Nonlinear Functional Analysis and Applications
Vol. 15, No. 3 (2010), pp. 345-353

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

SOME CONVERGENCE THEOREMS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN ARBITRARY REAL BANACH SPACES

Zhao-hong Sun[1], Yong-Qin Ni[2], Jong Kyu Kim[3] and Xin-ming Chen[4]

[1]Department of Computer Science
Zhong Kai University of Agriculture and Engineering,
24 Court Dongsha Street Henan-Fangzhi Road,
Guangzhou, guangdong 510225, P. R. China
e-mail: sunzh60@163.com

[2]Department of Physics, Yuxi Normal College
Yuxi, Yunnan 653100, P. R. China
e-mail: wdjy@yxtc.net

[3]Department of Mathematics Education, Kyungnam University
Masan, Kyungnam, 631-701, Korea
e-mail: jongkyuk@kyungnam.ac.kr

[4]Department of Computer Science
Zhong Kai University of Agriculture and Engineering,
24 Court Dongsha Street Henan-Fangzhi Road,
Guangzhou, guangdong 510225, P. R. China
e-mail: cxm@zhku.edu.cn

Abstract. In this paper we study some iterative processes for asymptotically quasi-nonexpansive mappings with error member in arbitrary Banach spaces and then we discuss strong convergence for the processes. Our results extend and improve some recent results.

[0]Received July 13, 2008. Revised July 20, 2009.
[0]2000 Mathematics Subject Classification: Primary 47H05. 47H10 . 47H15.
[0]Keywords: Asymptotically nonexpansive mapping, asymptotically pseudocontractive mapping, arbitrary Banach space, modified Ishikawa iterative sequence with errors.
[0]This work was supported by the National Science Foundation and Teachers Teaching Group Foundation of Zhong Kai University Of Agriculture and Engineering(G2360247).

1. Introduction and preliminaries

Throughout this paper, we assume that E is an arbitrary real Banach space and denote by J the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E.$$

Where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*.

Definition 1.1. Let D be a nonempty subset of E, $T : D \to D$ be a mapping.

(1) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}_0^\infty$ in $[1, +\infty)$ with $\lim\limits_{n\to+\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|$$

for all $x,\ y \in D$ and $n \geq 1$; if the sequence $\{k_n\}$ is a constant sequence $\{1\}$, then T is said to be nonexpansive.

(2) T is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}_0^\infty$ in $[1, +\infty)$ with $\lim\limits_{n\to+\infty} k_n = 1$ such that

$$\|T^n x - q\| \leq k_n\|x - q\|$$

for all $x \in D$ and for all $q \in F(T)$ ($F(T)$ denotes the set of fixed points of T) and $n \geq 1$.

It is well known that if T is nonexpansive, then T is asymptotically nonexpansive with invariant sequence $\{1\}_{n\geq1}$; if T is asymptotically nonexpansive, then T is uniformly L-Lipschitzian where $L = \sup\limits_{n\geq1}\{k_n\}$ and asymptotically quasi-nonexpansive. But the converse is not true in general and an asymptotically quasi-nonexpansive mapping needn't be continuous.

we are now in a position to introduce the following iterative processes for asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings in Banach spaces.

Definition 1.2. Let D be a nonempty convex subset of E and $T : D \to D$ be a mapping. Let $x \in D$ be a given point and $\{\alpha_n\}_0^\infty$, $\{\beta_n\}_0^\infty$, $\{\gamma_n\}_0^\infty$ and $\{\delta_n\}_0^\infty$ are real sequences in $[0, 1]$. Then the sequence $\{x_n\}_0^\infty$ defined by

$$\begin{cases} x \in D \\ x_{n+1} = \alpha_n x + (1 - \alpha_n - \gamma_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n + \gamma_n u_n \qquad n \geq 0 \\ \quad y_n = \beta_n x_n + (1 - \beta_n - \delta_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n + \delta_n v_n \end{cases} \tag{1.1}$$

is called the first type of modified Reich-Takahashi iterative sequence with errors of T, where $\{u_n\}_0^\infty, \{v_n\}_0^\infty$ are arbitrary bounded sequences in D. Especially, if $\beta_n = 1, \delta_n = 0$ for all $n \geq 0$, then $y_n = x_n$ and $\{x_n\}_0^\infty$ defined by

$$\begin{cases} x \in D \\ x_{n+1} = \alpha_n x + (1 - \alpha_n - \gamma_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n + \gamma_n u_n \qquad n \geq 0 \end{cases} \tag{1.2}$$

is called the second type of modified Reich-Takahashi iterative sequence with errors of T.

