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COMMON FIXED POINT THEOREMS FOR TWO MAPPINGS IN S-METRIC SPACES

Atena Javaheri[1], Shaban Sedghi[2] and Ho Geun Hyun[3]

[1]Department of Mathematics, Qaemshahr Branch
Islamic Azad University, Qaemshahr, Iran
e-mail: Javaheri.a91@gmail.com

[2]Department of Mathematics, Qaemshahr Branch
Islamic Azad University, Qaemshahr, Iran
e-mail: sedghi_gh@yahoo.com

[3]Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam 51767, Korea
e-mail: hyunhg8285@kyungnam.ac.kr

Abstract. In this paper, we present some definitions of S-metric spaces and prove a common fixed point theorem for two mappings under the condition of weakly compatible mappings in complete S-metric spaces. Also we improved some fixed point theorems in complete S-metric spaces.

1. Introduction

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways.

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[0]Corresponding author: H. G. Hyun(hyunhg8285@kyungnam.ac.kr),
S. Sedghi(sedghi_gh@yahoo.com).

In [9], Jungck introduced more generalized commuting mappings, called *compatible* mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems (see, e.g.,([1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 14]).

Dhage [5] introduced the notion of generalized metric or D-metric spaces and claimed that D-metric defines a Hausdorff topology and that D-metric is sequentially continuous in all the three variables. Many authors used these claims for proving some fixed point theorems in D-metric spaces.

Rhoades [9] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [13] introduced the concept of D-compatibility of maps in D-metric space and proved some fixed point theorems using a contractive condition. Unfortunately, almost all theorems in D-metric spaces are not valid (see [15, 16, 17]).

Recently, Sedghi et al [18, 23] introduced D^*-metric which is a modification of the definition of D-metric introduced by Dhage [5] and prove some basic properties in D^*-metric spaces. Also, Sedghi et al [21] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space [19, 20, 22].

2. Preliminaries

We begin by briefly recalling some basic definitions and results for S-metric spaces that will be needed in the sequel.

Definition 2.1. ([21]) Let X be a nonempty set. An S-metric on X is a function $S : X \times X \times X \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space.

Immediate examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X.
(2) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, then $S(x, y, z) = ||x - z|| + ||y - z||$ is an S-metric on X.

(3) Let X be a nonempty set, d is ordinary metric on X, then $S(x,y,z) = d(x,y) + d(y,z)$ is an S-metric on X.

Lemma 2.2. ([21]) *In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.*

Definition 2.3. ([21]) Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x,r)$ and closed ball $B_S\ [x,r]$ with center x and radius r as follows, respectively:

$$\begin{aligned} B_s(x,r) &= \{y \in X : S(y,y,x) < r\}, \\ B_s[x,r] &= \{y \in X : S(y,y,x) \leq r\}. \end{aligned}$$

Example 2.4. ([21]) Let $X = \mathbb{R}$. Denote $S(x,y,z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_s(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} \\ &= \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} \\ &= (0,2). \end{aligned}$$

Definition 2.5. ([21]) Let (X,S) be an S-metric space and $A \subset X$.

(1) If for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$, then the subset A is called open subset of X.
(2) Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x, y \in A$.
(3) A sequence $\{x_n\}$ in X converges to x if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \Longrightarrow S(x_n,x_n,x) < \varepsilon,$$

and we denote by $\lim_{n\to\infty} x_n = x$.
(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n, m \geq n_0$.
(5) The S-metric spaces (X,S) is said to be complete if every Cauchy sequence is convergent.
(6) Let τ be the set of all $A \subset X$ with $x \in A$ if there exists $r > 0$ such that $B_S(x,r) \subset A$. Then τ is a topology on X(induced by the S-metric S).

Definition 2.6. ([22]) Let (X,S) and (X',S') be two S-metric spaces, and let $f : (X,S) \to (X',S')$ be a function. Then f is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X, $S(x_n,x_n,a) \to 0$ implies $S'(f(x_n),f(x_n),f(a)) \to 0$. A function f is continuous on X if it is continuous at all $a \in X$.

3. Main results

Theorem 3.1. *Let f and g be self-mappings of a complete S-metric space (X, S) satisfying the following conditions:*

(i) *$g(X) \subseteq f(X)$, and $f(X)$ is closed subset of X,*
(ii) *the pair (f, g) is weakly compatible,*
(iii) *$S(gx, gy, gz) \leq \Psi(S(fx, fy, fz))$, for every $x, y, z \in X$,*

where $\Psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\sum_{n=1}^{\infty} \Psi^n(t)$ is convergent for each $t > 0$. Then f and g have a unique common fixed point in X.

