Nonlinear Functional Analysis and Applications
Vol. 24, No. 2 (2019), pp. 407-415
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

APPROXIMATION PROPERTIES OF THE PERTURBED DIFFUSION EQUATION ON THE HALF-LINE

Ki-Suk Lee[1] **and Soon-Mo Jung**[2]

[1]Department of Mathematics Education
Korea National University of Education, 28173 Cheongju, Republic of Korea
e-mail: `ksleeknue@gmail.com`

[2]Mathematics Section, College of Science and Technology
Hongik University, 30016 Sejong, Republic of Korea
e-mail: `smjung@hongik.ac.kr`

Abstract. We prove the generalized Hyers-Ulam stability of the diffusion equation, $u_t(x,t) - ku_{xx}(x,t) = 0$, on the half-line for a class of radially symmetric scalar functions $u : (0,\infty) \times (0,\infty) \to \mathbb{R}$ which are twice continuously differentiable.

1. Introduction

The stability problems for the functional equations and (ordinary or partial) differential equations originate from the question of Ulam [22]:

Under what conditions does there exist an additive function near an approximately additive function?

In 1941, Hyers [8] answered the question of Ulam in the affirmative for the Banach space cases. Indeed, Hyers' theorem states that the following statement is true for all $\varepsilon \geq 0$: if a function f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all x, then there exists an exact additive function F such that $\|f(x) - F(x)\| \leq \varepsilon$ for all x. In that case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have (satisfy) the Hyers-Ulam stability.

---

[0]Received November 23, 2018. Revised March 6, 2019.
[0]2010 Mathematics Subject Classification: 35K05, 35R45, 35B20, 35A23, 39B82.
[0]Keywords: Diffusion equation, perturbation, partial differential inequality, generalized Hyers-Ulam stability.
[0]Corresponding author: K.-S. Lee(ksleeknue@gmail.com).

Given a normed space X and an open interval I of $\mathbb{R}$, assume that the following statement is true for all $\varepsilon \geq 0$: for any function $y : I \to X$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \leq \varepsilon$$

for all $x \in I$, there exists a function $y_0 : I \to X$ satisfying the differential equation

$$a_n(x)y_0^{(n)}(x) + a_{n-1}(x)y_0^{(n-1)}(x) + \cdots + a_1(x)y_0'(x) + a_0(x)y_0(x) + h(x) = 0$$

and $\|y(x) - y_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is dependent on ε only. Then the differential equation is said to have the Hyers-Ulam stability.

When the above statement is true even if we replace ε and $K(\varepsilon)$ by $\varphi(x)$ and $\Phi(x)$, where φ, $\Phi : I \to [0, \infty)$ are functions not depending on y and y_0 explicitly, the corresponding differential equation is said to have the generalized Hyers-Ulam stability. (This type of stability is sometimes called the Hyers-Ulam-Rassias stability.)

These terminologies will also be applied for other differential equations and partial differential equations. For more detailed definitions of these terminologies, refer to [1, 2, 3, 4, 8, 9, 11, 15, 20, 22].

To the best of our knowledge, Obłoza was the first author who investigated the Hyers-Ulam stability of differential equations (see [16, 17]): Assume that $g, r : (a, b) \to \mathbb{R}$ are continuous functions with $\int_a^b |g(x)|dx < \infty$. Suppose ε is an arbitrary positive real number. Obłoza proved that there exists a constant $\delta > 0$ such that $|y(x) - y_0(x)| \leq \delta$ for all $x \in (a, b)$ whenever a differentiable function $y : (a, b) \to \mathbb{R}$ satisfies $|y'(x) + g(x)y(x) - r(x)| \leq \varepsilon$ for all $x \in (a, b)$ and a function $y_0 : (a, b) \to \mathbb{R}$ satisfies $y_0'(x)+g(x)y_0(x) = r(x)$ for all $x \in (a, b)$ and $y(\tau) = y_0(\tau)$ for some $\tau \in (a, b)$. Since then, a number of mathematicians have dealt with this subject (see [1, 14, 21]).

Prástaro and Rassias seem to be the first authors who investigated the Hyers-Ulam stability of partial differential equations (see [19]). Thereafter, Jung and Lee [13] proved the Hyers-Ulam stability of the first-order linear partial differential equation of the form

$$au_x(x, y) + bu_y(x, y) + cu(x, y) + d = 0,$$

where $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ are constants with $\Re(c) \neq 0$.