The conception of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [2] in 1972, they proved that every asymptotically nonexpansive self-mapping defined on a nonempty bounded closed convex subset of an uniformly convex Banach space has a fixed point.

The iterative algorithm for fixed points of maps is one of the main aspects in studying nonlinear analysis. Many authors have considered some special cases of sequence (1.1) and (1.2).

(1) If $E = H$ is a Hilbert space and $T : D \to D$ is a nonexpansive mapping, then the sequence $\{x_n\}$ defined by (1.2) was introduced and studied in Shimizu and Takahashi [5]. They showed that if $F(T) \neq \emptyset$ and if $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1, \alpha_n \to 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ converges strongly to some point in $F(T)$ which is nearest to x in $F(T)$.

(2) Let $E = H$ is a Hilbert space and $T : D \to D$ is a nonexpansive mapping and $\{x_n\}$ be the sequence defined by

$$\begin{cases} x \in D \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n \qquad n \geq 0 \end{cases} \tag{1.3}$$

Wittmann [8] showed that if $\{\alpha_n\}$ satisfies the following condition:

$$0 \leq \alpha_n \leq 1, \alpha_n \to 0, \sum_{n=0}^{\infty} \alpha_n = \infty, and \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then the sequence $\{x_n\}$ converges strongly to some fixed point of T in D.

Recently, Chang in [1] has proved the following theorem:

Theorem 1.3. *Let E be a real Banach space whose norm is Gâteaux differentiable, D be a nonempty closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty), k_n \to 1, \sum_{n=0}^{\infty}(e_n - 1) < \infty$, where*

$$e_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j \geq 1, \qquad \forall n \geq 0 \tag{1.4}$$

and let $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:

$$\alpha_n \to 0 (n \to \infty) \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$$

for any given $x \in D$ and for any $n \geq 1$ define a contractive mapping $S_n : D \to D$ as follows:

$$S_n(z) = (1 - d_n)x + d_n T^n z$$

where

$$d_n = \frac{t_n}{k_n}, n = 1, 2, \cdots, t_n \in (0, 1), \ t_n \to 1 \quad (as\ n \to \infty).$$

Let z_n be the unique fixed point of S_n, i.e., z_n satisfies the following:

$$z_n = S_n(z_n) = (1 - d_n)x + d_n T^n z_n, \qquad n \geq 1.$$

If $\{z_n\}$ converges strongly to some $z \in F(T)$ as $n \to \infty$, then the first type of modified Reich-Takahashi sequence $\{x_n\}$ defined by (1.1) converges strongly to the fixed point z if and only if $\{y_n\}$ defined by (1.1) is bounded.

Liu [3] studied the Ishikawa iterative approximation problems of fixed points for asymptotically quasi-nonexpansive mappings with error member in Banach spaces. He gave the necessary and sufficient condition for the Ishikawa iterative sequences to converge to fixed points of these mappings. Inspired by their idea, in this paper we will continue studying these problems and to extend the results of [1] from a real Banach space whose norm is Gâteaux differentiable to arbitrary Banach spaces and of [3] from the modified Ishikawa and Mann iterative sequences with error members to the first type of Reich-Takahashi iterative sequence with errors of T. We will give some necessary and sufficient condition for the sequence to converge to fixed points in arbitrary real Banach spaces. The method proving the main result in this paper is also quite different and of a little more succinct. Then results presented in this paper thus extend and improve the main results in [1-8].

2. Main results

Our results are the following.