Proof. From the conditions on Ψ, it is clear that $\lim_{n\to\infty} \Psi^n(t) = 0$ and $\Psi(t) < t$. Let x_0 be arbitrary point in X. By (i), we can choose a point x_1 in X such that $y_0 = gx_0 = fx_1$ and $y_1 = gx_1 = fx_2$. There exists a sequence $\{y_n\}$ such that, $y_n = gx_n = fx_{n+1}$, for $n = 0, 1, 2, \cdots$. For every $n \in \mathbb{N}$, we prove that the sequence $\{y_n\}$ is a Cauchy sequence. Consider

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &= S(gx_n, gx_n, gx_{n+1}) \\ &\leq \Psi(S(fx_n, fx_n, fx_n)) \\ &= \Psi(S(y_{n-1}, y_{n-1}, y_n)). \end{aligned}$$

Therefore we have

$$S(y_n, y_n, y_{n+1}) \leq \Psi(S(y_{n-1}, y_{n-1}, y_n)),$$

and so $S(y_n, y_n, y_{n+1}) \leq \Psi^n(S(y_0, y_0, y_1))$. Thus for $m > n$ with $n \in \{0, 1, \cdots\}$,

$$\begin{aligned} S(y_n, y_n, y_m) &\leq 2\sum_{i=n}^{m-2} S(y_i, y_i, y_{i+1}) + S(y_{m-1}, y_{m-1}, y_m) \\ &\leq 2\sum_{i=n}^{m-2} \Psi^i\ (S(y_0, y_0, y_1)) + \Psi^{m-1}\ (S(y_0, y_0, y_1)) \\ &\leq 2\sum_{i=n}^{m-1} \Psi^i\ (S(y_0, y_0, y_1)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \Psi^n(t)$ is convergent for each $t > 0$, $\{y_n\}$ is a Cauchy sequence in the $S-$metric space (X, S). By the completeness of X, there exists a $u \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_{n+1} = u.$$

Let $f(X)$ is closed. Then there exist $v \in X$ such that $fv = u$. Now we show that $gv = u$. From inequality (iii) we have that

$$S(gx_n, gx_n, gv) \leq \Psi(S(fx_n, fx_n, fv)).$$

Taking $n \to \infty$, we get

$$S(u, u, gv) \leq \Psi(S(0)) = 0,$$

it implies $gv = u$. Since the pair (f, g) are weakly compatible, we get, $gfv = fgv$. Thus $fu = gu$. Now we prove that $gu = u$. If set x_n, x_n, u replacing x, y, z respectively, in inequality (iii) we get

$$S(gx_n, gx_n, gu) \leq \Psi(S(fx_n, fx_n, fu)).$$

Taking $n \to \infty$, we get

$$S(u, u, gu) \leq \Psi(S(u, u, gu)).$$

If $gu \neq u$, then $S((u, u, gu) < S(u, u, gu)$, is a contradiction. Therefore, $fu = gu = u$.

For the uniqueness, let u and u' be common fixed points of f, g. Taking $x = y = u$ and $z = u'$ in (iii), we have

$$\begin{aligned} S(u, u, u') &= S(gu, gu, gu') \\ &\leq \Psi(S(fu, fu, fu')) = \Psi(S(u, u, u')) \\ &< S(u, u, u'), \end{aligned}$$

which is a contradiction. Thus we have $u = u'$. □

Corollary 3.2. *Let f, g and h be self-mappings of a complete S-metric space (X, S) satisfying the following conditions:*

(i) *$g(X) \subseteq fh(X)$, and $fh(X)$ is closed subset of X,*

(ii) *the pair (fh, g) is weakly compatible and $fh = hf$, $gh = hg$*

(iii) *$S(gx, gy, gz) \leq \Psi(S(fhx, fhy, fhz))$,
for every $x, y, z \in X$, where $\Psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\sum_{n=1}^{\infty} \Psi^n(t)$ is convergent for each $t > 0$.*

Then f, g and h have a unique common fixed point in X.

Proof. By Theorem 3.1, there exist a fixed point $u \in X$ such that $fhu = gu = u$. Now, we prove that $hu = u$. If $hu \neq u$, then in (iii), we have

$$\begin{aligned} S(hu, u, u) &= S(hgu, gu, gu) \\ &= S(ghu, gu, gu) \\ &\leq \Psi(S(fhhu, fhu, fhu)) \\ &= \Psi(S(hu, u, u)) \\ &< S(hu, u, u), \end{aligned}$$

which is a contradiction. Thus we have $hu = u$. Therefore, $fu = fhu = u = hu = gu$. □

Corollary 3.3. *Let g be a self-mapping of a complete S-metric space (X, S) satisfying the following condition:*

$$S(g^n x, g^n y, g^n z) \leq \Psi(S(x, y, z)),$$

for every $x, y, z \in X$ and $n \in \mathbb{N}$, where $\Psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\sum_{n=1}^{\infty} \Psi^n(t)$ is convergent for each $t > 0$. Then g have a unique fixed point in X.

