFYING CONTRACTIVE INEQUALITIES OF INTEGRAL TYPE

Zeqing Liu[1], Miao He[2] and Chahn Yong Jung[3]

[1]Department of Mathematics, Liaoning Normal University
Dalian, Liaoning 116029, China
e-mail: zeqingliu@163.com

[2]Department of Mathematics, Liaoning Normal University
Dalian, Liaoning 116029, China
e-mail: miaohe130@163.com

[3]Department of Business Administration, Gyeongsang National University
Jinju 52828, Korea
e-mail: bb5734@gnu.ac.kr

Abstract. The existence and uniqueness of common fixed point for two pairs of mappings satisfying contractive inequalities of integral type in metric spaces are proved. Two examples are included to illustrate that the results presented in this paper generalize indeed or differ from some known results in the literature.

1. Introduction and preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, (X, d) is a metric space and

$\Phi_1 = \big\{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0\big\}$,

[0]Received November 22, 2018. Revised March 5, 2019.
[0]2010 Mathematics Subject Classification: 54H25.
[0]Keywords: Common fixed points, contractive inequalities of integral type, weakly compatible mappings.
[0]Corresponding Author: C. Y. Jung(bb5734@gnu.ac.kr).

$\Phi_2 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that $\liminf_{n\to\infty} \varphi(a_n) > 0 \Leftrightarrow \liminf_{n\to\infty} a_n > 0$ for each $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+\}$,

$\Phi_3 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous$\}$,

$\Phi_4 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semi-continuous with $\varphi(t) = 0$ if and only if $t = 0\}$,

$\Phi_5 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$.

Dutta and Choudhuty [4] and Branciari [3] extended the Banach fixed point theorem and proved the following results for (ψ, φ)-weakly contractive mappings and contractive mapping of integral type, respectively.

Theorem 1.1. ([4]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{1.1}$$

where $\psi, \varphi \in \Phi_5$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Theorem 1.2. ([3]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.2}$$

where $c \in (0, 1)$ is a constant and $\varphi \in \Phi_1$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$.

Recently, the researchers [1, 2, 5, 6, 8–17] proved a lot of fixed and common fixed point theorems for various (ψ, φ)-weakly contractive mappings and contractive mappings of integral type. In particular, Liu et al. [9] and Hosseini [6] extended the results of Branciari [3] and Dutta and Choudhuty [4] and got the following theorems.

Theorem 1.3. ([9]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying for all $x, y \in X$,*

$$\psi\left(\int_0^{d(Tx,Ty)} \varphi(t)dt\right) \leq \psi\left(\int_0^{d(x,y)} \varphi(t)dt\right) - \varphi\left(\int_0^{d(x,y)} \varphi(t)dt)\right), \tag{1.3}$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_5$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$.

Theorem 1.4. ([6]) *Let T and S be two mappings from a complete metric space (X,d) into itself satisfying for all $x,y \in X$,*

$$\psi\left(\int_0^{d(Tx,Sy)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \varphi\left(\int_0^{M(x,y)} \varphi(t)dt)\right), \quad (1.4)$$

where $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_4 \times \Phi_5$ and for all $x,y \in X$

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(y,Tx) + d(x,Sy)]\right\}.$$

Then T and S have a unique common fixed point $a \in X$.

The purpose of this article is to study the existence and uniqueness of common fixed point for certain four mappings satisfying contractive inequalities of integral type in metric spaces. Our results extend Theorems 1.1 and 1.2, and are different from Theorems 1.3 and 1.4. Two examples are included.

Definition 1.5. ([7]) A pair of self mappings f and g in a metric space (X,d) are said to be *weakly compatible* if for all $t \in X$ the equality $ft = gt$ implies $fgt = gft$.

Lemma 1.6. ([10]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty}\int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 1.7. ([9]) *Let $\varphi \in \Phi_2$. Then $\varphi(t) > 0$ if and only if $t > 0$.*

2. Common fixed point theorems

Our main results are as follows.

Theorem 2.1. *Let A, B, S and T be mappings from a metric space (X,d) into itself satisfying*

$$\{A,T\} \text{ and } \{B,S\} \text{ are weakly compatible;} \quad (2.1)$$

$$T(X) \subseteq B(X) \text{ and } S(X) \subseteq A(X); \quad (2.2)$$

$$\text{one of } A(X), B(X), S(X) \text{ and } T(X) \text{ is complete}; \quad (2.3)$$

$$\psi\left(\int_0^{d(Tx,Sy)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M_1(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M_1(x,y)} \varphi(t)dt\right) \quad (2.4)$$

for all $x,y \in X$, where $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and for all $x,y \in X$,

$$M_1(x,y) = \Big\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy)+d(Tx,By)], \frac{1+2d(Ax,Sy)}{2(1+d(Ax,By))}d(Tx,By), \frac{1+2d(Tx,By)}{2(1+d(Ax,By))}d(Ax,Sy), \frac{1+d(Ax,Sy)}{1+2d(Ax,By)}d(Ax,Tx), \frac{1+d(Tx,By)}{1+2d(Ax,By)}d(By,Sy)\Big\}. \tag{2.5}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. (2.2) ensures that there exist two sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}_0}$ in X such that

$$y_{2n+1} = Bx_{2n+1} = Tx_{2n}, \quad y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.6}$$

Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. Suppose that $d_{2n} < d_{2n+1}$ for some $n \in \mathbb{N}$. It is clear that

$$\begin{aligned} d_{2n+1} - \frac{d(y_{2n}, y_{2n+2})}{2(1+d_{2n})} &\geq d_{2n+1} - \frac{d_{2n}+d_{2n+1}}{2(1+d_{2n})} \\ &= \frac{d_{2n+1}+2d_{2n+1}d_{2n}-d_{2n}}{2(1+d_{2n})} > 0 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} d_{2n+1} - \frac{1+d_{2n}+d_{2n+1}}{1+2d_{2n}}d_{2n} &= \frac{d_{2n+1}+d_{2n}d_{2n+1}-d_{2n}-d_{2n}^2}{1+2d_{2n}} \\ &> 0. \end{aligned} \tag{2.8}$$

In view of (2.4)-(2.8), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we know that

$$\begin{aligned} &M_1(x_{2n}, x_{2n+1}) \\ &= \max\Big\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n+1}, Sx_{2n+1}), \\ &\quad \frac{1}{2}[d(Ax_{2n}, Sx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})], \\ &\quad \frac{1+2d(Ax_{2n}, Sx_{2n+1})}{2(1+d(Ax_{2n}, Bx_{2n+1}))}d(Tx_{2n}, Bx_{2n+1}), \\ &\quad \frac{1+2d(Tx_{2n}, Bx_{2n+1})}{2(1+d(Ax_{2n}, Bx_{2n+1}))}d(Ax_{2n}, Sx_{2n+1}), \\ &\quad \frac{1+d(Ax_{2n}, Sx_{2n+1})}{1+2d(Ax_{2n}, Bx_{2n+1})}d(Ax_{2n}, Tx_{2n}), \\ &\quad \frac{1+d(Tx_{2n}, Bx_{2n+1})}{1+2d(Ax_{2n}, Bx_{2n+1})}d(Bx_{2n+1}, Sx_{2n+1})\Big\} \end{aligned}$$

$$\begin{aligned}
&= \max\Big\{ d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}),\\
&\quad \frac{1}{2}[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})],\\
&\quad \frac{1+2d(y_{2n}, y_{2n+2})}{2(1+d(y_{2n}, y_{2n+1}))} d(y_{2n+1}, y_{2n+1}),\\
&\quad \frac{1+2d(y_{2n+1}, y_{2n+1})}{2(1+d(y_{2n}, y_{2n+1}))} d(y_{2n}, y_{2n+2}),\\
&\quad \frac{1+d(y_{2n}, y_{2n+2})}{1+2d(y_{2n}, y_{2n+1})} d(y_{2n}, y_{2n+1}),\\
&\quad \frac{1+d(y_{2n+1}, y_{2n+1})}{1+2d(y_{2n}, y_{2n+1})} d(y_{2n+1}, y_{2n+2}) \Big\}\\
&= \max\Big\{ d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2} d(y_{2n}, y_{2n+2}), 0, \frac{d(y_{2n}, y_{2n+2})}{2(1+d_{2n})},\\
&\quad \frac{1+d(y_{2n}, y_{2n+2})}{1+2d_{2n}} d_{2n}, \frac{d_{2n+1}}{1+2d_{2n}} \Big\}\\
&= \max\{d_{2n}, d_{2n+1}\}\\
&= d_{2n+1}
\end{aligned}$$

and

$$\begin{aligned}
\psi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg) &= \psi\bigg(\int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt\bigg)\\
&\le \psi\bigg(\int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt\bigg)\\
&= \psi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg)\\
&< \psi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg),
\end{aligned}$$

which is a contradiction. Hence

$$d_{2n+1} \le d_{2n} = M_1(x_{2n}, x_{2n+1}), \quad \forall n \in \mathbb{N}.$$

Similarly,

$$d_{2n} \le d_{2n-1} = M_1(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}.$$

