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STRONG CONVERGENCE OF GENERAL ITERATIVE ALGORITHMS FOR PSEUDOCONTRACTIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce general iterative algorithms for finding a fixed point of a continuous pseudocontractive mapping in a Hilbert space. Then we establish strong convergence of sequences generated by the proposed iterative algorithms to a fixed point of the mapping, which is the unique solution of a certain variational inequality.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be a self-mapping on C. We denote by $Fix(T)$ the set of fixed points of T.

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall ([2, 3]) that a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be k*-strictly pseudocontractive* ([3]) if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. The class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is *nonexpansive* (*i.e.*, $\|Tx - Ty\| \le \|x - y\|$, $\forall x,\ y \in C$) if and only if T is

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0-strictly pseudocontractive. Clearly, the class of k-strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudocontractive mappings, see, for example, [1, 7, 8, 9, 11, 14, 22] and the references therein.

In 2010, by combining Yamada's method [20] and Marino and Xu's method [12], Tian [17] considered the following iterative algorithm for a nonexpansive mapping S:

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) S x_n, \quad \forall n \geq 0,$$

where $F : H \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$, $\eta > 0$, $V : H \to H$ is an l-Lipschitzian mapping with a constant $l \geq 0$, $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau = \mu(\eta - \frac{\mu\rho^2}{2})$.

In 2011, Ceng *et al.* [5] also proposed the following iterative algorithm for the nonexpansive mapping S:

$$x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) S x_n], \quad \forall n \geq 0,$$

where P_C is the metric projection of H onto C; $F : C \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$; $V : C \to H$ is an l-Lipschitzian mapping with a constant $l \geq 0$; $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.

In 2015, Jung [9] devised the following iterative algorithm for a k-strictly pseudocontractive mapping T for some $0 \leq k < 1$:

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n, \quad \forall n \geq 0,$$

where $F : H \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$, $\eta > 0$, $V : H \to H$ is an l-Lipschitzian mapping with a constant $l \geq 0$, $0 < \mu < \frac{2\eta}{\rho^2}$, $0 \leq \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$, $T_n : H \to H$ is a mapping defined by $T_n x = \lambda_n x + (1 - \lambda_n) T x$ for all $x \in H$, with $0 \leq k \leq \lambda_n \leq \lambda < 1$ and $\lim_{n\to\infty} \lambda_n = \lambda$. His results improved results of Tian [17] and Ceng *et al.* [5] from the class of the nonexpansive mappings to the class of strictly pseudocontractive mappings. The following problem arises:

Question. Can we extend the class of nonexpansive mappings in [5, 17] or the class of strictly pseudocontractive mappings in [9] to the more general class of pseudocontractive mappings?

In this paper, in order to give an affirmative answer to the above question, we introduce implicit and explicit iterative algorithms for a continuous pseudocontractive mapping T in a Hilbert space. Under suitable control conditions, we establish strong convergence of sequences generated by the proposed iterative algorithms to a fixed point of T, which is a solution of a certain variational inequality, where the constrained set is $Fix(T)$. The results in this paper improve and develop the corresponding results in [5, 7, 8, 9, 12, 15, 17, 18] and references therein.

2. Preliminaries and lemmas

We recall ([2]) that a mapping F of C into H is called

(i) *Lipschitzian* if there exists a constant $\kappa \geq 0$ such that

$$\|Fx - Fy\| \leq \kappa \|x - y\| \quad \forall x, y \in C;$$

(ii) *monotone* if $\langle x - y, Fx - Fy\rangle \geq 0, \quad \forall x,\ y \in C$;

(iii) η-*strongly monotone* if there exists a positive real number η such that

$$\langle x - y, Fx - Fy\rangle \geq \eta \|x - y\|^2, \quad \forall x,\ y \in C.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and that for $x \in H$,

$$z = P_C x \text{ if and only if } \langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

The lemma can be derived easily from the inner product (see [2]).

Lemma 2.1. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.2. ([21]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$;*

(iv) *$Fix(T)$ is a closed convex subset of C.*

We also need the following lemmas for the proof of our main results. The following Lemma 2.3 is essentially Lemma 2 in [10].

Lemma 2.3. ([10, 19]). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\beta_n)s_n + \beta_n\delta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\beta_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$,
(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=0}^{\infty}\beta_n|\delta_n| < \infty$,
(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. ([16]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in* [0,1] *which satisfies the following condition:*

$$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1.$$

Suppose that $x_{n+1} = \beta_n x_n + (1-\beta_n)l_n$, $n \geq 0$, and

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Then $\lim_{n\to\infty}\|l_n - x_n\| = 0$.

The following lemmas can be easily proven, and therefore, we omit the proofs (see [20]).

Lemma 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $V : C \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and $F : C \to H$ be a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x-y\rangle \geq (\mu\eta - \gamma l)\|x-y\|^2, \quad \forall x,\ y \in C.$$

That is, $\mu F - \gamma V$ is strongly monotone with a constant $\mu\eta - \gamma l$.