As a further step, Hegyi and Jung [6] investigated the generalized Hyers-Ulam stability of the Laplace's equation for spherically symmetric scalar functions. Moreover, they also proved the generalized Hyers-Ulam stability of the diffusion equation on the restricted domain or with an initial condition (see [7, 12]).

In this paper, applying an idea from [6, 7], we investigate the generalized Hyers-Ulam stability of the diffusion equation

$$u_t(x,t) - ku_{xx}(x,t) = 0 \tag{1.1}$$

for radially symmetric scalar functions, where $k > 0$ is a constant, $x > 0$, and $t > 0$.

The diffusion equation is sometimes called a heat equation or a continuity equation and it plays an important role in a number of fields of science. For example, the diffusion equation is strongly related to the Brownian motion in probability theory and it is also connected with chemical diffusion.

The main advantage of this present paper over the previous ones [7, 12] is that this paper can describe the behavior of relevant functions in the vicinity of origin while the existing works can only deal with domains not including the vicinity of origin.

2. Preliminaries

Before starting with our main theorem in the next section, we will modify the theorem ([10, Theorem 1]) to be suitable for its application to the proof of our main theorem (*cf.* [18, Theorem 2.2]). Indeed, the hypotheses and conditions of the original theorem [10, Theorem 1] were formulated with a instead of a_0 which impose a constraint on its usability. It is sufficient for us to follow the lines of the proof of [10, Theorem 1] for the proof of Theorem 2.1 below. Hence, we omit the proof.

Theorem 2.1. ([10, Theorem 1, Remark 3]) *Assume that X is a real Banach space and $I = (a,b)$ is an open interval for arbitrary constants $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$. Let $p : I \to \mathbb{R}$ and $q : I \to X$ be continuous functions such that there exists a constant $a_0 \in [a,b)$ with the properties:*

(i) $\int_{a_0}^t p(s)ds$ *exists for each* $t \in I$*;*

(ii) $\int_{a_0}^t q(y) \exp\big\{ \int_{a_0}^y p(s)ds \big\} dy$ *exists for any* $t \in I$*.*

Moreover, assume that $\varphi : I \to [0,\infty)$ is a function such that

(iii) $\int_{a_0}^b \varphi(y) \exp\big\{ \int_{a_0}^y p(s)ds \big\} dy$ *exists.*

If a continuously differentiable function $v : I \to X$ satisfies the differential inequality

$$\|v'(t) + p(t)v(t) + q(t)\| \le \varphi(t)$$

for all $t \in I$, then there exists a unique continuously differentiable function $v_0 : I \to X$ such that $v_0'(t) + p(t)v_0(t) + q(t) = 0$ for all $t \in I$ and

$$\|v(t) - v_0(t)\| \le \exp\left\{ -\int_{a_0}^t p(s)ds \right\} \int_t^b \varphi(y) \exp\left\{ \int_{a_0}^y p(s)ds \right\} dy$$

for all $t \in I$*.*

3. Main results

If $u(x,t)$ is a solution to the diffusion equation (1.1) and a is a positive constant, then the dilated function $w(x,t) := u(\sqrt{a}\,x, at)$ is also a solution to the diffusion equation (1.1). This property is called the invariance under dilation. Hence, it would be a nice idea to search for approximate solutions to (1.1), which are scalar functions of the form

$$u(x,t) = g\left(\frac{x}{\sqrt{4kt}}\right),$$

where g is a twice continuously differentiable function. That is, $u(x,t)$ depends on x and t primarily through the term $x/\sqrt{4kt}$. We note that the purpose of inclusion of the factor $1/\sqrt{4k}$ in the above formula is to simplify our formulations later.

Based on this argument, we define

$$\begin{aligned} U := \Bigg\{ & u : (0,\infty)\times(0,\infty) \to \mathbb{R} \mid \text{there exists a twice continuously} \\ & \text{differentiable function } g : (0,\infty) \to \mathbb{R} \text{ with } u(x,t) = g\left(\frac{x}{\sqrt{4kt}}\right) \\ & \text{for all } x > 0 \text{ and } t > 0 \Bigg\}. \end{aligned}$$