That is,

$$d_{n+1} \le d_n,\ \ d_{2n} = M_1(x_{2n}, x_{2n+1}),\ \ d_{2n-1} = M_1(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}, \tag{2.9}$$

which means that $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. Obviously, there exists $c \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = c$. Suppose that $c > 0$. By (2.4), (2.9),

$(\varphi,\phi,\psi)\in\Phi_1\times\Phi_2\times\Phi_3$ and Lemma 1.6, we obtain that

$$\begin{aligned}
&\psi\bigg(\int_0^c\varphi(t)dt\bigg)\\
&\quad=\limsup_{n\to\infty}\psi\bigg(\int_0^{d_{2n+1}}\varphi(t)dt\bigg)\\
&\quad=\limsup_{n\to\infty}\psi\bigg(\int_0^{d(Tx_{2n},Sx_{2n+1})}\varphi(t)dt\bigg)\\
&\quad\le\limsup_{n\to\infty}\bigg[\psi\bigg(\int_0^{M_1(x_{2n},x_{2n+1})}\varphi(t)dt\bigg)-\phi\bigg(\int_0^{M_1(x_{2n},x_{2n+1})}\varphi(t)dt\bigg)\bigg]\\
&\quad=\limsup_{n\to\infty}\bigg[\psi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg)-\phi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg)\bigg]\\
&\quad\le\limsup_{n\to\infty}\psi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg)-\liminf_{n\to\infty}\phi\bigg(\int_0^{d_{2n}}\varphi(t)dt\bigg)\\
&\quad<\psi\bigg(\int_0^c\varphi(t)dt\bigg),
\end{aligned}$$

which is absurd. Hence $c=0$, which yields that

$$\lim_{n\to\infty}d_n=0. \tag{2.10}$$

Next we show that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. According to (2.10), we have to prove that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. It follows that there exists $\varepsilon>0$ such that for each $k\in\mathbb{N}$ there exist positive integers $2m(k)$ and $2n(k)$ with $2m(k)>2n(k)>2k$ satisfying

$$d(y_{2n(k)},y_{2m(k)})\ge\varepsilon, \tag{2.11}$$

where $2m(k)$ is the least integer exceeding $2n(k)$ satisfying (2.11). It follows that

$$d(y_{2n(k)},y_{2m(k)-2})<\varepsilon,\quad\forall k\in\mathbb{N}. \tag{2.12}$$

In view of (2.11), (2.12) and the triangle inequality, we know that

$$\begin{aligned}
\varepsilon&\le d(y_{2n(k)},y_{2m(k)})\\
&\le d(y_{2n(k)},y_{2m(k)-2})+d(y_{2m(k)-2},y_{2m(k)-1})+d(y_{2m(k)-1},y_{2m(k)})\\
&<\varepsilon+d_{2m(k)-2}+d_{2m(k)-1},\quad\forall k\in\mathbb{N}
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
&|d(y_{2n(k)},y_{2m(k)-1})-d(y_{2n(k)},y_{2m(k)})|\le d_{2m(k)-1},\quad\forall k\in\mathbb{N};\\
&|d(y_{2n(k)+1},y_{2m(k)})-d(y_{2n(k)},y_{2m(k)})|\le d_{2n(k)},\quad\forall k\in\mathbb{N};\\
&|d(y_{2n(k)+1},y_{2m(k)-1})-d(y_{2n(k)},y_{2m(k)-1})|\le d_{2n(k)},\quad\forall k\in\mathbb{N}.
\end{aligned}\tag{2.14}$$

Clearly, (2.10), (2.13) and (2.14) guarantee that

$$\begin{aligned}\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) &= \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1})\\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)})\\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1})\\ &= \varepsilon.\end{aligned} \tag{2.15}$$

In light of (2.4), (2.5), (2.15), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.6, we infer that

$$\begin{aligned}&M_1(x_{2n(k)}, x_{2m(k)-1})\\ &= \max\Big\{d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2n(k)}),\\ &\quad d(Bx_{2m(k)-1}, Sx_{2m(k)-1}),\\ &\quad \frac{1}{2}[d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Tx_{2n(k)}, Bx_{2m(k)-1})],\\ &\quad \frac{1+2d(Ax_{2n(k)}, Sx_{2m(k)-1})}{2(1+d(Ax_{2n(k)}, Bx_{2m(k)-1}))} d(Tx_{2n(k)}, Bx_{2m(k)-1}),\\ &\quad \frac{1+2d(Tx_{2n(k)}, Bx_{2m(k)-1})}{2(1+d(Ax_{2n(k)}, Bx_{2m(k)-1}))} d(Ax_{2n(k)}, Sx_{2m(k)-1}),\\ &\quad \frac{1+d(Ax_{2n(k)}, Sx_{2m(k)-1})}{1+2d(Ax_{2n(k)}, Bx_{2m(k)-1})} d(Ax_{2n(k)}, Tx_{2n(k)}),\\ &\quad \frac{1+d(Tx_{2n(k)}, Bx_{2m(k)-1})}{1+2d(Ax_{2n(k)}, Bx_{2m(k)-1})} d(Bx_{2m(k)-1}, Sx_{2m(k)-1})\Big\}\\ &= \max\Big\{d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)-1}, y_{2m(k)}),\\ &\quad \frac{1}{2}[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})],\\ &\quad \frac{1+2d(y_{2n(k)}, y_{2m(k)})}{2(1+d(y_{2n(k)}, y_{2m(k)-1}))} d(y_{2n(k)+1}, y_{2m(k)-1}),\\ &\quad \frac{1+2d(y_{2n(k)+1}, y_{2m(k)-1})}{2(1+d(y_{2n(k)}, y_{2m(k)-1}))} d(y_{2n(k)}, y_{2m(k)}),\\ &\quad \frac{1+d(y_{2n(k)}, y_{2m(k)})}{1+2d(y_{2n(k)}, y_{2m(k)-1})} d(y_{2n(k)}, y_{2n(k)+1}),\\ &\quad \frac{1+d(y_{2n(k)+1}, y_{2m(k)-1})}{1+2d(y_{2n(k)}, y_{2m(k)-1})} d(y_{2m(k)-1}, y_{2m(k)})\Big\}\end{aligned}$$

$$\begin{aligned}&\to \max\left\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon+\varepsilon), \frac{1+2\varepsilon}{2(1+\varepsilon)}\varepsilon, \frac{1+2\varepsilon}{2(1+\varepsilon)}\varepsilon, 0, 0\right\}\\ &= \varepsilon \quad \text{as } k\to\infty\end{aligned}$$