Theorem 2.1. *Suppose that E be an arbitrary real Banach space, D be a nonempty closed convex subset of E, $T : D \to D$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty), k_n \to 1$, $\sum_{n=0}^{\infty}(e_n - 1) < \infty$, where*

$$e_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j \geq 1, \qquad \forall n \geq 0 \tag{2.1}$$

(T need not be continuous), and let $F(T) \neq \emptyset$. Suppose $\{\alpha_n\}_0^\infty$, $\{\beta_n\}_0^\infty$, $\{\gamma_n\}_0^\infty$ and $\{\delta_n\}_0^\infty$ are real sequence in $[0, 1]$ satisfying the following conditions:

(a) $\alpha_n + \gamma_n \leq 1, \quad \beta_n + \delta_n \leq 1, \qquad n = 0, 1, 2, \cdots;$

(b) $\sum_{n=0}^{\infty} \alpha_n < +\infty, \quad \sum_{n=0}^{\infty} \gamma_n < +\infty, \quad \sum_{n=0}^{\infty} \delta_n < +\infty.$

Then the first type of modified Reich-Takahashi sequence $\{x_n\}$ defined by (1.1) converges strongly to the fixed point p of T if and only if $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$, where $d(y, C)$ denotes the distance of y to set C; i.e., $d(y, C) = \inf_{\forall x \in C} d(y, x)$.

Theorem 2.2. *Suppose that E be an arbitrary real Banach space, D be a nonempty closed convex subset of E, $T : D \to D$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty), k_n \to 1$, $\sum_{n=0}^{\infty}(e_n - 1) < \infty$, and let $F(T) \neq \emptyset$. Suppose $\{\alpha_n\}_0^\infty$, $\{\gamma_n\}_0^\infty$ are real sequence in $[0, 1]$ satisfying the following conditions:*

(a) $\alpha_n + \gamma_n \leq 1, \qquad n = 0, 1, 2, \cdots;$

(b) $\sum_{n=0}^{\infty} \alpha_n < +\infty, \quad \sum_{n=0}^{\infty} \gamma_n < +\infty.$

Then the second type of modified Reich-Takahashi sequence $\{x_n\}$ defined by (1.2) converges strongly to the fixed point p of T if and only if $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$.

Theorem 2.3. *Suppose that E be an arbitrary real Banach space, D be a nonempty closed convex subset of E, $T : D \to D$ be a non-expansive mapping and let $F(T) \neq \emptyset$. Suppose the conditions for the iterative parameters as same as in Theorem2.1. Then the modified Reich-Takahashi sequence $\{x_n\}$ defined by (1.1) or (1.2) converges strongly to the fixed point p of T if and only if $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$.*

Theorem 2.4. *Suppose that E be an arbitrary real Banach space, D be a nonempty closed convex subset of E, $T : D \to D$ be a quasi-nonexpansive* *mapping* (T *need not be continuous*) *and let* $F(T) \neq \emptyset$. *Suppose the conditions for the iterative parameters as same as in Theorem2.1. Then the modified Reich-Takahashi sequence* $\{x_n\}$ *defined by* (1.1) *or* (1.2) *converges strongly to the fixed point* p *of* T *if and only if* $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$.

In order to prove the above theorem, the following lemma plays an important role.

Lemma 2.5. ([3]) *Let* $\{a_n\}_0^\infty, \{b_n\}_0^\infty$, *and* $\{t_n\}_0^\infty$ *be nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 + t_n)a_n + b_n,$$

for all $n \in N$, *and* $\sum_{n=0}^{\infty} t_n < +\infty$, $\sum_{n=0}^{\infty} b_n < +\infty$. *Then*

(a) $\lim_{n\to+\infty} a_n$ *exists.*

(b) *If* $\liminf_{n\to+\infty} a_n = 0$, *then* $\lim_{n\to+\infty} a_n = 0$.