Lemma 2.6. *Let C be a closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 < t < \varsigma \leq 1$. Then $S := \varsigma I - t\mu F : C \to H$ is a contractive mapping with a constant $\varsigma - t\tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\rho^2)}$.*

The following lemma is a variant of a Minty lemma (see [13]).

Lemma 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $G : C \to H$ is monotone and weakly continuous along segments, that is, $G(x + ty) \to G(x)$ weakly as $t \to 0$. Then the variational inequality*

$$\widetilde{x} \in C, \quad \langle G\widetilde{x}, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C,$$

is equivalent to the dual variational inequality

$$\widetilde{x} \in C, \quad \langle Gp, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C.$$

In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ means that $\{x_n\}$ converges strongly to x.

3. Iterative algorithms

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed convex subset of H;
- $T : C \to C$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $T_{r_t} : H \to C$ is a mapping defined by $T_{r_t}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_t}\langle y - z, (1 + r_t)z - x\rangle \leq 0, \quad \forall y \in C\right\}$ for $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $T_{r_n} : H \to C$ is a mapping defined by $T_{r_n}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0, \quad \forall y \in C\right\}$ for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $V : C \to C$ is an l-Lipschitzian mapping with constant $l \in [0, \infty)$;
- $F : C \to C$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- Constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- $Fix(T) \neq \emptyset$;
- P_C is the metric projection of H onto C.

By Lemma 2.2, T_{r_t} and T_{r_n} are nonexpansive and $Fix(T) = Fix(T_{r_t}) = Fix(T_{r_n})$.

In this section, first, we consider the following iterative algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = P_C[t\gamma V x_t + (I - t\mu F)T_{r_t}x_t], \quad t \in (0, 1). \tag{3.1}$$

Indeed, for $t \in (0,1)$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = P_C[t\gamma V x + (I - t\mu F)T_{r_t} x], \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - t(\tau - \gamma l)$. Indeed, by Lemma 2.6, we have

$$\begin{aligned}\|Q_t x - Q_t y\| &\le t\gamma\|Vx - Vy\| + \|(I - t\mu F)T_{r_t} x - (I - t\mu F)T_{r_t} y\| \\ &\le t\gamma l\|x - y\| + (1 - t\tau)\|x - y\| \\ &= (1 - t(\tau - \gamma l))\|x - y\|.\end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation (3.1).

Now, we establish the strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality:

$$\langle(\mu F - \gamma V)\widetilde{x}, \widetilde{x} - p\rangle \le 0, \quad \forall p \in Fix(T). \tag{3.2}$$

Theorem 3.1. *The net* $\{x_t\}$ *defined by* (3.1) *converges strongly to a fixed point* $\widetilde{x}$ *of* T *as* $t \to 0$*, which solves the variational inequality* (3.2)*. Equivalently, we have* $P_{Fix(T)}(I - \mu F + \gamma V)\widetilde{x}) = \widetilde{x}$.

Proof. We first show the uniqueness of a solution of the variational inequality (3.2), which is indeed a consequence of the strong monotonicity of $\mu F - \gamma V$. In fact, noting that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau$ if and only if $\rho \ge \eta$, it follows from Lemma 2.5 that

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \ge (\mu\eta - \gamma l)\|x - y\|^2.$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau \le \mu\eta$. Suppose that $\widetilde{x} \in Fix(T)$ and $\widehat{x} \in Fix(T)$ both are solutions to (3.2). Then we have

$$\langle(\mu F - \gamma V)\widetilde{x}, \widetilde{x} - \widehat{x}\rangle \le 0 \tag{3.3}$$

and

$$\langle(\mu F - \gamma V)\widehat{x}, \widehat{x} - \widetilde{x}\rangle \le 0. \tag{3.4}$$

Adding up (3.3) and (3.4) yields

$$\langle(\mu F - \gamma V)\widetilde{x} - (\mu F - \gamma V)\widehat{x}, \widetilde{x} - \widehat{x}\rangle \le 0.$$

The strong monotonicity of $\mu F - \gamma V$ implies that $\widetilde{x} = \widehat{x}$ and the uniqueness is proved.

Next, we prove that $x_t \to \widetilde{x}$ as $t \to 0$. To this end, we divide its proof into four steps.

Step 1. We show that $\{x_t\}$ is bounded, and so $\{Vx_t\}$, $\{Tx_t\}$, $\{T_{r_t}x_t\}$, $\{Fx_t\}$ and $\{FT_{r_t}x_t\}$ are bounded. Observing $Fix(T) = Fix(T_{r_t})$ by Lemma 2.2, from (3.1), we derive that

$$\begin{aligned}\|x_t - p\| &\le \|t\gamma Vx_t + (I - t\mu F)T_{r_t}x_t - p\|\\ &= \|t(\gamma Vx_t - \mu Fp) + (I - t\mu F)T_{r_t}x_t - (I - t\mu F)p\|\\ &\le (1 - t\tau)\|x_t - p\| + t\|\gamma Vx_t - \mu Fp\|,\end{aligned}$$

and hence

$$\begin{aligned}\|x_t - p\| &\le \frac{1}{\tau}\|\gamma Vx_t - \mu Fp\|\\ &\le \frac{1}{\tau}[\|\gamma Vx_t - \gamma Vp\| + \|\gamma Vp - \mu Fp\|]\\ &\le \frac{1}{\tau}[\gamma l\|x_t - p\| + \|\gamma Vp - \mu Fp\|].\end{aligned}$$

This implies that

$$\|x_t - p\| \le \frac{1}{\tau - \gamma l}\|\gamma Vp - \mu Fp\|.$$

Hence $\{x_t\}$, $\{Vx_t\}$, $\{Tx_t\}$, $\{T_{r_t}x_t\}$, $\{Fx_t\}$ and $\{FT_{r_t}x_t\}$ are bounded.