and

$$\begin{aligned}\psi\left(\int_0^{\varepsilon}\varphi(t)dt\right) &= \limsup_{k\to\infty}\psi\left(\int_0^{d(y_{2n(k)+1}, y_{2m(k)})}\varphi(t)dt\right)\\ &= \limsup_{k\to\infty}\psi\left(\int_0^{d(Tx_{2n(k)}, Sx_{2m(k)-1})}\varphi(t)dt\right)\\ &\le \limsup_{k\to\infty}\left[\psi\left(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})}\varphi(t)dt\right)\right.\\ &\quad\left. -\phi\left(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})}\varphi(t)dt\right)\right]\\ &\le \limsup_{k\to\infty}\psi\left(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})}\varphi(t)dt\right)\\ &\quad -\liminf_{k\to\infty}\phi\left(\int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})}\varphi(t)dt\right)\\ &< \psi\left(\int_0^{\varepsilon}\varphi(t)dt\right),\end{aligned}$$

which is impossible. Hence $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Now we show that A, B, S and T have a unique common fixed point. Assume that $A(X)$ is complete. It is clear that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Therefore, there exists $(z, w) \in A(X)\times X$ such that

$$\lim_{n\to\infty} y_{2n} = z = Aw.$$

Obviously

$$\begin{aligned}z &= \lim_{n\to\infty} y_n\\ &= \lim_{n\to\infty} Tx_{2n}\\ &= \lim_{n\to\infty} Bx_{2n+1}\\ &= \lim_{n\to\infty} Sx_{2n-1}\\ &= \lim_{n\to\infty} Ax_{2n}.\end{aligned} \tag{2.16}$$

Suppose that $Tw \neq z$. Notice that (2.4), (2.5), (2.16), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.6 yield that

$$
\begin{aligned}
& M_1(w, x_{2n+1}) \\
&= \max\Big\{ d(Aw, Bx_{2n+1}), d(Aw, Tw), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\qquad \frac{1}{2}[d(Aw, Sx_{2n+1}) + d(Tw, Bx_{2n+1})], \\
&\qquad \frac{1 + 2d(Aw, Sx_{2n+1})}{2(1 + d(Aw, Bx_{2n+1}))} d(Tw, Bx_{2n+1}), \\
&\qquad \frac{1 + 2d(Tw, Bx_{2n+1})}{2(1 + d(Aw, Bx_{2n+1}))} d(Aw, Sx_{2n+1}), \\
&\qquad \frac{1 + d(Aw, Sx_{2n+1})}{1 + 2d(Aw, Bx_{2n+1})} d(Aw, Tw), \\
&\qquad \frac{1 + d(Tw, Bx_{2n+1})}{1 + 2d(Aw, Bx_{2n+1})} d(Bx_{2n+1}, Sx_{2n+1}) \Big\} \\
&\to \max\Big\{ d(Aw, z), d(Aw, Tw), d(z, z), \frac{1}{2}[d(Aw, z) + d(Tw, z)], \\
&\qquad \frac{1 + 2d(Aw, z)}{2(1 + d(Aw, z))} d(Tw, z), \frac{1 + 2d(Tw, z)}{2(1 + d(Aw, z))} d(Aw, z), \\
&\qquad \frac{1 + d(Aw, z)}{1 + 2d(Aw, z)} d(Aw, Tw), \frac{1 + d(Tw, z)}{1 + 2d(Aw, z)} d(z, z) \Big\} \\
&= \max\Big\{ 0, d(z, Tw), 0, \frac{1}{2} d(Tw, z), \frac{1}{2} d(Tw, z), 0, d(z, Tw), 0 \Big\} \\
&= d(Tw, z) \quad \text{as } n \to \infty
\end{aligned}
$$

and

$$
\begin{aligned}
& \psi\Big(\int_0^{d(Tw, z)} \varphi(t)dt \Big) \\
&\quad = \limsup_{n\to\infty} \psi\Big(\int_0^{d(Tw, Sx_{2n+1})} \varphi(t)dt \Big) \\
&\quad \leq \limsup_{n\to\infty} \Big[\psi\Big(\int_0^{M_1(w, x_{2n+1})} \varphi(t)dt \Big) - \phi\Big(\int_0^{M_1(w, x_{2n+1})} \varphi(t)dt \Big) \Big] \\
&\quad \leq \limsup_{n\to\infty} \psi\Big(\int_0^{M_1(w, x_{2n+1})} \varphi(t)dt \Big) - \liminf_{n\to\infty} \phi\Big(\int_0^{M_1(w, x_{2n+1})} \varphi(t)dt \Big) \\
&\quad < \psi\Big(\int_0^{d(Tw, z)} \varphi(t)dt \Big),
\end{aligned}
$$

which is absurd. Thus, $Tw = z$. It follows from (2.2) that there exists a point $u \in X$ with $z = Bu = Tw$.

Suppose that $Su \neq z$. Making use of (2.4), (2.5), (2.16), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.6, we conclude that

$$
\begin{aligned}
& M_1(x_{2n}, u) \\
&= \max\Big\{ d(Ax_{2n}, Bu), d(Ax_{2n}, Tx_{2n}), d(Bu, Su), \\
&\qquad \frac{1}{2}[d(Ax_{2n}, Su) + d(Tx_{2n}, Bu)], \\
&\qquad \frac{1 + 2d(Ax_{2n}, Su)}{2(1 + d(Ax_{2n}, Bu))} d(Tx_{2n}, Bu), \frac{1 + 2d(Tx_{2n}, Bu)}{2(1 + d(Ax_{2n}, Bu))} d(Ax_{2n}, Su), \\
&\qquad \frac{1 + d(Ax_{2n}, Su)}{1 + 2d(Ax_{2n}, Bu)} d(Ax_{2n}, Tx_{2n}), \frac{1 + d(Tx_{2n}, Bu)}{1 + 2d(Ax_{2n}, Bu)} d(Bu, Su) \Big\} \\
&\to \max\Big\{ d(z, Bu), d(z, z), d(Bu, Su), \frac{1}{2}[d(z, Su) + d(z, Bu)], \\
&\qquad \frac{1 + 2d(z, Su)}{2(1 + d(z, Bu))} d(z, Bu), \frac{1 + 2d(z, Bu)}{2(1 + d(z, Bu))} d(z, Su), \\
&\qquad \frac{1 + d(z, Su)}{1 + 2d(z, Bu)} d(z, z), \frac{1 + d(z, Bu)}{1 + 2d(z, Bu)} d(Bu, Su) \Big\} \\
&= \max\Big\{ 0, 0, d(z, Su), \frac{1}{2} d(z, Su), 0, \frac{1}{2} d(z, Su), 0, d(z, Su) \Big\} \\
&= d(z, Su) \quad \text{as } n \to \infty
\end{aligned}
$$

and

$$
\begin{aligned}
& \psi\Big(\int_0^{d(z,Su)} \varphi(t)dt \Big) \\
&\qquad = \limsup_{n\to\infty} \psi\Big(\int_0^{d(Tx_{2n},Su)} \varphi(t)dt \Big) \\
&\qquad \le \limsup_{n\to\infty} \Big[\psi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt \Big) - \phi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt \Big) \Big] \\
&\qquad \le \limsup_{n\to\infty} \psi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt \Big) - \liminf_{n\to\infty} \phi\Big(\int_0^{M_1(x_{2n},u)} \varphi(t)dt \Big) \\
&\qquad < \psi\Big(\int_0^{d(z,Su)} \varphi(t)dt \Big),
\end{aligned}
$$

which is impossible. That is, $Su = z$. It follows from (2.1) that $Az = ATw = TAw = Tz$ and $Bz = BSu = SBu = Sz$.