Theorem 3.1. *Assume that* $\varphi, \psi : (0,\infty) \to [0,\infty)$ *are functions satisfying the conditions*

$$\int_0^\infty e^{-y^2} \int_y^\infty \varphi(z) e^{z^2} dz dy < \infty \tag{3.1}$$

and

$$c := \inf_{t>0} 4t\psi(t) > 0. \tag{3.2}$$

For any twice continuously differentiable function $u \in U$ *satisfying the inequality*

$$\big| u_t(x,t) - k u_{xx}(x,t) \big| \le \varphi\left(\frac{x}{\sqrt{4kt}}\right)\psi(t) \tag{3.3}$$

for all $x > 0$ *and* $t > 0$*, there exists a solution* $u_0 : (0,\infty)\times(0,\infty) \to \mathbb{R}$ *of the diffusion equation* (1.1) *such that* $u_0 \in U$ *and*

$$|u(x,t) - u_0(x,t)| \le c \int_{x/\sqrt{4kt}}^\infty e^{-y^2} \int_y^\infty \varphi(z) e^{z^2} dz dy \tag{3.4}$$

for all $x > 0$ *and* $t > 0$.

Proof. Our assumption implies that there exists a twice continuously differentiable function $g : (0, \infty) \to \mathbb{R}$ such that

$$u(x,t) = g(r)$$

for any $x > 0$ and $t > 0$, where $r = x/\sqrt{4kt}$. Using this notation, we compute

$$\frac{\partial r}{\partial t} = -\frac{r}{2t} \quad \text{and} \quad \frac{\partial r}{\partial x} = \frac{1}{\sqrt{4kt}}$$

and we further apply chain rule to evaluate u_t and u_{xx} as follows:

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial r} g(r) \frac{\partial r}{\partial t} = -\frac{r}{2t} g'(r), \\ u_x(x,t) &= \frac{\partial}{\partial r} g(r) \frac{\partial r}{\partial x} = \frac{1}{\sqrt{4kt}} g'(r), \\ u_{xx}(x,t) &= \frac{1}{\sqrt{4kt}} \frac{\partial}{\partial r} g'(r) \frac{\partial r}{\partial x} = \frac{1}{4kt} g''(r). \end{aligned}$$

So we have

$$u_t(x,t) - k u_{xx}(x,t) = -\frac{1}{4t}\big(g''(r) + 2rg'(r)\big) \tag{3.5}$$

for any $x > 0$, $t > 0$, and hence for all $r > 0$.

Moreover, it follows from the last equality and (3.3) that

$$|u_t(x,t) - k u_{xx}(x,t)| = \frac{1}{4t}\big|g''(r) + 2rg'(r)\big| \le \varphi(r)\psi(t)$$

or

$$\big|g''(r) + 2rg'(r)\big| \le \varphi(r)\{4t\psi(t)\}$$

for all $r > 0$ and $t > 0$. In view of (3.2), we have a differential inequality

$$\big|g''(r) + 2rg'(r)\big| \le c\varphi(r)$$

for any $r > 0$. If we set $h(r) := g'(r)$ in the last inequality, then we have

$$\big|h'(r) + 2rh(r)\big| \le c\varphi(r) \tag{3.6}$$

for any $r > 0$.

We can now apply Theorem 2.1 to our inequality (3.6) by considering the substitutions as we see in the following table.

Theorem 2.1	X	t	a	b	a_0	$v(t)$	$p(t)$	$q(t)$	$\varphi(t)$
(3.6)	$\mathbb{R}$	r	0	∞	0	$h(r)$	$2r$	0	$c\varphi(r)$

By considering the table above, we evaluate

$$\int_0^r 2sds = r^2$$

which implies that the condition (i) of Theorem 2.1 is satisfied. Trivially, the condition (ii) of Theorem 2.1 is fulfilled. Furthermore, it follows from (3.1) that

$$\int_0^\infty c\varphi(y) \exp\left\{ \int_0^y 2sds \right\} dy = \int_0^\infty c\varphi(y) e^{y^2} dy < \infty$$

which means that the condition (iii) of Theorem 2.1 is satisfied.

Due to Theorem 2.1 and inequality (3.6), there exists a unique continuously differentiable function $h_0 : (0, \infty) \to \mathbb{R}$ such that

$$h_0'(r) + 2rh_0(r) = 0 \tag{3.7}$$

for any $r > 0$ and

$$\begin{aligned} \left| h(r) - h_0(r) \right| &\leq \exp\left\{ -\int_0^r 2sds \right\} \int_r^\infty c\varphi(y) \exp\left\{ \int_0^y 2sds \right\} dy \\ &= ce^{-r^2} \int_r^\infty \varphi(y) e^{y^2} dy \end{aligned}$$

or equivalently

$$\left| g'(r) - h_0(r) \right| \leq ce^{-r^2} \int_r^\infty \varphi(y) e^{y^2} dy \tag{3.8}$$

for all $r > 0$. In view of (3.7), there exists a real constant α with $h_0(r) = \alpha e^{-r^2}$.