Step 2. We show that $\lim_{t\to 0}\|x_t - w_t\| = 0$, where $w_t = T_{r_t}x_t$. In fact, it follows that

$$\begin{aligned}\lim_{t\to 0}\|x_t - w_t\| &= \lim_{t\to 0}\|P_C[t\gamma Vx_t + (I - t\mu F)T_{r_t}x_t] - P_C(T_{r_t}x_t)\|\\ &\le \lim_{t\to 0}\|t\gamma Vx_t + (I - t\mu F)T_{r_t}x_t - T_{r_t}x_t\|\\ &= \lim_{t\to 0} t\|\gamma Vx_t - \mu FT_{r_t}x_t\|\\ &= 0.\end{aligned} \tag{3.5}$$

Step 3. We show that $\{x_t\}$ is relatively norm compact as $t \to 0$. To this end, let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$, $w_n = w_{t_n}$ and $r_n := r_{t_n}$. Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $q \in C$. First, we prove that $q \in Fix(T)$. In fact, from the definition of $w_n = T_{r_n}x_n$, we have

$$\langle y - w_n, Tw_n\rangle - \frac{1}{r_n}\langle y - w_n, (1 + r_n)w_n - x_n\rangle \le 0, \quad \forall y \in C. \tag{3.6}$$

Put $v_\tau = \tau v + (1 - \tau)q$ for all $\tau \in (0, 1]$ and $v \in C$. Then $v_\tau \in C$, and from (3.6) and pseudocontractivity of T, it follows that

$$\begin{aligned}&\langle w_n - v_\tau, Tv_\tau\rangle\\ &\ge \langle w_n - v_\tau, Tv_\tau\rangle + \langle v_\tau - w_n, Tw_n\rangle - \frac{1}{r_n}\langle v_\tau - w_n, (1 + r_n)w_n - x_n\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\langle v_\tau - w_n, Tv_\tau - Tw_n\rangle - \frac{1}{r_n}\langle v_\tau - w_n, w_n - x_n\rangle - \langle v_\tau - w_n, w_n\rangle \\
&\geq -\|v_\tau - w_n\|^2 - \frac{1}{r_n}\langle v_\tau - w_n, w_n - x_n\rangle - \langle v_\tau - w_n, w_n\rangle \qquad (3.7)\\
&= -\langle v_\tau - w_n, v_\tau\rangle - \langle v_\tau - w_n, \frac{w_n - x_n}{r_n}\rangle.
\end{aligned}
$$

By Step 2, we get $\frac{w_n - x_n}{r_n} \to 0$ as $n \to \infty$. Moreover, since $x_n \rightharpoonup q$, by Step 2, we have $w_n \rightharpoonup q$ as $n \to \infty$. Therefore, from (3.7), as $n \to \infty$, it follows that

$$\langle q - v_\tau, Tv_\tau\rangle \geq \langle q - v_\tau, v_\tau\rangle,$$

and hence

$$-\langle v - q, Tv_\tau\rangle \geq -\langle v - q, v_\tau\rangle, \quad \forall v \in C.$$

Letting $\tau \to 0$ and using the fact that T is continuous, we get

$$-\langle v - q, Tq\rangle \geq -\langle v - q, q\rangle, \quad \forall v \in C.$$

Putting $v = Tq$ attains $q = Tq$, that is, $q \in Fix(T)$.

Now, from (3.1), we write for $p \in Fix(T)$

$$
\begin{aligned}
x_t - p &= x_t - y_t + y_t - p \\
&= x_t - y_t + (t\gamma Vx_t + (I - t\mu F)T_{r_t}x_t - p) \\
&= x_t - y_t + t(\gamma Vx_t - \mu Fp) + (I - t\mu F)T_{r_t}x_t - (I - t\mu F)p,
\end{aligned}
$$

where $y_t = t\gamma Vx_t + (I - t\mu F)T_{r_t}x_t$. From (2.1), we derive

$$
\begin{aligned}
\|x_t - p\|^2 &= \langle x_t - y_t, x_t - p\rangle + t\langle \gamma Vx_t - \mu Fp, x_t - p\rangle \\
&\quad + \langle (I - t\mu F)T_{r_t}x_t - (I - t\mu F)p, x_t - p\rangle \\
&\leq (1 - t\tau)\|x_t - p\|^2 + t\gamma l\|x_t - p\|^2 + t\langle \gamma Vp - \mu Fp, x_t - p\rangle,
\end{aligned}
$$

and henc

$$\|x_t - p\|^2 \leq \frac{1}{\tau - \gamma l}\langle \gamma Vp - \mu Fp, x_t - p\rangle. \qquad (3.8)$$