Suppose that $Tz \neq Sz$. In light of (2.4), (2.5), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we know that

$$\begin{aligned} M_1(z,z) &= \max\Big\{ d(Az,Bz), d(Az,Tz), d(Bz,Sz), \\ &\quad \frac{1}{2}[d(Az,Sz)+d(Tz,Bz)], \\ &\quad \frac{1+2d(Az,Sz)}{2(1+d(Az,Bz))} d(Tz,Bz), \\ &\quad \frac{1+2d(Tz,Bz)}{2(1+d(Az,Bz))} d(Az,Sz), \\ &\quad \frac{1+d(Az,Sz)}{1+2d(Az,Bz)} d(Az,Tz), \\ &\quad \frac{1+d(Tz,Bz)}{1+2d(Az,Bz)} d(Bz,Sz) \Big\} \\ &= \max\Big\{ d(Tz,Sz), 0, 0, \frac{1}{2}[d(Tz,Sz)+d(Tz,Sz)], \\ &\quad \frac{1+2d(Tz,Sz)}{2(1+d(Tz,Sz))} d(Tz,Sz), \\ &\quad \frac{1+2d(Tz,Sz)}{2(1+d(Tz,Sz))} d(Tz,Sz), 0, 0 \Big\} \\ &= d(Tz,Sz) \end{aligned}$$

and

$$\begin{aligned} &\psi\Big(\int_0^{d(Tz,Sz)} \varphi(t)dt\Big) \\ &\quad \le \psi\Big(\int_0^{M_1(z,z)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(z,z)} \varphi(t)dt\Big) \\ &\quad = \psi\Big(\int_0^{d(Tz,Sz)} \varphi(t)dt\Big) - \phi\Big(\int_0^{d(Tz,Sz)} \varphi(t)dt\Big) \\ &\quad < \psi\Big(\int_0^{d(Tz,Sz)} \varphi(t)dt\Big), \end{aligned}$$

which is absurd. Thus, $Tz = Sz$.

Suppose that $Tz \neq z$. According to (2.4), (2.5), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we have

$$\begin{aligned}
&M_1(z,u)\\
&= \max\Big\{d(Az,Bu), d(Az,Tz), d(Bu,Su),\\
&\qquad \frac{1}{2}[d(Az,Su)+d(Tz,Bu)],\\
&\qquad \frac{1+2d(Az,Su)}{2(1+d(Az,Bu))}d(Tz,Bu), \frac{1+2d(Tz,Bu)}{2(1+d(Az,Bu))}d(Az,Su),\\
&\qquad \frac{1+d(Az,Su)}{1+2d(Az,Bu)}d(Az,Tz), \frac{1+d(Tz,Bu)}{1+2d(Az,Bu)}d(Bu,Su)\Big\}\\
&= \max\Big\{d(Tz,z), 0, 0, \frac{1}{2}[d(Tz,z)+d(Tz,z)],\\
&\qquad \frac{1+2d(Tz,z)}{2(1+d(Tz,z))}d(Tz,z),\\
&\qquad \frac{1+2d(Tz,z)}{2(1+d(Tz,z))}d(Tz,z), 0, 0\Big\}\\
&= d(Tz,z)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big)\\
&\quad = \psi\Big(\int_0^{d(Tz,Su)}\varphi(t)dt\Big)\\
&\quad \le \psi\Big(\int_0^{M_1(z,u)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(z,u)}\varphi(t)dt\Big)\\
&\quad = \psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big)\\
&\quad < \psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big),
\end{aligned}$$

which is ridiculous. Therefore, $Tz = z$, that is, z is a common fixed point of A, B, S and T.

Suppose that A, B, S and T have a common fixed point $b \in X \setminus \{z\}$. Taking account of (2.4), (2.5), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we attain that

$$\begin{aligned}
&M_1(b,z)\\
&= \max\Big\{d(Ab,Bz), d(Ab,Tb), d(Bz,Sz), \frac{1}{2}[d(Ab,Sz)+d(Tb,Bz)],\\
&\qquad \frac{1+2d(Ab,Sz)}{2(1+d(Ab,Bz))}d(Tb,Bz), \frac{1+2d(Tb,Bz)}{2(1+d(Ab,Bz))}d(Ab,Sz),\\
&\qquad \frac{1+d(Ab,Sz)}{1+2d(Ab,Bz)}d(Ab,Tb), \frac{1+d(Tb,Bz)}{1+2d(Ab,Bz)}d(Bz,Sz)\Big\}\\
&= \max\Big\{d(b,z), 0, 0, \frac{1}{2}[d(b,z)+d(b,z)], \frac{1+2d(b,z)}{2(1+d(b,z))}d(b,z),\\
&\qquad \frac{1+2d(b,z)}{2(1+d(b,z))}d(b,z), 0, 0\Big\}\\
&= d(b,z)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)\\
&\quad = \psi\Big(\int_0^{d(Tb,Sz)}\varphi(t)dt\Big)\\
&\quad \le \psi\Big(\int_0^{M_1(b,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(b,z)}\varphi(t)dt\Big)\\
&\quad = \psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big) - \phi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)\\
&\quad < \psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big),
\end{aligned}$$

which is a contradiction. Hence A, B, S and T have a unique common fixed point in X.

Similarly we infer that A, B, S and T have a unique common fixed point in X if one of $B(X), S(X)$ and $T(X)$ is complete. This completes the proof. □

Theorem 2.2. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying* (2.1)-(2.3) *and for all $x, y \in X$,*

$$\psi\Big(\int_0^{d(Tx,Sy)}\varphi(t)dt\Big) \le \psi\Big(\int_0^{M_2(x,y)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_2(x,y)}\varphi(t)dt\Big), \quad (2.17)$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and

$$
\begin{aligned}
&M_2(x,y)\\
&=\Big\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy)+d(Tx,By)],\\
&\quad \frac{1+d(By,Sy)}{1+d(Ax,Tx)}d(Ax,By), \frac{1+d(Ax,Tx)}{1+d(By,Sy)}d(Ax,By),\\
&\quad \frac{d^2(Ax,By)}{1+d(Tx,Sy)}, \frac{d(Ax,Sy)d(Tx,By)}{1+d(Tx,Sy)}\Big\}, \quad \forall x,y \in X.
\end{aligned}
\tag{2.18}
$$

Then A, B, S and T have a unique common fixed point in X.

Proof. Let x be an arbitrary point in X. It follows from (2.2) that there exist two sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}_0}$ in X satisfying (2.6). Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. Suppose that $d_{2n} < d_{2n+1}$ for some $n \in \mathbb{N}$. It is easy to see that

$$
d_{2n+1} - \frac{d_{2n}^2}{1+d_{2n+1}} = \frac{d_{2n+1}+d_{2n+1}^2-d_{2n}^2}{1+d_{2n+1}} > 0,
\tag{2.19}
$$

$$
d_{2n+1} - \frac{1+d_{2n+1}}{1+d_{2n}}d_{2n} = \frac{d_{2n+1}-d_{2n}}{1+d_{2n}} > 0.
\tag{2.20}
$$

By (2.6), (2.17)-(2.20), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we infer that

$$
\begin{aligned}
&M_2(x_{2n}, x_{2n+1})\\
&=\max\Big\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n+1}, Sx_{2n+1}),\\
&\quad \frac{1}{2}[d(Ax_{2n}, Sx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})]\\
&\quad \frac{1+d(Bx_{2n+1}, Sx_{2n+1})}{1+d(Ax_{2n}, Tx_{2n})}d(Ax_{2n}, Bx_{2n+1}),\\
&\quad \frac{1+d(Ax_{2n}, Tx_{2n})}{1+d(Bx_{2n+1}, Sx_{2n+1})}d(Ax_{2n}, Bx_{2n+1}),\\
&\quad \frac{d^2(Ax_{2n}, Bx_{2n+1})}{1+d(Tx_{2n}, Sx_{2n+1})}, \frac{d(Ax_{2n}, Sx_{2n+1})d(Tx_{2n}, Bx_{2n+1})}{1+d(Tx_{2n}, Sx_{2n+1})}\Big\}
\end{aligned}
$$