We again apply Theorem 2.1 to our inequality (3.8) by considering the substitutions as we see in the following table.

Theorem 2.1	X	t	a	b	a_0	$v(t)$	$p(t)$	$q(t)$	$\varphi(t)$
(3.8)	$\mathbb{R}$	r	0	∞	0	$g(r)$	0	$-h_0(r)$	$ce^{-r^2} \int_r^\infty \varphi(z) e^{z^2} dz$

By considering the table above, we note that the condition (i) of Theorem 2.1 is trivially satisfied. Since

$$\left| \int_0^r h_0(y) dy \right| \leq |\alpha| \int_0^\infty e^{-y^2} dy < \infty,$$

the condition (ii) of Theorem 2.1 is fulfilled. Moreover, in view of (3.1), we easily see that (iii) of Theorem 2.1 is also fulfilled.

On account of Theorem 2.1, there exists a unique continuously differentiable function $g_0 : (0, \infty) \to \mathbb{R}$ such that

$$g_0'(r) - h_0(r) = 0 \tag{3.9}$$

and

$$\big|g(r) - g_0(r)\big| \le c \int_r^\infty e^{-y^2} \int_y^\infty \varphi(z) e^{z^2} dz dy \tag{3.10}$$

for all $r > 0$.

Let us define a function $u_0 : (0,\infty) \times (0,\infty) \to \mathbb{R}$ by $u_0(x,t) := g_0(r)$. As we did in (3.5), by using (3.7) and (3.9), we compute

$$\frac{\partial}{\partial t} u_0(x,t) - k \frac{\partial^2}{\partial x^2} u_0(x,t) = -\frac{1}{4t}\big(h_0'(r) + 2rh_0(r)\big) = 0$$

for any $x > 0$ and $t > 0$, *i.e.*, u_0 is a solution of the diffusion equation (1.1) and $u_0 \in U$. By a direct calculation, we easily show that u_0 is explicitly given by

$$u_0(x,t) = \alpha_1 E\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right) + \alpha_2$$

for all $x > 0$ and $t > 0$, where

$$E\mathrm{rf}\,(r) = \frac{2}{\sqrt{\pi}} \int_0^r e^{-s^2} ds$$

is called the error function and α_1 and α_2 are real constants.

Finally, inequality (3.4) is an immediate consequence of (3.10). $\square$

In the following corollary, we introduce an explicit form of control function φ such that φ satisfies the condition (3.1).

Corollary 3.2. *Assume that $\varphi, \psi : (0,\infty) \to [0,\infty)$ are functions and $0 < \varepsilon < 1$ is a constant such that there exist constants c and θ with*

$$\varphi(r) \le \theta r^{-1-\varepsilon} e^{-r^2}, \ \forall r > 0 \tag{3.11}$$

and

$$c := \inf_{t>0} 4t\psi(t) > 0.$$

For each twice continuously differentiable function $u \in U$ satisfying the inequality

$$\big|u_t(x,t) - ku_{xx}(x,t)\big| \le \varphi\left(\frac{x}{\sqrt{4kt}}\right)\psi(t)$$

for all $x > 0$ and $t > 0$, there exists a solution $u_0 : (0,\infty)\times(0,\infty) \to \mathbb{R}$ to the diffusion equation (1.1) *such that $u_0 \in U$ and*

$$|u(x,t) - u_0(x,t)| \le \frac{c\theta}{\varepsilon}\int_{x/\sqrt{4kt}}^{\infty} y^{-\varepsilon}e^{-y^2}dy \tag{3.12}$$

for all $x > 0$ and $t > 0$.

Proof. It follows from (3.11) that

$$\int_y^\infty \varphi(z)e^{z^2}dz \le \int_y^\infty \theta z^{-1-\varepsilon}e^{-z^2}e^{z^2}dz = \frac{\theta}{\varepsilon}y^{-\varepsilon}$$

for all $y > 0$. Moreover, it holds that

$$\begin{aligned}\int_0^\infty e^{-y^2}\int_y^\infty \varphi(z)e^{z^2}dzdy &\le \frac{\theta}{\varepsilon}\int_0^\infty y^{-\varepsilon}e^{-y^2}dy \\ &\le \frac{\theta}{\varepsilon}\int_0^1 y^{-\varepsilon}dy + \frac{\theta}{\varepsilon}\int_1^\infty e^{-y^2}dy \\ &< \infty.\end{aligned}$$