We substitute q for p in (3.8) to obtain

$$\|x_n - q\|^2 \leq \frac{1}{\tau - \gamma l}\langle \gamma Vq - \mu Fq, x_n - q\rangle. \qquad (3.9)$$

Note that $x_n \rightharpoonup q$. This fact and the inequality (3.9) imply that $x_n \to q$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

Step 4. We show that that q is a solution of the variational inequality (3.2). Indeed, putting x_{t_n} in place of x_t in (3.8) and taking the limit as $t_n \to 0$, we obtain

$$\|q - p\|^2 \leq \frac{1}{\tau - \gamma l}\langle \gamma Vp - \mu Fp, q - p\rangle, \quad \forall p \in Fix(T).$$

In particular, q solves the following variational inequality

$$q \in Fix(T), \quad \langle \gamma Vp - \mu Fp, q - p\rangle \geq 0, \quad \forall p \in Fix(T).$$

or the equivalent dual variational inequality (see Lemma 2.7)

$$q \in Fix(T), \quad \langle \gamma Vq - \mu Fq, q - p\rangle \geq 0, \quad \forall p \in Fix(T).$$

That is, $q \in Fix(T)$ is a solution of the variational inequality (3.2), hence $q = \widetilde{x}$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\widetilde{x}$. Therefore, $x_t \to \widetilde{x}$ as $t \to 0$.

The variational inequality (3.2) can be rewritten as

$$\langle (I - \mu F + \gamma V)\widetilde{x} - \widetilde{x}, \widetilde{x} - p\rangle \geq 0, \quad \forall p \in Fix(T).$$

By (2.1), this is equivalent to the fixed point equation

$$P_{Fix(T)}(I - \mu F + \gamma V)\widetilde{x}) = \widetilde{x}.$$

This completes the proof. □

Now, we consider the following iterative algorithm which generates a sequence in an explicit way:

$$x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)T_{r_n} x_n], \quad \forall n \geq 0, \tag{3.11}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$ and $x_0 \in C$ is an arbitrary initial guess

Using Theorem 3.1, we establish strong convergence of the sequence generated by the explicit algorithm (3.11) to a fixed point $\widetilde{x}$ of T, which is the unique solution of the variational inequality (3.2).

Theorem 3.2. *Let $\{x_n\}$ be the sequence generated by the iterative algorithm* (3.11)*, where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(C3) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ *(the perturbed control condition).*
(C4) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ *and* $r_n > b > 0$ *for* $n \geq 1$.

Then $\{x_n\}$ converges strongly to $\widetilde{x} \in Fix(T)$, which is the unique solution of the variational inequality (3.2)*.*

Proof. First, note that from the condition (C1), without loss of generality, we assume that $\alpha_n \tau < 1$ and $\frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n \gamma l} < 1$ for all $n \geq 0$.

Let x_t be defined by (3.1), that is, $x_t = P_C[t\gamma V x_t + (I - t\mu F)T_{r_t} x_t]$ for $0 < t < 1$, and let $\lim_{t\to 0} x_t := \widetilde{x} \in Fix(T)$ (by Theorem 3.1). Then $\widetilde{x}$ is the unique solution of the variational inequality (3.2).

We divide the proof into several steps:

Step 1. We show that $\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\right\}$ for all $n \ge 0$ and all $p \in Fix(T)$. Indeed, let $p \in Fix(T)$. Noticing $p = T_{r_n}p$, we have

$$\begin{aligned}\|x_{n+1} - p\| &\le \|\alpha_n(\gamma Vx_n - \mu Fp) + (I - \alpha_n\mu F)T_{r_n}x_n - (I - \alpha_n\mu F)T_{r_n}p\| \\ &\le (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\|\gamma Vx_n - \mu Fp\| \\ &\le (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n(\|\gamma Vx_n - \gamma Vp\| + \|\gamma Vp - \mu Fp\|) \\ &\le [1 - (\tau - \gamma l)\alpha_n]\|x_n - p\| + (\tau - \gamma l)\alpha_n\frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l} \\ &\le \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\right\}.\end{aligned}$$

Using an induction, we have $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\}$. Hence $\{x_n\}$ is bounded, and so are $\{Vx_n\}$, $\{Tx_n\}$ $\{T_{r_n}x_n\}$, $\{FT_{r_n}x_n\}$, and $\{Fx_n\}$.