$$\begin{aligned}
&= \max\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}),\\
&\quad \frac{1}{2}[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})],\\
&\quad \frac{1+d(y_{2n+1}, y_{2n+2})}{1+d(y_{2n}, y_{2n+1})} d(y_{2n}, y_{2n+1}),\\
&\quad \frac{1+d(y_{2n}, y_{2n+1})}{1+d(y_{2n+1}, y_{2n+2})} d(y_{2n}, y_{2n+1}),\\
&\quad \frac{d^2(y_{2n}, y_{2n+1})}{1+d(y_{2n+1}, y_{2n+2})}, \frac{d(y_{2n}, y_{2n+2})d(y_{2n+1}, y_{2n+1})}{1+d(y_{2n+1}, y_{2n+2})}\Big\}\\
&= \max\Big\{d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2}d(y_{2n}, y_{2n+2}), \frac{1+d_{2n+1}}{1+d_{2n}} d_{2n},\\
&\quad \frac{1+d_{2n}}{1+d_{2n+1}} d_{2n}, \frac{d_{2n}^2}{1+d_{2n+1}}, 0\Big\}\\
&= \max\{d_{2n}, d_{2n+1}\} = d_{2n+1}
\end{aligned}$$

and

$$\begin{aligned}
&\psi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg)\\
&\quad = \psi\bigg(\int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt\bigg)\\
&\quad \le \psi\bigg(\int_0^{M_2(x_{2n}, x_{2n+1})} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_2(x_{2n}, x_{2n+1})} \varphi(t)dt\bigg)\\
&\quad = \psi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg)\\
&\quad < \psi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg),
\end{aligned}$$

which is a contradiction. Hence

$$d_{2n+1} \le d_{2n} = M_2(x_{2n}, x_{2n+1}), \quad \forall n \in \mathbb{N}.$$

In a similar way, we get that

$$d_{2n} \le d_{2n-1} = M_2(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}.$$

That is,

$$\begin{aligned}
d_{n+1} \le d_n, \quad d_{2n} = M_2(x_{2n}, x_{2n+1}),\\
d_{2n-1} = M_2(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N},
\end{aligned} \tag{2.21}$$

which implies that $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. Clearly, there exists $c \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = c$. Suppose that $c > 0$. It follows from (2.17), (2.21), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.6 that

$$\begin{aligned}
&\psi\bigg(\int_0^c \varphi(t)dt\bigg)\\
&\quad = \limsup_{n\to\infty} \psi\bigg(\int_0^{d_{2n+1}} \varphi(t)dt\bigg)\\
&\quad = \limsup_{n\to\infty} \psi\bigg(\int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt\bigg)\\
&\quad \le \limsup_{n\to\infty} \bigg[\psi\bigg(\int_0^{M_2(x_{2n}, x_{2n+1})} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M_2(x_{2n}, x_{2n+1})} \varphi(t)dt\bigg)\bigg]\\
&\quad = \limsup_{n\to\infty} \bigg[\psi\bigg(\int_0^{d_{2n}} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{d_{2n}} \varphi(t)dt\bigg)\bigg]\\
&\quad \le \limsup_{n\to\infty} \psi\bigg(\int_0^{d_{2n}} \varphi(t)dt\bigg) - \liminf_{n\to\infty} \phi\bigg(\int_0^{d_{2n}} \varphi(t)dt\bigg)\\
&\quad < \psi\bigg(\int_0^c \varphi(t)dt\bigg),
\end{aligned}$$

which is a contradiction. Hence $c = 0$, that is, (2.10) holds.

Next we show that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By virtue of (2.10), we only have to prove that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. It follows that there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there exist positive integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ satisfying (2.11)-(2.15). By means of (2.15), (2.17), (2.18), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.6, we conclude that

$$\begin{aligned}
&M_2(x_{2n(k)}, x_{2m(k)-1})\\
&= \max\bigg\{d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2n(k)}), d(Bx_{2m(k)-1}, Sx_{2m(k)-1}),\\
&\qquad \frac{1}{2}[d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Tx_{2n(k)}, Bx_{2m(k)-1})],\\
&\qquad \frac{1 + d(Bx_{2m(k)-1}, Sx_{2m(k)-1})}{1 + d(Ax_{2n(k)}, Tx_{2n(k)})} d(Ax_{2n(k)}, Bx_{2m(k)-1}),\\
&\qquad \frac{1 + d(Ax_{2n(k)}, Tx_{2n(k)})}{1 + d(Bx_{2m(k)-1}, Sx_{2m(k)-1})} d(Ax_{2n(k)}, Bx_{2m(k)-1}),\\
&\qquad \frac{d^2(Ax_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Tx_{2n(k)}, Sx_{2m(k)-1})},\\
&\qquad \frac{d(Ax_{2n(k)}, Sx_{2m(k)-1}) d(Tx_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Tx_{2n(k)}, Sx_{2m(k)-1})}\bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{ d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)-1}, y_{2m(k)}),\\
&\qquad \frac{1}{2}[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})],\\
&\qquad \frac{1+d(y_{2m(k)-1}, y_{2m(k)})}{1+d(y_{2n(k)}, y_{2n(k)+1})} d(y_{2n(k)}, y_{2m(k)-1}),\\
&\qquad \frac{1+d(y_{2n(k)}, y_{2n(k)+1})}{1+d(y_{2m(k)-1}, y_{2m(k)})} d(y_{2n(k)}, y_{2m(k)-1}),\\
&\qquad \frac{d^2(y_{2n(k)}, y_{2m(k)-1})}{1+d(y_{2n(k)+1}, y_{2m(k)})}, \frac{d(y_{2n(k)}, y_{2m(k)})d(y_{2n(k)+1}, y_{2m(k)-1})}{1+d(y_{2n(k)+1}, y_{2m(k)})}\Big\}\\
&\to \max\Big\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon+\varepsilon), \varepsilon, \varepsilon, \frac{\varepsilon^2}{1+\varepsilon}, \frac{\varepsilon^2}{1+\varepsilon}\Big\}\\
&= \varepsilon \quad \text{as } k \to \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi\Big(\int_0^{\varepsilon} \varphi(t)dt\Big)\\
&\quad = \limsup_{k\to\infty} \psi\Big(\int_0^{d(y_{2n(k)+1}, y_{2m(k)})} \varphi(t)dt\Big)\\
&\quad = \limsup_{k\to\infty} \psi\Big(\int_0^{d(Tx_{2n(k)}, Sx_{2m(k)-1})} \varphi(t)dt\Big)\\
&\quad \le \limsup_{k\to\infty} \Big[\psi\Big(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt\Big)\\
&\qquad - \phi\Big(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt\Big)\Big]\\
&\quad \le \limsup_{k\to\infty} \psi\Big(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt\Big)\\
&\qquad - \liminf_{k\to\infty} \phi\Big(\int_0^{M_2(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt\Big)\\
&\quad < \psi\Big(\int_0^{\varepsilon} \varphi(t)dt\Big),
\end{aligned}
$$

which is absurd. Therefore, $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Now we show that A, B, S and T have a unique common fixed point. Assume that $A(X)$ is complete. It is clear that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Therefore, there exists $(z, w) \in A(X) \times X$ with $\lim_{n\to\infty} y_{2n} = z = Aw$. It is obvious that (2.16) holds. Suppose that $Tw \ne z$. It follows from (2.16)-(2.18),