According to Theorem 3.1, there exists a solution $u_0 \in U$ of the diffusion equation (1.1) such that inequality (3.12) holds for all $x > 0$ and $t > 0$. □

Remark 3.3. For all $x > 0$ and $t > 0$ with $0 < x/\sqrt{4kt} < 1$, inequality (3.12) becomes

$$\begin{aligned}|u(x,t) - u_0(x,t)| &\le \frac{c\theta}{\varepsilon}\int_{x/\sqrt{4kt}}^\infty y^{-\varepsilon}e^{-y^2}dy \\ &\le \frac{c\theta}{\varepsilon}\int_{x/\sqrt{4kt}}^1 y^{-\varepsilon}dy + \frac{c\theta}{\varepsilon}\int_1^\infty e^{-y^2}dy \\ &= \frac{c\theta}{\varepsilon(1-\varepsilon)}\left(1 - \left(\frac{x}{\sqrt{4kt}}\right)^{1-\varepsilon}\right) + \frac{\sqrt{\pi}c\theta}{2\varepsilon}\big(1 - E\mathrm{rf}\,(1)\big).\end{aligned}$$

For all $x > 0$ and $t > 0$ with $x/\sqrt{4kt} \ge 1$, inequality (3.12) becomes

$$\begin{aligned}|u(x,t) - u_0(x,t)| &\le \frac{c\theta}{\varepsilon}\int_{x/\sqrt{4kt}}^\infty y^{-\varepsilon}e^{-y^2}dy \\ &\le \frac{c\theta}{\varepsilon}\int_{x/\sqrt{4kt}}^\infty e^{-y^2}dy \\ &= \frac{\sqrt{\pi}c\theta}{2\varepsilon}\left(1 - E\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right).\end{aligned}$$

References

[1] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl., **2** (1998), 373–380.

[2] J. Brzdęk, D. Popa and I. Raşa, *Hyers-Ulam stability with respect to gauges*, J. Math. Anal. Appl., **453** (2017), 620–628.

[3] J. Brzdęk, D. Popa, I. Raşa and B. Xu, *Ulam Stability of Operators*, Mathematical Analysis and Its Applications, Vol. 1, Academic Press, Elsevier, 2018.

[4] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[5] L. C. Evans, Partial Differential Equations, vol. 19 of Graduate Studies in Mathematics, *American Mathematical Society*, 1998.

[6] B. Hegyi and S.-M. Jung, *On the stability of Laplace's equation*, Appl. Math. Lett., **26** (2013), 549–552.

[7] B. Hegyi and S.-M. Jung, *On the stability of heat equation*, Abstr. Appl. Anal., **2013** (2013), Article ID 202373, 4 pages.

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224.

[9] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.

[10] S.-M. Jung, *Hyers-Ulam stability of linear differential equations of first order, II*, Appl. Math. Lett. **19** (2006), 854–858.

[11] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optimization and Its Applications Vol. 48, Springer, New York, 2011.

[12] S.-M. Jung, *On the stability of the heat equation with an initial condition*, J. Inequal. Appl., **2013** (2013), no. 475, 6 pages.

[13] S.-M. Jung and K.-S. Lee, *Hyers-Ulam stability of first order linear partial differential equations with constant coefficients*, Math. Inequal. Appl., **10** (2007), 261–266.

[14] T. Miura, S.-M. Jung and S.-E. Takahasi, *Hyers-Ulam-Rassias stability of the Banach space valued linear differential equations* $y' = \lambda y$, J. Korean Math. Soc., **41** (2004), 995–1005.

[15] Z. Moszner, *Stability has many names*, Aequationes Math., **90** (2016), 983–999.

[16] M. Obłoza, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat., **13** (1993), 259–270.

[17] M. Obłoza, *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Mat., **14** (1997), 141–146.

[18] D. Popa and I. Raşa, *On the Hyers-Ulam stability of the linear differential equation*, J. Math. Anal. Appl., **381** (2011), 530–537.

[19] A. Prástaro and Th. M. Rassias, *Ulam stability in geometry of PDE's*, Nonlinear Funct. Anal. Appl., **8** (2003), 259–278.

[20] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.

[21] S.-E. Takahasi, T. Miura and S. Miyajima, *On the Hyers-Ulam stability of the Banach space-valued differential equation* $y' = \lambda y$, Bull. Korean Math. Soc., **39** (2002), 309–315.

[22] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, 1964.