Step 2. We show that $\|w_n - w_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|K_1$, where $w_n := T_{r_n}x_n$ and $K_1 = \sup\{\|w_n - x_n\| : n \ge 1\}$. Indeed, let $w_n = T_{r_n}x_n$ and $w_{n-1} = T_{r_{n-1}}x_{n-1}$. Then we get

$$\langle y - w_{n-1}, Tw_{n-1}\rangle - \frac{1}{r_{n-1}}\langle y - w_{n-1}, (1 + r_{n-1})w_{n-1} - x_{n-1}\rangle \le 0, \quad \forall y \in C, \tag{3.12}$$

and

$$\langle y - w_n, Tw_n\rangle - \frac{1}{r_n}\langle y - w_n, (1 + r_n)w_n - x_n\rangle \le 0, \quad \forall y \in C. \tag{3.13}$$

Putting $y = w_n$ in (3.12) and $y = w_{n-1}$ in (3.13), we obtain

$$\langle w_n - w_{n-1}, Tw_{n-1}\rangle - \frac{1}{r_{n-1}}\langle w_n - w_{n-1}, (1 + r_{n-1})w_{n-1} - x_{n-1}\rangle \le 0, \tag{3.14}$$

and

$$\langle w_{n-1} - w_n, Tw_n\rangle - \frac{1}{r_n}\langle w_{n-1} - w_n, (1 + r_n)w_n - x_n\rangle \le 0. \tag{3.15}$$

Adding up (3.14) and (3.15), we have

$$\begin{aligned}&\langle w_n - w_{n-1}, Tw_{n-1} - Tw_n\rangle \\ &\qquad - \langle w_n - w_{n-1}, \frac{(1 + r_{n-1})w_{n-1} - x_{n-1}}{r_{n-1}} - \frac{(1 + r_n)w_n - x_n}{r_n}\rangle \le 0,\end{aligned}$$

which implies that

$$\begin{aligned}&\langle w_n - w_{n-1}, (w_n - Tw_n) - (w_{n-1} - Tw_{n-1})\rangle \\ &\qquad\qquad - \langle w_n - w_{n-1}, \frac{w_{n-1} - x_{n-1}}{r_{n-1}} - \frac{w_n - x_n}{r_n}\rangle \le 0.\end{aligned}$$

Now, using the fact that T is pseudocontractive, we deduce

$$\langle w_n - w_{n-1}, \frac{w_{n-1} - x_{n-1}}{r_{n-1}} - \frac{w_n - x_n}{r_n} \rangle \geq 0,$$

and hence

$$\langle w_n - w_{n-1}, w_{n-1} - w_n + w_n - x_{n-1} - \frac{r_{n-1}}{r_n}(w_n - x_n) \rangle \geq 0. \tag{3.16}$$

Since $r_n > b > 0$ for $n \geq 1$, by (3.16), we have

$$\begin{aligned} \|w_n - w_{n-1}\|^2 &\leq \langle w_n - w_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(w_n - x_n) \rangle \\ &\leq \|w_n - w_{n-1}\| \left[\|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}| \|w_n - x_n\| \right], \end{aligned}$$

which implies

$$\|w_n - w_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}| K_1, \tag{3.17}$$

where $K_1 = \sup\{\|w_n - x_n\| : n \geq 1\}$.

Step 3. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. In fact, by Step 1, there exists a constant $K_2 > 0$ such that for all $n \geq 0$,

$$\mu \|F T_{r_n} x_n\| + \gamma \|V x_n\| \leq K_2.$$

Then, by Step 2, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|(I - \alpha_n \mu F) T_{r_n} x_n - (I - \alpha_n \mu F) T_{r_{n-1}} x_{n-1} \\ &\quad + \mu(\alpha_n - \alpha_{n-1}) F T_{r_{n-1}} x_{n-1} \\ &\quad + \gamma[\alpha_n (V x_n - V x_{n-1}) + V x_{n-1}(\alpha_n - \alpha_{n-1})]\| \\ &\leq (1 - \alpha_n \tau) \|T_{r_n} x_n - T_{r_{n-1}} x_{n-1}\| \\ &\quad + \mu |\alpha_n - \alpha_{n-1}| \|F T_{r_{n-1}} x_{n-1}\| \\ &\quad + \gamma[\alpha_n [l \|x_n - x_{n-1}\| + \|V x_{n-1}\| |\alpha_n - \alpha_{n-1}|] \\ &\leq (1 - \alpha_n(\tau - \gamma l)) \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}| K_1 \\ &\quad + |\alpha_n - \alpha_{n-1}| K_2 \\ &\leq (1 - \alpha_n(\tau - \gamma l)) \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}| K_1 \\ &\quad + (o(\alpha_n) + \sigma_{n-1}) K_2. \end{aligned} \tag{3.18}$$

By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\beta_n = \alpha_n(\tau - \gamma l)$, $\beta_n \delta_n = o(\alpha_n) K_2$ and $\gamma_n = \sigma_{n-1} K_2 + \frac{1}{b}|r_n - r_{n-1}| K_1$, from (3.18), we derive

$$s_{n+1} \leq (1 - \beta_n) s_n + \beta_n \delta_n + \gamma_n.$$

Hence, by (C2), (C3), (C4) and Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|x_n - w_n\| = 0$, where $w_n := T_{r_n} x_n$. Indeed, from the condition (C1) and Step 3, it follows that

$$\begin{aligned}
\|x_n - w_n\| &= \|x_n - T_{r_n} x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n} x_n\| \\
&\le \|x_n - x_{n+1}\| + \|\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} x_n - T_{r_n} x_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n \|\gamma V x_n - \mu F T_{r_n} x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n K_3 \to 0 \quad (\text{as } n \to \infty),
\end{aligned}$$

where $K_3 = \sup\{\|\gamma V x_n \ - \mu F T_{r_n} x_n\| : n \ge 0\}$.