$(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.6 that

$$\begin{aligned}
& M_2(w, x_{2n+1}) \\
&= \max\Big\{ d(Aw, Bx_{2n+1}), d(Aw, Tw), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\qquad \frac{1}{2}[d(Aw, Sx_{2n+1}) + d(Tw, Bx_{2n+1})], \\
&\qquad \frac{1 + d(Bx_{2n+1}, Sx_{2n+1})}{1 + d(Aw, Tw)} d(Aw, Bx_{2n+1}), \\
&\qquad \frac{1 + d(Aw, Tw)}{1 + d(Bx_{2n+1}, Sx_{2n+1})} d(Aw, Bx_{2n+1}), \\
&\qquad \frac{d^2(Aw, Bx_{2n+1})}{1 + d(Tw, Sx_{2n+1})}, \frac{d(Aw, Sx_{2n+1}) d(Tw, Bx_{2n+1})}{1 + d(Tw, Sx_{2n+1})} \Big\} \\
&\to \max\Big\{ d(Aw, z), d(Aw, Tw), d(z, z), \frac{1}{2}[d(Aw, z) + d(Tw, z)], \\
&\qquad \frac{1 + d(z, z)}{1 + d(Aw, Tw)} d(Aw, z), \frac{1 + d(Aw, Tw)}{1 + d(z, z)} d(Aw, z), \\
&\qquad \frac{d^2(Aw, z)}{1 + d(Tw, z)}, \frac{d(Aw, z) d(Tw, z)}{1 + d(Tw, z)} \Big\} \\
&= \max\Big\{ 0, d(z, Tw), 0, \frac{1}{2} d(Tw, z), 0, 0, 0, 0 \Big\} \\
&= d(Tw, z) \quad \text{as } n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
& \psi\Big(\int_0^{d(Tw, z)} \varphi(t) dt \Big) \\
&\quad = \limsup_{n\to\infty} \psi\Big(\int_0^{d(Tw, Sx_{2n+1})} \varphi(t) dt \Big) \\
&\quad \le \limsup_{n\to\infty} \Big[\psi\Big(\int_0^{M_2(w, x_{2n+1})} \varphi(t) dt \Big) - \phi\Big(\int_0^{M_2(w, x_{2n+1})} \varphi(t) dt \Big) \Big] \\
&\quad \le \limsup_{n\to\infty} \psi\Big(\int_0^{M_2(w, x_{2n+1})} \varphi(t) dt \Big) \\
&\qquad - \liminf_{n\to\infty} \phi\Big(\int_0^{M_2(w, x_{2n+1})} \varphi(t) dt \Big) \\
&\quad < \psi\Big(\int_0^{d(Tw, z)} \varphi(t) dt \Big),
\end{aligned}$$

which is impossible. Thus, $Tw = z$. It follows from (2.2) that there exists a point $u \in X$ with $z = Bu = Tw$. Suppose that $Su \neq z$. Taking account of (2.16)-(2.18), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.6, we derive that

$$\begin{aligned}
&M_2(x_{2n}, u)\\
&= \max\Big\{d(Ax_{2n}, Bu), d(Ax_{2n}, Tx_{2n}), d(Bu, Su),\\
&\qquad \frac{1}{2}[d(Ax_{2n}, Su) + d(Tx_{2n}, Bu)], \frac{1 + d(Bu, Su)}{1 + d(Ax_{2n}, Tx_{2n})} d(Ax_{2n}, Bu),\\
&\qquad \frac{1 + d(Ax_{2n}, Tx_{2n})}{1 + d(Bu, Su)} d(Ax_{2n}, Bu), \frac{d^2(Ax_{2n}, Bu)}{1 + d(Tx_{2n}, Su)},\\
&\qquad \frac{d(Ax_{2n}, Su) d(Tx_{2n}, Bu)}{1 + d(Tx_{2n}, Su)}\Big\}\\
&\to \max\Big\{d(z, Bu), d(z, z), d(Bu, Su), \frac{1}{2}[d(z, Su) + d(z, Bu)],\\
&\qquad \frac{1 + d(Bu, Su)}{1 + d(z, z)} d(z, Bu), \frac{1 + d(z, z)}{1 + d(Bu, Su)} d(z, Bu),\\
&\qquad \frac{d^2(z, Bu)}{1 + d(z, Su)}, \frac{d(z, Su) d(z, Bu)}{1 + d(z, Su)}\Big\}\\
&= \max\Big\{0, 0, d(z, Su), \frac{1}{2} d(z, Su), 0, 0, 0, 0\Big\}\\
&= d(z, Su) \quad \text{as } n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(z,Su)} \varphi(t)dt\Big)\\
&\quad = \limsup_{n\to\infty} \psi\Big(\int_0^{d(Tx_{2n},Su)} \varphi(t)dt\Big)\\
&\quad \leq \limsup_{n\to\infty} \Big[\psi\Big(\int_0^{M_2(x_{2n},u)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_2(x_{2n},u)} \varphi(t)dt\Big)\Big]\\
&\quad \leq \limsup_{n\to\infty} \psi\Big(\int_0^{M_2(x_{2n},u)} \varphi(t)dt\Big) - \liminf_{n\to\infty} \phi\Big(\int_0^{M_2(x_{2n},u)} \varphi(t)dt\Big)\\
&\quad < \psi\Big(\int_0^{d(z,Su)} \varphi(t)dt\Big),
\end{aligned}$$

which is a contradiction. Hence $Su = z$. It follows from (2.1) that $Az = ATw = TAw = Tz$ and $Bz = BSu = SBu = Sz$. Suppose that $Tz \neq Sz$. In light of (2.17), (2.18), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we know that

$$\begin{aligned}
M_2(z, z) &= \max\Big\{ d(Az, Bz), d(Az, Tz), d(Bz, Sz), \frac{1}{2}[d(Az, Sz) + d(Tz, Bz)], \\
&\qquad \frac{1 + d(Bz, Sz)}{1 + d(Az, Tz)} d(Az, Bz), \frac{1 + d(Az, Tz)}{1 + d(Bz, Sz)} d(Az, Bz), \\
&\qquad \frac{d^2(Az, Bz)}{1 + d(Tz, Bz)}, \frac{d(Az, Sz) d(Tz, Bz)}{1 + d(Tz, Bz)} \Big\} \\
&= \max\Big\{ d(Tz, Sz), 0, 0, \frac{1}{2}[d(Tz, Sz) + d(Tz, Sz)], d(Tz, Sz), \\
&\qquad d(Tz, Sz), \frac{d^2(Tz, Sz)}{1 + d(Tz, Sz)}, \frac{d^2(Tz, Sz)}{1 + d(Tz, Sz)} \Big\} \\
&= d(Tz, Sz)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(Tz,Sz)} \varphi(t) dt \Big) \\
&\quad \le \psi\Big(\int_0^{M_2(z,z)} \varphi(t) dt \Big) - \phi\Big(\int_0^{M_2(z,z)} \varphi(t) dt \Big) \\
&\quad = \psi\Big(\int_0^{d(Tz,Sz)} \varphi(t) dt \Big) - \phi\Big(\int_0^{d(Tz,Sz)} \varphi(t) dt \Big) \\
&\quad < \psi\Big(\int_0^{d(Tz,Sz)} \varphi(t) dt \Big),
\end{aligned}$$

which is absurd. Hence $Tz = Sz$.

Suppose that $Tz \ne z$. Taking account of (2.17), (2.18), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we have

$$\begin{aligned}
M_2(z, u) &= \max\Big\{ d(Az, Bu), d(Az, Tz), d(Bu, Su), \frac{1}{2}[d(Az, Su) + d(Tz, Bu)], \\
&\qquad \frac{1 + d(Bu, Su)}{1 + d(Az, Tz)} d(Az, Bu), \frac{1 + d(Az, Tz)}{1 + d(Bu, Su)} d(Az, Bu), \\
&\qquad \frac{d^2(Az, Bu)}{1 + d(Tz, Su)}, \frac{d(Az, Su) d(Tz, Bu)}{1 + d(Tz, Su)} \Big\} \\
&= \max\Big\{ d(Tz, z), 0, 0, \frac{1}{2}[d(Tz, z) + d(Tz, z)], d(Tz, z), \\
&\qquad d(Tz, z), \frac{d^2(Tz, z)}{1 + d(Tz, z)}, \frac{d^2(Tz, z)}{1 + d(Tz, z)} \Big\} \\
&= d(Tz, z)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big)\\
&\quad=\psi\Big(\int_0^{d(Tz,Su)}\varphi(t)dt\Big)\\
&\quad\le\psi\Big(\int_0^{M_2(z,u)}\varphi(t)dt\Big)-\phi\Big(\int_0^{M_2(z,u)}\varphi(t)dt\Big)\\
&\quad=\psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big)-\phi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big)\\
&\quad<\psi\Big(\int_0^{d(Tz,z)}\varphi(t)dt\Big),
\end{aligned}$$

which is a contradiction. Therefore, $Tz = z$, that is, z is a common fixed point of A, B, S and T.