3. Proofs of the main results

Proof of Theorem 2.1. The necessity of the conditions is obvious. Thus we will only prove the sufficiency. For any $q \in F(T)$, from (1.1) and by the definition of normalized duality mapping we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|\alpha_n(x - q) + (1 - \alpha_n - \gamma_n)\frac{1}{n+1}\sum_{j=0}^{n}(T^j y_n - q) + \gamma_n(u_n - q)\|^2 \\
&= \langle \alpha_n(x - q) + (1 - \alpha_n - \gamma_n)\frac{1}{n+1}\sum_{j=0}^{n}(T^j y_n - q) + \gamma_n(u_n - q), J(x_{n+1} - q)\rangle \\
&= \langle \alpha_n(x - q), J(x_{n+1} - q)\rangle + (1 - \alpha_n - \gamma_n)\frac{1}{n+1}\sum_{j=0}^{n}\langle (T^j y_n - q, J(x_{n+1} - q)\rangle \\
&\quad + \gamma_n\langle u_n - q, J(x_{n+1} - q)\rangle \\
&\leq \alpha_n\|x - q\|\|x_{n+1} - q\| + e_n\|y_n - q\|\|x_{n+1} - q\| \\
&\quad + \gamma_n\|u_n - q\|\|x_{n+1} - q\| \qquad \forall n \geq 0.
\end{aligned}$$

No matter what $\|x_{n+1} - q\|$ is zero or not, from above inequality, we always obtain that

$$\|x_{n+1} - q\| \leq e_n\|y_n - q\| + \alpha_n\|x - q\| + \gamma_n\|u_n - q\|, \quad \forall n \geq 0. \tag{3.1}$$

Now we consider $\|y_n - q\|$ which follows from (1.1) that

$$\begin{aligned}
&\|y_n - q\| \\
&= \|(1-\beta_n-\delta_n)\frac{1}{n+1}\sum_{j=0}^{n}(T^j x_n - q) + \beta_n(x_n - q) + \delta_n(v_n - q)\| \\
&\leq (1-\beta_n-\delta_n)e_n\|x_n - q\| + \beta_n\|x_n - q\| + \delta_n\|v_n - q\| \\
&= [(1-\beta_n-\delta_n)e_n + \beta_n]\|x_n - q\| + \delta_n\|v_n - q\| \\
&\leq e_n\|x_n - q\| + \delta_n\|v_n - q\| \qquad \forall n \geq 0.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1) we have

$$\begin{aligned}
\|x_{n+1} - q\| &\leq e_n^2\|x_n - q\| + e_n\delta_n\|v_n - q\| + \alpha_n\|x - q\| + \gamma_n\|u_n - q\| \\
&= [1 + (e_n^2 - 1)]\|x_n - q\| + (\delta_n + \alpha_n + \gamma_n)M_1
\end{aligned}$$

where $M_1 = \max\{\|x - q\|, \sup_{n\geq 0} e_n\|v_n - q\|, \sup_{n\geq 0}\|u_n - q\|\}$.

Let $a_n = \|x_{n+1} - q\|, t_n = e_n^2 - 1, b_n = (\delta_n + \alpha_n + \gamma_n)M_1$, then $\sum\limits_{n=0}^{\infty} t_n < +\infty$, $\sum\limits_{n=0}^{\infty} b_n < +\infty$, and above inequality becomes $\|x_{n+1} - q\| \leq (1 + t_n)\|x_n - q\| + b_n$, so it is followed from lemma2.5 that $\lim\limits_{n\to+\infty}\|x_n - q\|$ exists and we have

$$\lim_{n\to+\infty} d(x_n, F(T)) = 0$$

by the hypothesis of theorem. By using inequality $1 + x \leq \exp x (\forall x \geq 0)$ and above inequality for all positive integer $m \geq 1$ we have

$$\begin{aligned}
\|x_{n+m} - q\| &\leq (1 + t_{n+m-1})\|x_{n+m-1} - q\| + b_{n+m-1} \\
&\leq \exp(t_{n+m-1})\|x_{n+m-1} - q\| + b_{n+m-1} \\
&\leq \exp(\sum_{k=n}^{n+m-1} t_k)\|x_n - q\| + \exp(\sum_{k=n}^{n+m-1} t_k)\sum_{k=n}^{n+m-1} b_k \\
&\leq M\|x_n - q\| + M\sum_{k=n}^{n+m-1} b_k
\end{aligned}$$

where $M = \exp(\sum\limits_{k=n}^{+\infty} t_k)$. Thus so far we have showed that

(c) $\|x_{n+m} - q\| \leq M\|x_n - q\| + M\sum\limits_{k=n}^{n+m-1} b_k$.