Step 5. We show that $\limsup_{n\to\infty} \langle \gamma V \widetilde{x} - \mu F \widetilde{x}, x_n - \widetilde{x} \rangle \le 0$, where $\widetilde{x}$ is a solution of the variational inequality (3.2). First we prove that

$$\limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, w_n - \widetilde{x} \rangle = \limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, T_{r_n} x_n - \widetilde{x} \rangle \le 0.$$

Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, w_n - \widetilde{x} \rangle = \lim_{i\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, w_{n_i} - \widetilde{x} \rangle. \tag{3.19}$$

Without loss of generality, we may assume that $\{x_{n_i}\}$ converges weakly to $q \in C$. From $\|w_n - x_n\| \to 0$ by Step 4, it follows that $w_{n_i} \rightharpoonup q$. Thus, by the same argument as in Step 3 of the proof of Theorem 3.1, we obtain $q \in Fix(T)$. So, from (3.19), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, w_n - \widetilde{x} \rangle &= \lim_{i\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, w_{n_i} - \widetilde{x} \rangle \\
&= \langle (\gamma V - \mu F)\widetilde{x}, q - \widetilde{x} \rangle \le 0.
\end{aligned} \tag{3.20}$$

Since $\lim_{n\to\infty} \|x_n - w_n\| = 0$ by Step 4, from (3.20), we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, x_n - \widetilde{x} \rangle \\
&\le \limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, x_n - w_n \rangle + \limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, w_n - \widetilde{x} \rangle \\
&\le \limsup_{n\to\infty} \|(\gamma V - \mu F)\widetilde{x}\| \|x_n - w_n\| + \limsup_{n\to\infty} \langle (\gamma V - \mu F)\widetilde{x}, w_n - \widetilde{x} \rangle \\
&\le 0.
\end{aligned}$$

Step 6. We show that $\lim_{n\to\infty} \|x_n - \widetilde{x}\| = 0$, where $\widetilde{x}$ is a solution of the variational inequality (3.2). To this end, let $y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} x_n$. Then, we obtain

$$y_n - \widetilde{x} = \alpha_n (\gamma V x_n - \mu F \widetilde{x}) + (I - \alpha_n \mu F) T_{r_n} x_n - (I - \alpha_n \mu F)\widetilde{x}$$

and

$$
\begin{aligned}
& x_{n+1} - \widetilde{x} \\
&= x_{n+1} - y_n + y_n - \widetilde{x} \\
&= x_{n+1} - y_n + \alpha_n(\gamma V x_n - \mu F \widetilde{x}) + (I - \alpha_n \mu F) T_{r_n} x_n - (I - \alpha_n \mu F)\widetilde{x}.
\end{aligned}
$$

Applying (2.1), Lemma 2.1 and Lemma 2.6, we derive

$$
\begin{aligned}
& \|x_{n+1} - \widetilde{x}\|^2 \\
&= \|x_{n+1} - y_n + (I - \alpha_n \mu F) T_{r_n} x_n - (I - \alpha_n \mu F)\widetilde{x} \\
&\quad + \alpha_n(\gamma V x_n - \mu F \widetilde{x})\|^2 \\
&\le \|(I - \alpha_n \mu F) T_{r_n} x_n - (I - \alpha_n \mu F) T_{r_n} \widetilde{x}\|^2 \\
&\quad + 2\langle x_{n+1} - y_n, x_{n+1} - \widetilde{x}\rangle + 2\alpha_n \langle \gamma V x_n - \mu F \widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\le (1 - \alpha_n \tau)^2 \|x_n - \widetilde{x}\|^2 + 2\alpha_n \langle \gamma V x_n - \gamma V \widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\quad + 2\alpha_n \langle \gamma V \widetilde{x} - \mu F \widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\le (1 - \alpha_n \tau)^2 \|x_n - \widetilde{x}\|^2 + \alpha_n \gamma l (\|x_n - \widetilde{x}\|^2 + \|x_{n+1} - \widetilde{x}\|^2) \\
&\quad + 2\alpha_n \langle \gamma V \widetilde{x} - \mu F \widetilde{x}, x_{n+1} - \widetilde{x}\rangle.
\end{aligned}
\tag{3.21}
$$