Suppose that A, B, S and T have a common fixed point $b \in X \setminus \{z\}$. Using (2.17), (2.18), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.7, we have

$$\begin{aligned}
M_2(b,z) &= \max\Big\{d(Ab,Bz), d(Ab,Tb), d(Bz,Sz), \frac{1}{2}[d(Ab,Sz)+d(Tb,Bz)],\\
&\qquad \frac{1+d(Bz,Sz)}{1+d(Ab,Tb)}d(Ab,Bz), \frac{1+d(Ab,Tb)}{1+d(Bz,Sz)}d(Ab,Bz),\\
&\qquad \frac{d^2(Ab,Bz)}{1+d(Tb,Sz)}, \frac{d(Ab,Sz)d(Tb,Bz)}{1+d(Tb,Sz)}\Big\}\\
&= \max\Big\{d(b,z), 0, 0, \frac{1}{2}[d(b,z)+d(b,z)], d(b,z), d(b,z),\\
&\qquad \frac{d^2(b,z)}{1+d(b,z)}, \frac{d^2(b,z)}{1+d(b,z)}\Big\}\\
&= d(b,z)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)=\psi\Big(\int_0^{d(Tb,Sz)}\varphi(t)dt\Big)\\
&\quad\le\psi\Big(\int_0^{M_2(b,z)}\varphi(t)dt\Big)-\phi\Big(\int_0^{M_2(b,z)}\varphi(t)dt\Big)\\
&\quad=\psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)-\phi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big)\\
&\quad<\psi\Big(\int_0^{d(b,z)}\varphi(t)dt\Big),
\end{aligned}$$

which is a contradiction. Hence z is a unique common fixed point of A, B, S and T in X.

Similarly we conclude that A, B, S and T have a unique common fixed point in X if one of $B(X), S(X)$ and $T(X)$ is complete. This completes the proof. □

3. Remark and examples

Remark 3.1. Theorems 2.1 and 2.2 extend Theorems 1.1 and 1.2, respectively. The follows examples reveal that Theorems 2.1 and 2.2 generalize proper Theorems 1.1 and 1.2, and differ from Theorems 1.3 and 1.4, respectively.

Example 3.2. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$. Define $A, B, S, T : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Ax = 16x, \quad Bx = x, \quad Sx = 0, \quad \forall x \in X, \quad Tx = \begin{cases} 0, & \forall x \in X \setminus \{\frac{1}{32}\}, \\ \frac{1}{64}, & x = \frac{1}{32}, \end{cases}$$

$$\psi(t) = \begin{cases} t, & \forall t \in [0, \frac{1}{64}], \\ 2t - \frac{1}{64}, & \forall t \in (\frac{1}{64}, +\infty), \end{cases} \quad \phi(t) = \begin{cases} \frac{1}{4}t, & \forall t \in [0, \frac{1}{64}], \\ \frac{1}{128}, & \forall t \in (\frac{1}{64}, +\infty), \end{cases}$$

$$\varphi(t) = 1, \quad \forall t \in \mathbb{R}^+.$$

Clearly, (2.1)-(2.3) hold, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, $\psi(t) \geq \phi(t)$ for each $t \in \mathbb{R}^+$, ψ is increasing and $\sup \phi(\mathbb{R}^+) \leq \frac{1}{128}$. Let $x, y \in X$. In order to prove (2.4), we need to consider two possible cases as follows:

Case 1. $x \in X \setminus \{\frac{1}{32}\}$. It follows that

$$\psi\Big(\int_0^{d(Tx,Sy)} \varphi(t)dt\Big) = 0 \leq \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big);$$

Case 2. $x = \frac{1}{32}$. Notice that

$$M_1(x, y) \geq d\Big(A\frac{1}{32}, T\frac{1}{32}\Big) = \Big|\frac{1}{2} - \frac{1}{64}\Big| = \frac{31}{64}$$

and

$$\begin{aligned} \psi\Big(\int_0^{d(Tx,Sy)} \varphi(t)dt\Big) &= \psi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big) = \psi\Big(\frac{1}{64}\Big) = \frac{1}{64} < \frac{121}{128} \\ &= \frac{61}{64} - \frac{1}{128} = \psi\Big(\int_0^{\frac{31}{64}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{31}{64}} \varphi(t)dt\Big) \\ &\leq \psi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_1(x,y)} \varphi(t)dt\Big). \end{aligned}$$

That is, (2.4) holds. It follows from Theorem 2.1 that the mappings A, B, S and T have a unique common fixed point $0 \in X$. But Theorems 1.1-1.4 cannot be used to prove the existence of fixed points of T and common fixed points of T and S in X.

Suppose that T satisfies the conditions of Theorem 1.1. That is, there exist ϕ and $\psi \in \Phi_5$ satisfying (1.1). It follows from (1.1) that

$$\begin{aligned}\psi\Big(\frac{1}{64}\Big) &= \psi\Big(d\Big(T\frac{1}{32}, T\frac{1}{64}\Big)\Big)\\ &\le \psi\Big(d\Big(\frac{1}{32}, \frac{1}{64}\Big)\Big) - \phi\Big(d\Big(\frac{1}{32}, \frac{1}{64}\Big)\Big)\\ &= \psi\Big(\frac{1}{64}\Big) - \phi\Big(\frac{1}{64}\Big) < \psi\Big(\frac{1}{64}\Big),\end{aligned}$$

which is absurd.

Suppose that T satisfies the conditions of Theorem 1.2. That is, there exist $c \in (0,1)$ and $\varphi \in \Phi_1$ satisfying (1.2). By (1.2), we get that

$$0 < \int_0^{\frac{1}{64}} \varphi(t)dt = \int_0^{d(T\frac{1}{32}, T\frac{1}{64})} \varphi(t)dt \le c\int_0^{d(\frac{1}{32}, \frac{1}{64})} \varphi(t)dt < \int_0^{\frac{1}{64}} \varphi(t)dt,$$

which is a contradiction.

Suppose that T satisfies the conditions of Theorem 1.3. That is, there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_5$ satisfying (1.3). Using (1.3), we gain that

$$\begin{aligned}&\psi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{d(T\frac{1}{32}, T\frac{1}{64})} \varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{d(\frac{1}{32}, \frac{1}{64})} \varphi(t)dt\Big) - \phi\Big(\int_0^{d(\frac{1}{32}, \frac{1}{64})} \varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big),\end{aligned}$$

which is impossible.

Suppose that T and S satisfy the conditions of Theorem 1.4. That is, there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_4 \times \Phi_5$ satisfying (1.4). It follows from (1.4) that

$$\begin{aligned}
&\psi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big) \\
&= \psi\Big(\int_0^{d(T\frac{1}{32}, S\frac{1}{64})} \varphi(t)dt\Big) \\
&\le \psi\Big(\int_0^{M(\frac{1}{32}, \frac{1}{64})} \varphi(t)dt\Big) - \phi\Big(\int_0^{M(\frac{1}{32}, \frac{1}{64})} \varphi(t)dt\Big) \\
&= \psi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big) \\
&< \psi\Big(\int_0^{\frac{1}{64}} \varphi(t)dt\Big),
\end{aligned}$$

which is absurd.