(d) $\lim\limits_{n\to+\infty} d(x_n, F(T)) = 0$.

So it is easy to show that $\{x_n\}_{n=1}^{\infty}$ must be Cauchy sequence. Because for all $\epsilon > 0$, since $\lim\limits_{n\to+\infty} d(x_n, F(T)) = 0$ and $\sum\limits_{n=0}^{\infty} b_n < \infty$, there must exist a constant N_1, such that when $n \geq N_1$,

$$d(x_n, F(T)) \leq \frac{\epsilon}{4M} \qquad \textit{and} \qquad \sum_{k=n}^{\infty} b_k \leq \frac{\epsilon}{6M},$$

and so $d(x_{N_1}, F(T)) \leq \frac{\epsilon}{4M}$. There must exist $\bar{p} \in F(T)$, such that

$$d(x_{N_1}, \bar{p}) \leq \frac{\epsilon}{3M}.$$

From (c) above it can be obtained that when $n \geq N_1$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - \bar{p}\| + \|\bar{p} - x_n\| \\ &\leq M\|x_{N_1} - \bar{p}\| + M\sum_{k=N_1}^{\infty} b_k + M\|x_{N_1} - \bar{p}\| + M\sum_{k=N_1}^{\infty} b_k \\ &\leq M\,\frac{\epsilon}{3M} + M\,\frac{\epsilon}{6M} + M\,\frac{\epsilon}{3M} + M\,\frac{\epsilon}{6M} \\ &= \epsilon.\end{aligned}$$

This implies $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. The space is complete, thus $\lim\limits_{n\to+\infty} x_n$ exists. Let $\lim\limits_{n\to+\infty} x_n = p$. We will show that p is a fixed point, i.e., $p \in F(T)$. For all $\epsilon > 0$, since $\lim\limits_{n\to+\infty} x_n = p$ and $\lim\limits_{n\to+\infty} d(x_n, F(T)) = 0$, there must exist a constant N, such that when $n \geq N$,

$$\|x_n - p\| \leq \frac{\epsilon}{2(k_1+1)} \qquad \textit{and} \qquad d(x_n, F(T)) \leq \frac{\epsilon}{3(k_1+1)}.$$

And so there must exist a $p_1 \in F(T)$, such that

$$d(x_N, p_1) \leq \frac{\epsilon}{2(k_1+1)}.$$

Hence we have

$$\begin{aligned}\|Tp - p\| &= \|Tp - p_1 + p_1 - p\| \leq (k_1+1)\|p - p_1\| \\ &\leq (k_1+1)[\|p - x_N\| + \|x_N - p_1\|] \\ &\leq \epsilon.\end{aligned}$$

Thus $Tp = p$, i.e., p is a fixed point by the arbitrary property of ϵ. With this, the proof of Theorem2.1 is completed. □

Theorem 2.2 can be proved by Theorem 2.1. Using the same method, Theorem 2.3 and Theorem 2.4 can be proven.

Remark 3.1. Our results extend the results of [1] from real Banach spaces whose norm is Gâteaux differentiable to arbitrary real Banach spaces. It is also easy to see that our results are significant extensions of the results of [2-8] to arbitrary real Banach spaces and to the more general classes of mappings considered here. Moreover, the boundedness of domain is removed, and the method proving the main result in this paper is new and succinct. It is worth to mention that the statement in our theorems are simpler than that in Theorems in [1] and that the restriction for the iterative parameter $\{\alpha_n\}$ which satisfies $\sum_{n=0}^{+\infty} \alpha_n = \infty$ in [1, theorems] is replaced by considering the condition that $\sum_{n=0}^{+\infty} \alpha_n < \infty$ and that the proof is completed in our paper only by using one single auxiliary lemma. We also mention that the iterative process in [1] is extended to that with error members.

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