Then, it follows from (3.21) that

$$
\begin{aligned}
& \|x_{n+1} - \widetilde{x}\|^2 \\
&\le \frac{(1 - \alpha_n \tau)^2 + \alpha_n \gamma l}{1 - \alpha_n \gamma l} \|x_n - \widetilde{x}\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma l} \langle \gamma V \widetilde{x} - \mu F \widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\le \left(1 - \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n \gamma l}\right) \|x_n - \widetilde{x}\|^2 \\
&\quad + \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n \gamma l} \left(\frac{1}{\tau - \gamma l} \langle \gamma V \widetilde{x} - \mu F \widetilde{x}, x_{n+1} - \widetilde{x}\rangle + \frac{\alpha_n \tau^2}{2(\tau - \gamma l)} K_4 \right),
\end{aligned}
\tag{3.22}
$$

where $K_4 = \sup\{\|x_n - \widetilde{x}\|^2 : n \ge 0\}$. Put

$$
\beta_n = \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n \gamma l} \quad \text{and}
$$

$$
\delta_n = \frac{1}{\tau - \gamma l} \langle \mu F \widetilde{x} - \gamma V \widetilde{x}, \widetilde{x} - x_{n+1}\rangle + \frac{\alpha_n \tau^2}{2(\tau - \gamma l)} K_3.
$$

From (C1), (C2) and Step 5, it follows that $\beta_n \to 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Since (3.22) reduces to

$$
\|x_{n+1} - \widetilde{x}\|^2 \le (1 - \beta_n)\|x_n - \widetilde{x}\|^2 + \beta_n \delta_n,
$$

from Lemma 2.3 with $\gamma_n = 0$, we conclude that $\lim_{n\to\infty} \|x_n - \widetilde{x}\| = 0$. This completes the proof. □

In case that $C = H$, we can dispense with the condition (C3) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (perturbed control condition) in Theorem 3.2. For this purpose, we propose the following iterative algorithm which generates a sequence in an explicit way:

$$x_{n+1} = \alpha_n \gamma V x_n + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)T_{r_n} x_n, \quad \forall n \ge 0, \tag{3.23}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$ and $x_0 \in C$ is an arbitrary initial guess.

Using Theorem 3.1, we also prove strong convergence of the sequence generated by the explicit algorithm (3.23).

Theorem 3.3. *Suppose that $C = H$. Let Let $\{x_n\}$ be the sequence generated by the iterative algorithm* (3.23)*, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(C3)$'$ $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*; and*
(C4)$'$ $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n > b > 0$ *for* $n \ge 1$.

Then $\{x_n\}$ converges strongly to $\widetilde{x} \in Fix(T)$, which is the unique solution of the variational inequality (3.2).

Proof. We only include the difference from the proof of Theorem 3.2. We also divide the proof into several steps:

Step 1. We show that $\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\right\}$ for all $n \ge 0$ and all $p \in Fix(T)$. Indeed, let $p \in Fix(T)$. Noticing $p = T_{r_n} p$, we have

$$\begin{aligned}
\|x_{n+1} - p\| = &\|\alpha_n(\gamma V x_n - \mu F p) + \beta_n(x_n - p) \\
&+ ((1-\beta_n)I - \alpha_n \mu F)T_{r_n} x_n - ((1-\beta_n)I - \alpha_n \mu F)T_{r_n} p\| \\
\le &(1 - \beta_n - \alpha_n \tau)\|x_n - p\| + \beta_n \|x_n - p\| + \alpha_n \|\gamma V x_n - \mu F p\| \\
\le &(1 - \alpha_n \tau)\|x_n - p\| + \alpha_n(\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu F p\|) \\
\le &(1 - \alpha_n \tau)\|x_n - p\| + \alpha_n(\gamma l \|x_n - p\| + \|\gamma V p - \mu F p\|) \\
\le &[1 - (\tau - \gamma l)\alpha_n]\|x_n - p\| + (\tau - \gamma l)\alpha_n \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l} \\
\le &\max\left\{\|x_n - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l}\right\}.
\end{aligned}$$

Using an induction, we have $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\}$. Hence $\{x_n\}$ is bounded, and so are $\{Vx_n\}$, $\{Tx_n\}$ $\{T_{r_n} x_n\}$, $\{FT_{r_n} x_n\}$, and $\{Fx_n\}$.

Step 2. From Step 2 in the proof of Theorem 3.2, we know that

$$\|w_{n+1} - w_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|K_1,$$

where $w_n := T_{r_n}x_n$ and $K_1 = \sup\{\|w_n - x_n\| : n \geq 1\}$.

Step 3. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. To this end, define

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)l_n, \forall n \geq 0.$$

Then, we deduce

$$\begin{aligned}
& l_{n+1} - l_n \\
&= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}\gamma V x_{n+1} + ((1 - \beta_{n+1})I - \alpha_{n+1}\mu F)T_{r_{n+1}}x_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma V x_n + ((1 - \beta_n)I - \alpha_n\mu F)T_{r_n}x_n}{1 - \beta_n} \\
&= T_{r_{n+1}}x_{n+1} - T_{r_n}x_n \\
&\quad + \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\gamma V x_{n+1} - \mu F T_{r_{n+1}}x_{n+1}) + \frac{\alpha_n}{1 - \beta_n}(\mu F T_{r_n}x_n - \gamma V x_n).
\end{aligned}$$

Thus, from Step 2, we obtain

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} + \frac{\alpha_n}{1 - \beta_n}\right)K_2 + \frac{1}{b}|r_{n+1} - r_n|K_1,$$

where $K_2 = \sup\{\mu\|FT_{r_n}x_n\| + \gamma\|Vx_n\| : n \geq 0\}$. From conditions (C1), (C3)$'$ and (C4)$'$, we derive