Proof. Replace f with I, the identity map, in Theorem 3.1. Hence the all conditions of Theorem 3.1 are hold and therefore there exists a unique $u \in X$ such that $g^n u = u$. Thus $g^n(gu) = g(g^n u) = gu$. Since u is unique, we have $gu = u$. □

Corollary 3.4. *Let f and g be self-mappings of a complete S-metric space (X, S) satisfying the following conditions:*

(i) *$g^n(X) \subseteq f^m(X)$, and $f^m(X)$ is closed subset of X,*
(ii) *the pair (f^m, g^n) is weakly compatible and $f^m g = g f^m$, $g^n f = f g^n$*
(iii) *$S(g^n x, g^n y, g^n z) \leq \Psi(S(f^m x, f^m y, f^m z))$, for every $x, y, z \in X$ and $n, m \in \mathbb{N}$, where $\Psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\sum_{n=1}^{\infty} \Psi^n(t)$ is convergent for each $t > 0$.*

Then f and g have a unique common fixed point in X.

Proof. By Theorem 3.1 there exist a fixed point $u \in X$ such that $f^m u = g^n u = u$. On the other hand, we have

$$gu = g(g^n u) = g^n(gu) \text{ and } gu = g(f^m u) = f^m(gu).$$

Since u is unique, we have $gu = u$. Similarly, we have $fu = u$. □

Corollary 3.5. *Let (X, S) be a complete S-metric space and let $f_1, f_2, \cdots, f_n, g : X \longrightarrow X$ be maps that satisfy the following conditions:*

(i) *$g(X) \subseteq f_1 f_2 \cdots f_n(X)$;*
(ii) *the pair $(f_1 f_2 \cdots f_n, g)$ is weak compatible, $f_1 f_2 \cdots f_n(X)$ is closed subset of X;*
(iii) *$S(gx, gy, gz) \leq \Psi(S(f_1 f_2 \cdots f_n(x), f_1 f_2 \cdots f_n(y), f_1 f_2 \cdots f_n(z)))$, for all $x, y, z \in X$ and $n \in \mathbb{N}$, where $\Psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\sum_{n=1}^{\infty} \Psi^n(t)$ is convergent for each $t > 0$;*

(iv) $g(f_2 \cdots f_n) = (f_2 \cdots f_n)g,$
$g(f_3 \cdots f_n) = (f_3 \cdots f_n)g,$
$\vdots$
$gf_n = f_n g,$
$f_1(f_2 \cdots f_n) = (f_2 \cdots f_n)f_1,$
$f_1 f_2(f_3 \cdots f_n) = (f_3 \cdots f_n)f_1 f_2,$
$\vdots$
$f_1 \cdots f_{n-1}(f_n) = (f_n)f_1 \cdots f_{n-1}.$

Then $f_1, f_2, \cdots, f_n, g$ have a unique common fixed point.

Proof. By Corollary 3.2, if set $f_1 f_2 \cdots f_n = f$ then f, g have a unique common fixed point in X. That is, there exists $x \in X$, such that $f_1 f_2 \cdots f_n(x) = g(x) = x$. We prove that $f_i(x) = x$, for $i = 1, 2, \cdots$. From (iii), we have

$$\begin{aligned}&S(g(f_2 \cdots f_n x), g(x), g(x))\\ &\le \Psi(S(f_1 f_2 \cdots f_n(f_2 \cdots f_n x), f_1 f_2 \cdots f_n(x), f_1 f_2 \cdots f_n(x))).\end{aligned}$$

By (iv), we get

$$S(f_2 \cdots f_n x, x, x) \le \Psi(S(f_2 \cdots f_n x, x, x)) < S(f_2 \cdots f_n x, x, x).$$

Hence, $f_2 \cdots f_n(x) = x$. Thus , $f_1(x) = f_1 f_2 \cdots f_n(x) = x$. Similarly, we have $f_2(x) = \cdots f_n(x) = x$. □

Now, we give one example to validate Theorem 3.1.

Example 3.6. Let (X, S) be a complete S-metric space, where $X = [0, 2]$ and

$$S(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Define self-maps f and g on X as follows: $fx = \frac{x+1}{2}$ and $gx = \frac{x+5}{6}$, for all $x \in X$. Let $\Psi(t) = \frac{1}{2}t$. Then , we have

$$\begin{aligned}S(gx, gy, gz) &= \frac{1}{6}(|x - y| + |y - z| + |z - x|)\\ &\le \frac{1}{4}(|x - y| + |y - z| + |x - z|)\\ &= \Psi(S(fx, fy, fz).\end{aligned}$$

That is, all conditions of Theorem 3.1 are hold and 1 is the unique common fixed point of f and g.

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