Example 3.3. Let $X = [0, 1]$ be endowed with the Euclidean metric $d = |\cdot|$. Define $A, B, S, T : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$Ax = 3x - \frac{5}{6}, \quad Bx = x, \quad Sx = \frac{5}{12}, \quad \forall x \in X, \quad Tx = \begin{cases} \frac{5}{12}, & \forall x \in [0, \frac{1}{2}], \\ \frac{1}{2}, & \forall x \in (\frac{1}{2}, 1], \end{cases}$$

$$\psi(t) = 100t, \quad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+, \quad \phi(t) = \begin{cases} t, & \forall t \in [0, \frac{1}{36}), \\ 1, & \forall t \in [\frac{1}{36}, +\infty). \end{cases}$$

Obviously, (2.1)-(2.3) hold, $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, $\psi(t) \ge \phi(t)$ for each $t \in \mathbb{R}^+$, ψ is increasing and $\sup \phi(\mathbb{R}^+) \le 1$. Let $x, y \in X$. In order to prove (2.17), we have to consider two possible cases as follows:

Case 1. $x \in [0, \frac{1}{2}]$. It is clear that

$$\psi\Big(\int_0^{d(Tx,Sy)} \varphi(t)dt\Big) = 0 \le \psi\Big(\int_0^{M_2(x,y)} \varphi(t)dt\Big) - \phi\Big(\int_0^{M_2(x,y)} \varphi(t)dt\Big);$$

Case 2. $x \in (\frac{1}{2}, 1]$. Note that

$$M_2(x, y) \ge d(Ax, Tx) = 3x - \frac{5}{6} - \frac{1}{2} \ge \frac{1}{6}$$

and

$$\begin{aligned}\psi\Big(\int_0^{d(Tx,Sy)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{\frac{1}{12}}\varphi(t)dt\Big) = \psi\Big(\frac{1}{144}\Big) = \frac{100}{144}\\ &< \frac{100}{36} - 1 = \psi\Big(\int_0^{\frac{1}{6}}\varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{6}}\varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{M_2(x,y)}\varphi(t)dt\Big) - \phi\Big(\int_0^{M_2(x,y)}\varphi(t)dt\Big).\end{aligned}$$

Hence, (2.17) holds. It follows from Theorem 2.2 that the mappings A, B, S and T have a unique common fixed point $\frac{5}{12} \in X$. But Theorems 1.1-1.4 cannot be used to prove the existence of fixed points of T and common fixed points of T and S in X.

Suppose that T satisfies the conditions of Theorem 1.1. That is, there exist ϕ and $\psi \in \Phi_5$ satisfying (1.1). It follows from (1.1) that

$$\begin{aligned}\psi\Big(\frac{1}{12}\Big) &= \psi\Big(d\Big(T\frac{7}{12}, T\frac{1}{2}\Big)\Big) \le \psi\Big(d\Big(\frac{7}{12}, \frac{1}{2}\Big)\Big) - \phi\Big(d\Big(\frac{7}{12}, \frac{1}{2}\Big)\Big)\\ &= \psi\Big(\frac{1}{12}\Big) - \phi\Big(\frac{1}{12}\Big) < \psi\Big(\frac{1}{12}\Big),\end{aligned}$$

which is absurd.

Suppose that T satisfies the conditions of Theorem 1.2. That is, there exist $c \in (0,1)$ and $\varphi \in \Phi_1$ satisfying (1.2). In light of (1.2), we obtain that

$$0 < \int_0^{\frac{1}{12}}\varphi(t)dt = \int_0^{d(T\frac{7}{12}, T\frac{1}{2})}\varphi(t)dt \le c\int_0^{d(\frac{7}{12},\frac{1}{2})}\varphi(t)dt < \int_0^{\frac{1}{12}}\varphi(t)dt,$$

which is a contradiction.

Suppose that T satisfies the conditions of Theorem 1.3. That is, there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_5$ satisfying (1.3). In view of (1.3), we know that

$$\begin{aligned}&\psi\Big(\int_0^{\frac{1}{12}}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{d(T\frac{7}{12}, T\frac{1}{2})}\varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{d(\frac{7}{12},\frac{1}{2})}\varphi(t)dt\Big) - \phi\Big(\int_0^{d(\frac{7}{12},\frac{1}{2})}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{\frac{1}{12}}\varphi(t)dt\Big) - \phi\Big(\int_0^{\frac{1}{12}}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{\frac{1}{12}}\varphi(t)dt\Big),\end{aligned}$$

which is impossible.

Suppose that T and S satisfy the conditions of Theorem 1.4. That is, there exists $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_4 \times \Phi_5$ satisfying (1.4). Taking advantage of (1.4), we conclude that

$$\begin{aligned}
&\psi\bigg(\int_0^{\frac{1}{12}} \varphi(t)dt\bigg) \\
&= \psi\bigg(\int_0^{d(T\frac{7}{12}, S\frac{1}{2})} \varphi(t)dt\bigg) \\
&\leq \psi\bigg(\int_0^{M(\frac{7}{12}, \frac{1}{2})} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M(\frac{7}{12}, \frac{1}{2})} \varphi(t)dt\bigg) \\
&= \psi\bigg(\int_0^{\frac{1}{12}} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{\frac{1}{12}} \varphi(t)dt\bigg) \\
&< \psi\bigg(\int_0^{\frac{1}{12}} \varphi(t)dt\bigg),
\end{aligned}$$

which is a contradiction.

Acknowledgement: This work was supported by the Gyeongsang National University Fund for professors on Sabbatical Leave, 2017.

References

[1] M. Abbas and D. Dorić, *Common fixed point theorem for four mappings satisfying generalized weak contractive condition*, Filomat, **24** (2010), 1–10.

[2] M. Abbas and M.A. Khan, *Common fixed point theorem of two mappings satisfying a generalized weak contractive condition*, Int. J. Math. Math. Sci., **2009** (2009), Article ID 131068, 9 pages.

[3] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29** (2002), 531–536.

[4] P. N. Dutta and B. S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed Point Theory Appl., **2008** (2008), Article ID 406368, 8 pages.

[5] A. Djoudi and F. Merghadi, *Common fixed point theorems for maps under a contractive condition of integral type*, J. Math. Anal. Appl., **341** (2008), 953–960.

[6] V. R. Hosseini, *Common fixed point for (φ, ψ)-weak contractions mappings condition*, Int. J. Math. Anal., **4** (2010), 1535–1543.

[7] G. Jungck and B. E. Rhoades, *Fixed point for set valued functions without continuity*, Indian J. Pure Appl. Math., **29** (1998), 227–238.

[8] J. Jachymski, *Remarks on contractive conditions of integral type*, Nonlinear Anal., **71** (2009), 1073–1081.

[9] Z. Liu, J. Li and S. M. Kang, *Fixed point theorems of contractive mappings of integral type*, Fixed Point Theory Appl., **2013** (2013), Paper No. 300, 17 pages.

[10] Z. Liu, X. Li, S. M. Kang and S. Y. Cho, *Fixed point theorems for mappings satisfying contractive conditions of integral type and applications*, Fixed Point Theory Appl., **2011** (2011), Paper No. 64, 18 pages.

[11] C. Mongkolkeha and P. Kumam, *Fixed point and common fixed point theorems for generalized weak contraction mappings of integral type in Modular spaces*, Int. J. Math. Math. Sci., **2011** (2011), Article ID 705943, 12 pages.

[12] B. E. Rhoades, *Two fixed-point theorems for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **2003** (2003), 4007–4013.

[13] W. Sintunavarat and P. Kumam, *Gregus-type common fixed point theorems for tangential multivalued mappings of integral type in metric spaces*, Int. J. Math. Math. Sci., **2011** (2011), Article ID 923458, 12 pages.

[14] W. Sintunavarat and P. Kumam, *Gregus type fixed points for a tangential multi-valued mappings satisfying contractive conditions of integral type*, J. Inequal. Appl., **2011** (2011), Paper No. 3, 12 pages.

[15] T. Suzuki, *Meir-Keeler contractions of integral type are still Meir-Keeler contractions*, Int. J. Math. Math. Sci., **2007** (2007), Article ID 39281, 6 pages.

[16] R. Shah, A. Zada and T. Li, *New common coupled fixed point results of integral type contraction in generalized metric spaces*, J. Anal. Num. Theor., **4** (2016), 145–152.

[17] P. Vijayaraju, B. E. Rhoades and R. Mohanraj, *A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **2005** (2005), 2359–2364.