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|l_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|l_n - x_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|x_n - w_n\| = 0$, where $w_n := T_{r_n}x_n$. Indeed, by (3.23), we have

$$\begin{aligned}
\|x_n - w_n\| &= \|x_n - T_{r_n}x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n}x_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n(\gamma\|Vx_n\| + \mu\|FT_{r_n}x_n\|) + \beta_n\|x_n - T_{r_n}x_n\|,
\end{aligned}$$

that is,

$$\|x_n - w_n\| = \|x_n - T_{r_n}x_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}K_2.$$

So, by conditions (C1) and (C3)$'$, and Step 3, we conclude

$$\lim_{n\to\infty}\|x_n - w_n\| = \lim_{n\to\infty}\|x_n - T_{r_n}x_n\| = 0.$$

Step 5. From Step 5 in proof of Theorem 3.2, we see that

$$\limsup_{n\to\infty}\langle\gamma V\widetilde{x} - \mu F\widetilde{x}, x_n - \widetilde{x}\rangle \le 0,$$

where $\widetilde{x}$ is a solution of the variational inequality (3.2).

Step 6. We show that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$, where $\widetilde{x}$ is a solution of the variational inequality (3.2). By using (3.23), we have

$$\begin{aligned} x_{n+1} - \widetilde{x} &= \alpha_n(\gamma V x_n - \mu F\widetilde{x}) + \beta_n(x_n - \widetilde{x}) \\ &\quad + ((1-\beta_n)I - \alpha_n\mu F)T_{r_n}x_n - ((1-\beta_n)I - \alpha_n\mu F)\widetilde{x}. \end{aligned}$$

Applying Lemma 2.1 and Lemma 2.6, we obtain

$$\begin{aligned} &\|x_{n+1} - \widetilde{x}\|^2 \\ &= \|((1-\beta_n)I - \alpha_n\mu F)T_{r_n}x_n - ((1-\beta_n)I - \alpha_n\mu F)\widetilde{x} \\ &\quad + \beta_n(x_n - \widetilde{x}) + \alpha_n(\gamma V x_n - \mu F\widetilde{x})\|^2 \\ &\le \|((1-\beta_n)I - \alpha_n\mu F)T_{r_n}x_n - ((1-\beta_n)I - \alpha_n\mu F)T_{r_n}\widetilde{x} + \beta_n(x_n - \widetilde{x})\|^2 \\ &\quad + 2\alpha_n\langle\gamma V x_n - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\le ((1-\beta_n-\alpha_n\tau)\|x_n - \widetilde{x}\| + \beta_n\|x_n - \widetilde{x}\|)^2 + 2\alpha_n\langle\gamma V x_n - \gamma V\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\quad + 2\alpha_n\langle\gamma V\widetilde{x} - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\le (1-\alpha_n\tau)^2\|x_n - \widetilde{x}\|^2 + 2\alpha_n\gamma l(\|x_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\|) \\ &\quad + 2\alpha_n\langle\gamma V\widetilde{x} - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\le (1-\alpha_n\tau)^2\|x_n - \widetilde{x}\|^2 + \alpha_n\gamma l(\|x_n - \widetilde{x}\|^2 + \|x_{n+1} - \widetilde{x}\|^2) \\ &\quad + 2\alpha_n\langle\gamma V\widetilde{x} - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle. \end{aligned}$$

The remainder follows from Step 6 in proof of Theorem 3.2. This completes the proof. □

Remark 3.4. (1) Theorem 3.1, Theorem 3.2 and Theorem 3.3 improve and develop the corresponding results in [5, 8, 9, 17] from the class of the strictly pseudocontractive mappings or the class of the nonexpansive mapping to the class of the pseudocontractive mappings.

(2) Theorem 3.1 includes the corresponding results of Tian [18], Marino and Xu [12] and Moudafi [15] as some special cases.

(3) Theorem 3.2 and Theorem 3.3 also generalizes the corresponding results of Cho *et al.* [7], Jung [8] and Marino and Xu [12] in following aspects:
   (a) A strongly positive bounded linear operator A in [7, 8, 12] is extended to the case of a ρ-Lipschitzian and η-strongly monotone operator F. (In fact, from the definitions, it follows that a strongly positive bounded linear operator A (i.e., there exists a constant $\overline{\gamma} > 0$ with the property: $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad x \in H$) is a $\|A\|$-Lipschitzian and $\overline{\gamma}$-strongly monotone operator).
   (b) The contractive mapping f with a constant $\alpha \in (0, 1)$ in [7, 8, 12] is extended to the case of a Lipschizian mapping V with a constant $l \geq 0$.
   (c) The nonexpansive mapping S in [7, 12] is extended to the case of the pseudocontractive mapping T.

(4) For iterative algorithms for systems of generalized equilibria, mixed equilibria, minimization, split inclusion and fixed point problems, we can also refer to [4, 6] and the references therein. By combining our methods in this paper and methods in [4, 6], we will consider new iterative algorithms for the above-mentioned problems coupled with fixed point problems of pseudocontractive mappings.

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